National University of Colombia
Department of Mathematics

Introduction to Optimization
Linear Matrix Inequality

Jorge Luis Castillo Orduz . . . .

Show that the solution set of a linear matrix inequality is convex.

Solution

Given a linear matrix polynomial $A : \mathbb{R}^n \to SM(\mathbb{R}^{mxm})$ defined by:

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A - n$$

where $SM(\mathbb{R}^{mxm})$ is the set of mxm symetric matrices of real numbers, and matrices $A_0, A_1, \cdots, A_n$ are symetric as well. We can form the following linear matrix inequality $A(x) \preceq B_m$ whose solution set is $S := \{x \in \mathbb{R}^n \mid A(x) \preceq B_m\}$. Then, this set S is the one to demonstrate is convex. Let $x, y \in S$ and $\lambda \in [0, 1]$. Hence:

$$\begin{aligned}
A(\lambda x + (1-\lambda)y) &\preceq B_m \\
A_0 + (\lambda x_1 + (1-\lambda)y_1)A_1 + \cdots + (\lambda x_n + (1-\lambda)y_n)A_n &\preceq B_m \\
A_0 + \lambda x_1 A_1 + (1-\lambda)yA_1 + \cdots + \lambda x_n A_n + (1-\lambda)yA_n &\preceq B_m \\
A_0 + \lambda(x_1 A_1 + \cdots + x_n A_n) + (1-\lambda)(y_1 A_1 + \cdots + y_n A_n) &\preceq B_m \\
\lambda A(x) + (1-\lambda)A(y) &\preceq B_m
\end{aligned}$$

Since $x, y \in S$ then $A(x) \preceq B_m$ and $A(y) \preceq B_m$. Since a convex combination of two positive semidefinite matrices is also positive semidefinite, we conclude that set S is convex.