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AZUL

I. PROBLEM STATEMENT

Azul is a board game in which the goal is to obtain the maximum number of points as a square is sequentially filled in. Let us suppose a 3x3 square, in which points will be obtained as follows:

There are 9 boxes that must be filled in with an X. Whenever a box is filled in, the player will obtain as many points as adjacent rows or columns are formed with that X.

II. DESCRIPTION AND HYPOTHESES

The model that we propose for solving the problem is a Mixed Integer Linear model. Each position in the square has a different range of possibilities for obtaining an amount of points depending on the number of X that are placed in the adyacent positions. The model that we propose not only takes into account the adyacent positions completed, but also the posible combinations of them that may affect to the outcome of points. To do this, we have not only created variables for points and completed positions, but also for combinations of positions completed. This is crucial, because it allows the model to give the correct amount of points in each turn.

III. MATHEMATICAL FORMULATION OF THE OPTIMIZATION PROBLEM

A. Sets

$i \in \{1, \ldots, 9\}$ turn

$j, k, l, m, n \in \{1, \ldots, 9\}$ position

B. Parameters

ε little positive ($> 0, \sim 0$) auxiliar parameter

M big positive ($\sim \infty$) auxiliar parameter

m big negative ($\sim -\infty$) auxiliar parameter

C. Variables

1. Integer variables

p_{ij} points obtained for completing position j in turn i

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2. Binary variables

c_{ij} position *j* completed in turn *i*

c_{ijk} one position *k* completed before position *j* is completed in turn *i*

c_{ijkl} two positions *k* and *l* completed before position *j* is completed in turn *i*

c_{ijklm} three positions *k*, *l* and *m* completed before position *j* is completed in turn *i*

c_{ijklmn} four positions *k*, *l*, *m* and *n* completed before position *j* is completed in turn *i*

D. Equations:

1. One turn per position

$$\sum_{i} c_{ij} = 1 \quad \forall j$$

2. One position per turn

$$\sum_{j} c_{ij} = 1 \quad \forall i$$

3. $\sum_{a=1}^{i-1} c_{ak} + c_{ij} \geq 2 \leftrightarrow c_{ijk} = 1$

$$\sum_{a=1}^{i-1} c_{ak} + c_{ij} \geq 2 + m \times \left(1 - c_{ijk}\right) \quad \forall i, j, k$$

$$\sum_{a=1}^{i-1} c_{ak} + c_{ij} \leq 2 - \varepsilon + (M + \varepsilon) \times c_{ijk} \quad \forall i, j, k$$

4. $\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + c_{ij} \geq 3 \leftrightarrow c_{ijkl} = 1$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + c_{ij} \geq 3 + m \times \left(1 - c_{ijkl}\right) \quad \forall i, j, k, l$$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + c_{ij} \leq 3 - \varepsilon + (M + \varepsilon) \times c_{ijkl} \quad \forall i, j, k, l$$

5. $\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + c_{ij} \geq 4 \leftrightarrow c_{ijklm} = 1$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + c_{ij} \geq 4 + m \times \left(1 - c_{ijklm}\right) \quad \forall i, j, k, l, m$$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + c_{ij} \leq 4 - \varepsilon + (M + \varepsilon) \times c_{ijklm} \quad \forall i, j, k, l, m$$

6. $\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + \sum_{a=1}^{i-1} c_{an} + c_{ij} \geq 5 \leftrightarrow c_{ijklmn} = 1$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + \sum_{a=1}^{i-1} c_{an} + c_{ij} \geq 5 + m \times \left(1 - c_{ijklmn}\right) \quad \forall i, j, k, l, m, n$$

$$\sum_{a=1}^{i-1} c_{ak} + \sum_{a=1}^{i-1} c_{al} + \sum_{a=1}^{i-1} c_{am} + \sum_{a=1}^{i-1} c_{an} + c_{ij} \leq 5 - \varepsilon + (M + \varepsilon) \times c_{ijklmn} \quad \forall i, j, k, l, m, n$$

7. Points per turn and position

$$p_{i1} = c_{i1} + c_{i12} + c_{i14} + c_{i123} + c_{i124} + c_{i147} \quad \forall i$$

$$p_{i2} = c_{i2} + c_{i21} + c_{i23} + c_{i25} + c_{i258} + c_{i215} + c_{i235} - c_{i2135} \quad \forall i$$

$$p_{i3} = c_{i3} + c_{i32} + c_{i36} + c_{i312} + c_{i369} + c_{i326} \quad \forall i$$

$$p_{i4} = c_{i4} + c_{i41} + c_{i45} + c_{i47} + c_{i456} + c_{i415} + c_{i457} - c_{i4157} \quad \forall i$$

$$p_{i5} = c_{i5} + c_{i52} + c_{i54} + c_{i56} + c_{i58} + c_{i524} + c_{i526} + c_{i548} + c_{i568} - c_{i5246} - c_{i5268} - c_{i5248} - c_{i5468} + c_{i52468} \quad \forall i$$

$$p_{i6} = c_{i6} + c_{i63} + c_{i65} + c_{i69} + c_{i645} + c_{i635} + c_{i659} - c_{i6359} \quad \forall i$$

$$p_{i7} = c_{i7} + c_{i74} + c_{i78} + c_{i714} + c_{i748} + c_{i789} \quad \forall i$$

$$p_{i8} = c_{i8} + c_{i85} + c_{i87} + c_{i89} + c_{i825} + c_{i857} + c_{i859} - c_{i8579} \quad \forall i$$

$$p_{i9} = c_{i9} + c_{i96} + c_{i98} + c_{i936} + c_{i968} + c_{i978} \quad \forall i$$

E. Objective Function

Maximize the total number of points

$$\max \left\{ \sum_{i,j} p_{ij} \right\}$$