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Introduction to Optimization
Convex Sets Exercises

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From the guide book solve the following exercises:

2.1 Let $C \subseteq \mathbb{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint*. Use induction on k.

Solution

We will use induction on k as the *Hint* suggests. It is important to highlight that the definition of convex sets shows that this is true for $k = 2$, so we will start our induction with the base case $k = 3$.

- **Base Case:** $k = 3$
  Let C be a convex set and let's suppose that $x_1, x_2, x_3 \in C$. Additionally, let $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ be the coefficients with $\theta_1 + \theta_2 + \theta_3 = 1$ and $\theta_1, \theta_2, \theta_3 \geq 0$. We will see that $y = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$. We have multiple cases, one for each case where $\theta_i = 1$, so the rest of the coefficients will be zero and $y = x_i$ for some $i = 1, 2, 3$. In any of these previous cases $y \in C$ because our hypothesis holds that $x_i \in C$ for $i = 1, 2, 3$. Now, we know that at least one of the $\theta_i \neq 1$, without loss of generality we can assume that $\theta_1 \neq 1$. We can write our expression like:

$$
\begin{aligned}
y &= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\
&= \theta_1 x_1 + (1 - \theta_1) \left(\frac{\theta_2 x_2 + \theta_3 x_3}{1 - \theta_1} \right) \\
&= \theta_1 x_1 + (1 - \theta_1) \left(\frac{\theta_2 x_2}{1 - \theta_1} + \frac{\theta_3 x_3}{1 - \theta_1} \right)
\end{aligned}
$$

  We can rename the coefficients in order to make calculations easier.

$$
\beta_2 = \frac{\theta_2}{1 - \theta_1} \qquad \beta_3 = \frac{\theta_3}{1 - \theta_1}
$$

So, we end up with:

$$y = \theta_1 x_1 + (1 - \theta_1)(\beta_2 x_2 + \beta_3 x_3). \tag{1}$$

Let's notice two important things here. As $\theta_1 \neq 1$ we can be sure that $\beta_2, \beta_3 \geq 0$. And moreover:

$$\beta_2 + \beta_3 = \frac{\theta_2}{1 - \theta_1} + \frac{\theta_3}{1 - \theta_1}$$
$$\beta_2 + \beta_3 = \frac{\theta_2 + \theta_3}{1 - \theta_1}$$

Remember that our hypothesis tells us that $\theta_1 + \theta_2 + \theta_3 = 1$, so $\theta_2 + \theta_3 = 1 - \theta_1$. So, replacing we will get:

$$\beta_2 + \beta_3 = \frac{1 - \theta_1}{1 - \theta_1} = 1$$

Thus, $\beta_2, \beta_3 \geq 0$, $\beta_2 + \beta_3 = 1$, $x_2, x_3 \in C$ and as C is a convex set, then the point $\boldsymbol{\beta_2 x_2 + \beta_3 x_3 \in C}$.

Now, if we take a look of y in the equation (1), we see:

- $x_1, (\beta_2 x_2 + \beta_3 x_3) \in C$
- $\theta_1, (1 - \theta_1) \geq 0$
- $\theta_1 + (1 - \theta_1) = 1$

Thus, as C is a convex set, and we have two points of C, due to the definition of convex set, then the point $\boldsymbol{y \in C}$.

- **Induction Hypothesis**

  Let's suppose that for an arbitrary $n \in \mathbb{N}$, it is true that given $x_1, \dots, x_n \in C$, $\theta_1, \dots, \theta_n \in \mathbb{R}$ with $\theta_i \geq 0$ and $\theta_1 + \cdots + \theta_n = 1$, then we have $\theta_1 x_1 + \cdots + \theta_n x_n \in C$.

- **Inductive Step**

  Let's see what happens with $n + 1$. Suppose that $x_1, \dots, x_{n+1} \in C$. Additionally, let $\theta_1, \dots, \theta_{n+1} \in \mathbb{R}$ be the coefficients with $\theta_1, \dots, \theta_{n+1} \geq 0$ and $\theta_1 + \cdots + \theta_{n+1} = 1$. We will see that $y = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_{n+1} x_{n+1} \in C$. As we did in the Base Case, we know that at least one of the $\theta_i \neq 1$.

Let's assume, without loss of generality, that $\theta_{n+1} \neq 1$ and write our expression like:

$$\begin{aligned}
y &= \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n + \theta_{n+1} x_{n+1} \\
&= \theta_{n+1} x_{n+1} + (1-\theta_{n+1}) \left(\frac{\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n}{1-\theta_{n+1}} \right) \\
&= \theta_{n+1} x_{n+1} + (1-\theta_{n+1}) \left(\frac{\theta_1 x_1}{1-\theta_{n+1}} + \frac{\theta_2 x_2}{1-\theta_{n+1}} + \cdots + \frac{\theta_n x_n}{1-\theta_{n+1}} \right)
\end{aligned}$$

Once again, lets rename the variables:

$$\beta_1 = \frac{\theta_1}{1-\theta_{n+1}} \qquad \beta_2 = \frac{\theta_2}{1-\theta_{n+1}} \qquad \cdots \qquad \beta_n = \frac{\theta_n}{1-\theta_{n+1}}$$

Then, we will get the following expression:

$$y = \theta_{n+1} x_{n+1} + (1-\theta_{n+1}) \left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \right) \tag{2}$$

We know that $\theta_{n+1} \neq 1$, which allows us to be sure that each $\beta_i \geq 0$. Additionally, remember that $\theta_1 + \cdots + \theta_n + \theta_{n+1} = 1$, which means that $\theta_1 + \cdots + \theta_n + = 1 - \theta_{n+1}$. Now, we can sum up all the β's:

$$\begin{aligned}
\beta_1 + \beta_2 + \cdots + \beta_n &= \frac{\theta_1}{1-\theta_{n+1}} + \frac{\theta_2}{1-\theta_{n+1}} + \cdots + \frac{\theta_n}{1-\theta_{n+1}} \\
&= \frac{\theta_1 + \theta_2 + \cdots + \theta_n}{1-\theta_{n+1}} \\
&= \frac{1-\theta_{n+1}}{1-\theta_{n+1}} \\
\beta_1 + \beta_2 + \cdots + \beta_n &= 1
\end{aligned}$$

So, for $i = 1, 2, \ldots, n$ we have $\beta_i \geq 0$ and $\beta_1 + \beta_2 + \cdots + \beta_n = 1$. Thus, thanks to our **induction hypothesis** we can affirm that $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \in C$, because we have n points of C. I'd like to mention that we can get this conclusion independently of the $\theta_i \neq 1$ we assume, because the number of points will always be n in the end. Now, if we take a look of y in the equation (2), we see:

- $x_{n+1}, (\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n) \in C$
- $\theta_{n+1}, (1-\theta_{n+1}) \geq 0$
- $\theta_{n+1} + (1-\theta_{n+1}) = 1$

Thus, as C is a convex set, and we have two points of C, due to the definition of convex set then the point $\boldsymbol{y} \in \boldsymbol{C}$.

$\square$

2.3 *Midpoint convexity.* A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Solution

Let C be a closed and midpoint convex set. We will show that for all $\theta \in [0,1]$ and for all $x, y \in C$, $\theta x + (1-\theta)y \in C$. Now, let's define $\theta^{(k)}$ as the binary representation of length k of a decimal number, this number will be of the form:

$$\theta^{(k)} = c_1 2^{-1} + c_2 2^{-2} + \cdots + c_k 2^{-k}$$

the coefficients $c_i \in \{0,1\}$, $k \in \mathbb{N}$ and $\theta^{(k)}$ will be the closest number of this form to θ. It is important to highlight that $0 \leq \theta^{(k)} \leq 1$, and now we can replace this representation and get:

$$\begin{aligned}
z &= \theta x + (1-\theta)y \\
z^{(k)} &= \theta^{(k)} x + (1-\theta^{(k)})y \\
z^{(k)} &= (c_1 2^{-1} + c_2 2^{-2} + \cdots + c_k 2^{-k})x + (1 - (c_1 2^{-1} + c_2 2^{-2} + \cdots + c_k 2^{-k}))y \\
z^{(k)} &= \left(\frac{c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k}{2^k}\right) x + \left(1 - \left(\frac{c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k}{2^k}\right)\right) y \\
z^{(k)} &= \left(\frac{c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k}{2^k}\right) x + \left(\frac{2^k - c_1 2^{(k-1)} - c_2 2^{(k-2)} - \cdots - c_k}{2^k}\right) y
\end{aligned}$$

We can conclude several things from this new point of view. First:

$$\left(\frac{c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k}{2^k}\right) + \left(\frac{2^k - c_1 2^{(k-1)} - c_2 2^{(k-2)} - \cdots - c_k}{2^k}\right) = \frac{2^k}{2^k} = 1$$

Second:

$$\left(\frac{c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k}{2^k}\right) \geq 0 \qquad \left(\frac{2^k - c_1 2^{(k-1)} - c_2 2^{(k-2)} - \cdots - c_k}{2^k}\right) \geq 0$$

And last but not least, we can see this expression as the midpoint convexity applied recursively k times:

$$(c_1 2^{(k-1)} + c_2 2^{(k-2)} + \cdots + c_k)\frac{x}{2^k} + (2^k - c_1 2^{(k-1)} - c_2 2^{(k-2)} - \cdots - c_k)\frac{y}{2^k}$$

Thus, as C is a midpoint convex set, then $z^{(k)} = \theta^{(k)}x + (1 - \theta^{(k)})y \in C$. Now we will use the fact that C is closed, by making k tend to infinity, which means:

$$\lim_{k\to\infty} \theta^{(k)}x + (1 - \theta^{(k)})y = \theta x + (1 - \theta)y$$

We can state this because C is closed and moreover, we can see this process as applying midpoint convexity infinite times. Finally, we can affirm that $z = \theta x + (1 - \theta)y \in C$. Thus, C is convex.

$\square$

2.4 Show that the convex hull of a set S is the intersection of all convex sets that contain S. (The same method can be used to show that the conic, or affine, or linear hull of a set S is the intersection of all conic sets, or affine sets, or subspaces that contain S.)

Solution

Let S be a set and C its convex hull. Now, let D be the intersection of all convex sets that contains S:

$$D = \bigcap\{E \mid E \text{ is convex and } S \subseteq E\}$$

We will show that $C = D$ using the axiom of extension.

- $C \subseteq D$
  Let $\boldsymbol{x \in C}$ be an arbitrary element of the convex hull of S, which means that x is a convex combination of some points $x_1, \ldots, x_n \in S$. Now, let E be any convex set that belongs to D. As $E \in D$, then $S \subseteq E$, which means that $x_1, \ldots, x_n \in E$. Let's remember that E is convex by definition as well, meaning that any convex combination of $x_1, \ldots, x_n$ belongs to E, so we can affirm that $x \in E$. As we can say the same for any convex set E that contains S, then x is in the intersection of all sets E and $\boldsymbol{x \in D}$.
- $D \subseteq C$
  Since C is convex by definition and $S \subseteq C$, then we must have that $C = E$ for some E of the construction of D. Additionally, the convex hull C is the smallest convex set that contains S, which means that the intersection of all convex sets will be contained in C.

Thus, we conclude that $C = D$

$\square$

2.5 What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x = b_2\}$?

Solution

Let's name our hyperplanes:

$$H_1 = \{x \in \mathbb{R}^n \mid a^T x = b_1\} \qquad H_2 = \{x \in \mathbb{R}^n \mid a^T x = b_2\}$$

From the curse of Linear Algebra we know that the distance between two parallel hyperplanes is given by:

$$d(H_1, H_2) = \frac{|b_1 - b_2|}{\|a\|}$$

where $\|a\|$ is the Euclidean norm $\|a\|_2$.

2.8 Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x \mid Ax \preceq b,\ Fx = g\}$.

Solution

a) $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1,\ -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbb{R}^n$.

The definition of polyhedron states that it is the intersection of a finite number of halfspaces and hyperplanes. We can see this set as the intersection of 3 sets:

- S_1: the plane defined by a_1 and a_2
- $S_2 = \{z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T = 0,\ -1 \leq y_1 \leq 1\}$
- $S_3 = \{z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T = 0,\ -1 \leq y_2 \leq 1\}$

Thus, the set is a polyhedron.

b) $S = \{x \in \mathbb{R}^n \mid x \succeq 0,\ 1^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

We can clearly see that $x \succeq 0$ is a halfspace. Additionally $1^T x = 1$, $\sum_{i=1}^n x_i a_i = b_1$ and $\sum_{i=1}^n x_i a_i^2 = b_2$ are hyperplanes. Thus, this set is a polyhedron as well.

c) $S = \{x \in \mathbb{R}^n \mid x \succeq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$.

First, let's take a look of $x^T y \leq 1$ and $\|y\|_2 = 1$. We know from the Cauchy-Schwarz inequality:

$$|x\ y| \leq \|x\|\|y\|$$
$$|x\ y| \leq \|x\|$$

But this is constrained by $x^T y \leq 1$, which implies that $\|x\| \leq 1$ and this is a euclidean ball. We intersect this ball with the nonnegative orthant $\mathbb{R}^n_+$. We will end up with the section of a ball, but to represent that section we need infinite number of intersected hyperplanes. However, our definition is limited to finite number of halfspaces and hyperplanes. Thus, this is not a polyhedron.

d) $S = \{x \in \mathbb{R}^n \mid x \succeq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1\}$.

Once again, we are intersecting the nonnegative orthant $\mathbb{R}^n_+$ with other sets, but this time the constraint will be given by $max_i\ |x_i| \leq 1$, which means we are working with the norm $\|x\|_\infty$. Thus, the ball resulting from this norm has the shape of a square, i.e. the ball can be expressed as the intersection of a finite amount of halfspaces and thus, the set S will be a polyhedron.

2.9 *Voronoi sets and polyhedral decomposition.* Let $x_0, \ldots, x_K \in \mathbb{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, *i.e.*,

$$V = \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - xi\|_2,\ i = 1, \ldots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \ldots, x_K$.

Solution

a) Show that V is a polyhedron. Express V in the form $V = x \mid Ax \preceq b$.

Let's start with the definition. We know that x is the closest point to x_0, if for all $i = 1, \ldots k$

$$\|x - x_0\|_2 \leq \|x - x_i\|$$
$$\sqrt{(x_1 - x_{0_1})^2 + (x_2 - x_{0_2})^2 \ldots (x_n - x_{0_n})^2} \leq \sqrt{(x_1 - x_{i_1})^2 + (x_2 - x_{i_2})^2 \ldots (x_n - x_{i_n})^2}$$
$$(x_1 - x_{0_1})^2 + (x_2 - x_{0_2})^2 \ldots (x_n - x_{0_n})^2 \leq (x_1 - x_{i_1})^2 + (x_2 - x_{i_2})^2 \ldots (x_n - x_{i_n})^2$$

We can do this because the terms $(x_n - x_{i_n})^2$ are always positive due to the square. Now, notice that the resulting expression can be rewritten as:

$$
\begin{aligned}
(x - x_0)^T (x - x_0) &\leq (x - x_i)^T (x - x_i) \\
x^T x - 2x_0^T x + x_0^T x_0 &\leq x^T x - 2x_i^T x + x_i^T x_i \\
2(x_i - x_0)^T x &\leq x_i^T x_i - x_0^T x_0
\end{aligned}
$$

If we name $A = 2(x_i - x_0)^T$ and $b = x_i^T x_i - x_0^T x_0$ then we can express V as $V = \{x \mid Ax \preceq b\}$, which defines a halfspace. Thus, V is a polyhedron.

b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \ldots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \ldots, x_K$.

Let $V = \{x \mid Ax \preceq b\}$ be a polyhedron with nonempty interior, with $A \in \mathbb{R}^{K \times n}$ and $b \in \mathbb{R}^K$. We can pick any point from this set and name it as x_0, and after that, we can construct K points x_i. As this $x_0 \in \{x \mid Ax \preceq b\}$, then the rest of the points can be represented as $x_i = x_0 + \lambda a_i$, where λ is chosen in such a way that the distance of x_i to the hyperplane defined by $a_i^T x = b_i$ is equal to the distance of x_0 to the hyperplane:

$$
\begin{aligned}
b_i - a_i^T x_0 &= a_i^T x_i - b_i \\
b_i - a_i^T x_0 &= a_i^T (x_0 + \lambda a_i) - b_i \\
b_i - a_i^T x_0 &= a_i^T x_0 + a_i^T \lambda a_i - b_i \\
2b_i - 2a_i^T x_0 &= a_i^T \lambda a_i \\
\frac{2(b_i - a_i^T x_0)}{a_i^T a_i} &= \lambda \\
\frac{2(b_i - a_i^T x_0)}{\|a_i\|^2} &= \lambda
\end{aligned}
$$

Thus, we can represent the x_i points as:

$$
x_i = x_0 + \frac{2(b_i - a_i^T x_0)}{\|a_i\|^2} a_i
$$

c) We can also consider the sets

$$
V_k = \{x \in \mathbb{R}^n \mid \|x - x_k\|_2 \leq \|x - x_i\|_2, \ i \neq k\}.
$$

The set V_k consists of points in $\mathbb{R}^n$ for which the closest point in the set $\{x_0, \ldots, x_K\}$

is x_k.

The sets $V_0, \ldots, V_K$ give a polyhedral decomposition of $\mathbb{R}^n$. More precisely, the sets V_k are polyhedra, $\bigcup_{k=0}^K V_k = \mathbb{R}^n$, and $\textbf{int}\ V_i \cap \textbf{int}\ V_j = \emptyset$ for $i \neq j$, *i.e.*, V_i and V_j intersect at most along a boundary.
Suppose that $P_1, \ldots, P_m$ are polyhedra such that $\bigcup_{i=1}^m P_i = \mathbb{R}^n$, and $\textbf{int}\ P_i \cap \textbf{int}\ P_j = \emptyset$ for $i \neq j$. Can this polyhedral decomposition of $\mathbb{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

asd

2.10 *Solution set of a quadratic inequality.* Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n \mid x^T Ax + b^T x + c \leq 0\},$$

with $A \in S^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

a) Show that C is convex if $A \succeq 0$.

b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

2.13 *Conic hull of outer products.* Consider the set of rank-k *outer products*, defined as $\{XX^T \mid X \in \mathbb{R}^{n\times k},\ \text{rank } X = k\}$. Describe its conic hull in simple terms.

2.14 *Expanded and restricted sets.* Let $S \subseteq \mathbb{R}^n$, and let $\| \,.\, \|$ be a norm on $\mathbb{R}^n$.

a) For $a \geq 0$ we define S_a as $\{x \mid \mathbf{dist}(x, S) \leq a\}$, where $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$. We refer to S_a as S *expanded* or *extended* by a. Show that if S is convex, then S_a is convex.

b) For $a \geq 0$ we define $S_{-a} = \{x \mid B(x, a) \subseteq S\}$, where $B(x, a)$ is the ball (in the norm $\| \,.\, \|$), centered at x, with radius a. We refer to S_{-a} as S *shrunk* or *restricted* by a, since S_{-a} consists of all points that are at least a distance a from $\mathbb{R}^n \setminus S$. Show that if S is convex, then S_{-a} is convex.

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \ldots < a_n$. Of course $p \in \mathbb{R}^n$ lies in the standard probability simplex $P = \{p \mid 1^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

a) $\alpha \leq Ef(x) \leq \beta$, where $\mathbf{E}f(x)$ is the expected value P of $f(x)$, i.e., $\mathbf{E}f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbb{R} \to \mathbb{R}$ is given.)

b) $\mathbf{prob}(x > \alpha) \leq \beta$.

c) $\mathbf{E}|x^3| \leq \alpha\ \mathbf{E}\ |x|$.

d) $\mathbf{E}x^2 \leq \alpha$.

e) $\mathbf{E}x^2 \geq \alpha$.

f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}x)^2$ is the variance of x.

g) $\mathbf{var}(x) \geq \alpha$

h) $\mathbf{quartile}(x) \geq \alpha$, where $\mathbf{quartile}(x) = \mathbf{inf}\{\beta \mid \mathrm{prob}(x \leq \beta) \geq 0{,}25\}$.

i) $\mathbf{quartile}(x) \leq \alpha$

2.16 Show that if S_1 and S_2 are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m,\ y_1, y_2 \in \mathbb{R}^n,\ (x, y_1) \in S_1,\ (x, y_2) \in S_2\}.$$

2.17 *Image of polyhedral sets under perspective function.* In this problem we study the image of hyperplanes, halfspaces, and polyhedra under the perspective function $P(x, t) = x/t$, with $\mathbf{dom}\ P = \mathbb{R}^n \times \mathbb{R}_{++}$. For each of the following sets C, give a simple description of

$$P(C) = \{v/t \mid (v, t) \in C,\ t > 0\}.$$

a) The polyhedron $C = \mathbf{conv}\ \{(v_1, t_1), \ldots, (v_K, t_K)\}$ where $v_i \in \mathbb{R}^n$ and $t_i > 0$.

b) The hyperplane $C = \{(v, t) \mid f^T v + gt = h\}$ (with f and g not both zero).

c) The halfspace $C = \{(v, t) \mid f^T v + gt \le h\}$ (with f and g not both zero).

d) The polyhedron $C = \{(v, t) \mid Fv + gt \preceq h\}$.

2.18 *Invertible linear-fractional functions.* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad dom f = \{x \mid c^T x + d > 0\}.$$

Suppose the matrix

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$$

is nonsingular. Show that f is invertible and that f^{-1} is a linear-fractional mapping. Give an explicit expression for f^{-1} and its domain in terms of A, b, c, and d. *Hint.* It may be easier to express f^{-1} in terms of Q.