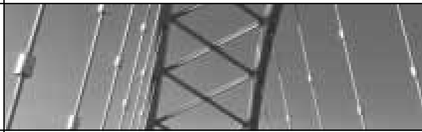


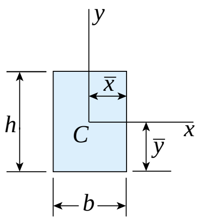
# D

# Properties of Plane Areas

Notation: $A$ = area
$\bar{x}, \bar{y}$ = distances to centroid $C$
$I_x, I_y$ = moments of inertia with respect to the $x$ and $y$ axes, respectively
$I_{xy}$ = product of inertia with respect to the $x$ and $y$ axes
$I_P = I_x + I_y$ = polar moment of inertia with respect to the origin of the $x$ and $y$ axes
$I_{BB}$ = moment of inertia with respect to axis $B$-$B$

---

**1**



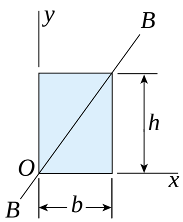

**Rectangle** (Origin of axes at centroid)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0 \qquad I_P = \frac{bh}{12}(h^2 + b^2)$$

---

**2**



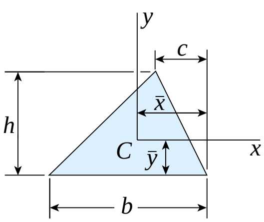

**Rectangle** (Origin of axes at corner)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3} \qquad I_{xy} = \frac{b^2h^2}{4} \qquad I_P = \frac{bh}{3}(h^2 + b^2)$$

$$I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

---

**3**




**Triangle** (Origin of axes at centroid)

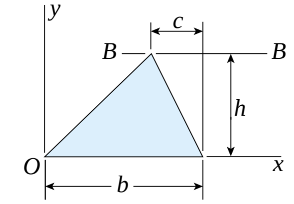
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_P = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

---

**4**

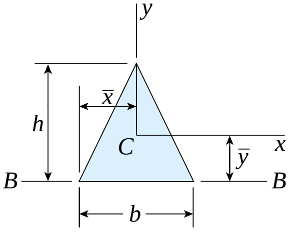



**Triangle** (Origin of axes at vertex)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

**5**




**Isosceles triangle** (Origin of axes at centroid)

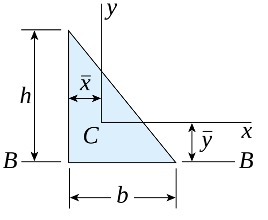
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_P = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(*Note:* For an equilateral triangle, $h = \sqrt{3}\, b/2$.)

**6**



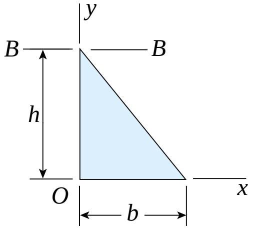

**Right triangle** (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_P = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

**7**

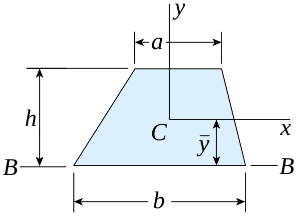



**Right triangle** (Origin of axes at vertex)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

$$I_P = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

**8**

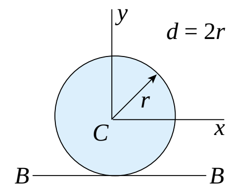



**Trapezoid** (Origin of axes at centroid)

$$A = \frac{h(a + b)}{2} \qquad \bar{y} = \frac{h(2a + b)}{3(a + b)}$$

$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

**9**

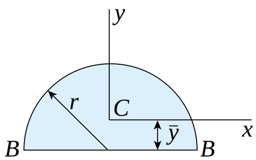



**Circle** (Origin of axes at center)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$I_{xy} = 0 \qquad I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

**10**

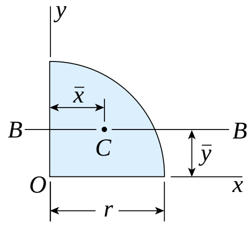



**Semicircle** (Origin of axes at centroid)

$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_y = \frac{\pi r^4}{8} \qquad I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

**11**

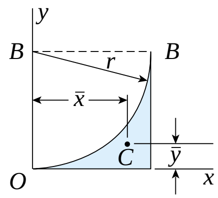



**Quarter circle** (Origin of axes at center of circle)

$$A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$$

$$I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8} \qquad I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488r^4$$

**12**

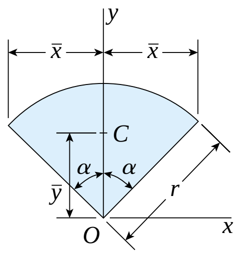



**Quarter-circular spandrel** (Origin of axes at point of tangency)

$$A = \left(1 - \frac{\pi}{4}\right)r^2 \qquad \bar{x} = \frac{2r}{3(4 - \pi)} \approx 0.7766r \qquad \bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)} \approx 0.2234r$$

$$I_x = \left(1 - \frac{5\pi}{16}\right)r^4 \approx 0.01825r^4 \qquad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right)r^4 \approx 0.1370r^4$$

**13**



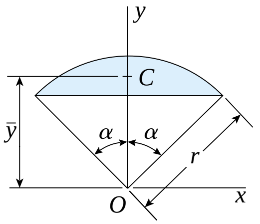

**Circular sector** (Origin of axes at center of circle)

$\alpha$ = angle in radians $\quad (\alpha \leq \pi/2)$

$$A = \alpha r^2 \qquad \bar{x} = r \sin \alpha \qquad \bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

$$I_x = \frac{r^4}{4}(\alpha + \sin \alpha \cos \alpha) \qquad I_y = \frac{r^4}{4}(\alpha - \sin \alpha \cos \alpha) \qquad I_{xy} = 0 \qquad I_P = \frac{\alpha r^4}{2}$$

**14**




**Circular segment** (Origin of axes at center of circle)

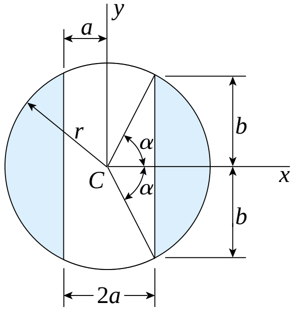
$\alpha$ = angle in radians $\quad (\alpha \leq \pi/2)$

$$A = r^2(\alpha - \sin\alpha\cos\alpha) \qquad \bar{y} = \frac{2r}{3}\left(\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}\right)$$

$$I_x = \frac{r^4}{4}(\alpha - \sin\alpha\cos\alpha + 2\sin^3\alpha\cos\alpha) \qquad I_{xy} = 0$$

$$I_y = \frac{r^4}{12}(3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha\cos\alpha)$$

**15**



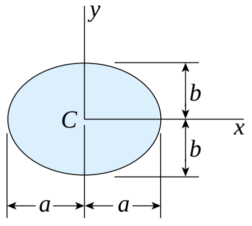

**Circle with core removed** (Origin of axes at center of circle)

$\alpha$ = angle in radians $\quad (\alpha \leq \pi/2)$

$$\alpha = \arccos\frac{a}{r} \qquad b = \sqrt{r^2 - a^2} \qquad A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$$

$$I_x = \frac{r^4}{6}\left(3\alpha - \frac{3ab}{r^2} - \frac{2ab^3}{r^4}\right) \qquad I_y = \frac{r^4}{2}\left(\alpha - \frac{ab}{r^2} + \frac{2ab^3}{r^4}\right) \qquad I_{xy} = 0$$

**16**




**Ellipse** (Origin of axes at centroid)

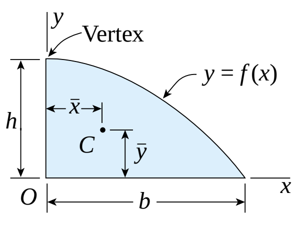
$$A = \pi ab \qquad I_x = \frac{\pi ab^3}{4} \qquad I_y = \frac{\pi ba^3}{4}$$

$$I_{xy} = 0 \qquad I_P = \frac{\pi ab}{4}(b^2 + a^2)$$

Circumference $\approx \pi[1.5(a + b) - \sqrt{ab}\,] \qquad (a/3 \leq b \leq a)$

$\approx 4.17b^2/a + 4a \qquad (0 \leq b \leq a/3)$

**17**

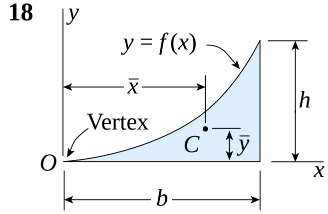



**Parabolic semisegment** (Origin of axes at corner)

$$y = f(x) = h\left(1 - \frac{x^2}{b^2}\right)$$

$$A = \frac{2bh}{3} \qquad \bar{x} = \frac{3b}{8} \qquad \bar{y} = \frac{2h}{5}$$

$$I_x = \frac{16bh^3}{105} \qquad I_y = \frac{2hb^3}{15} \qquad I_{xy} = \frac{b^2h^2}{12}$$

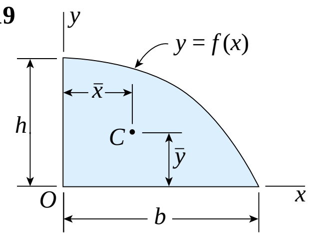
**18** 


**Parabolic spandrel** (Origin of axes at vertex)

$$y = f(x) = \frac{hx^2}{b^2}$$

$$A = \frac{bh}{3} \qquad \bar{x} = \frac{3b}{4} \qquad \bar{y} = \frac{3h}{10}$$

$$I_x = \frac{bh^3}{21} \qquad I_y = \frac{hb^3}{5} \qquad I_{xy} = \frac{b^2h^2}{12}$$

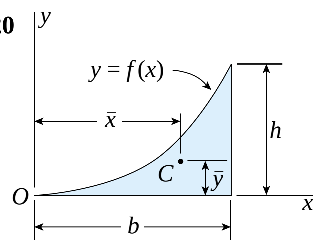
**19** 


**Semisegment of *n*th degree** (Origin of axes at corner)

$$y = f(x) = h\left(1 - \frac{x^n}{b^n}\right) \qquad (n > 0)$$

$$A = bh\left(\frac{n}{n+1}\right) \qquad \bar{x} = \frac{b(n+1)}{2(n+2)} \qquad \bar{y} = \frac{hn}{2n+1}$$

$$I_x = \frac{2bh^3n^3}{\quad} \qquad I_y = \frac{hb^3n}{3(n+3)} \qquad I_{xy} = \frac{b^2h^2n^2}{4(n+1)(n+2)}$$

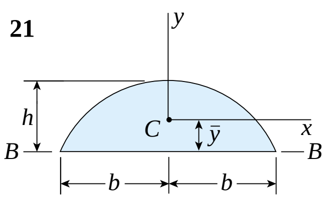
**20** 


**Spandrel of *n*th degree** (Origin of axes at point of tangency)

$$y = f(x) = \frac{hx^n}{b^n} \qquad (n > 0)$$

$$A = \frac{bh}{n+1} \qquad \bar{x} = \frac{b(n+1)}{n+2} \qquad \bar{y} = \frac{h(n+1)}{2(2n+1)}$$

$$I_x = \frac{bh^3}{3(3n+1)} \qquad I_y = \frac{hb^3}{n+3} \qquad I_{xy} = \frac{b^2h^2}{4(n+1)}$$

**21** 

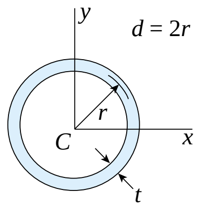

**Sine wave** (Origin of axes at centroid)

$$A = \frac{4bh}{\pi} \qquad \bar{y} = \frac{\pi h}{8}$$

$$I_x = \left(\frac{8}{9\pi} - \frac{\pi}{16}\right)bh^3 \approx 0.08659bh^3 \qquad I_y = \left(\frac{4}{\pi} - \frac{32}{\pi^3}\right)hb^3 \approx 0.2412hb^3$$

$$I_{xy} = 0 \qquad I_{BB} = \frac{8bh^3}{9\pi}$$

**22**



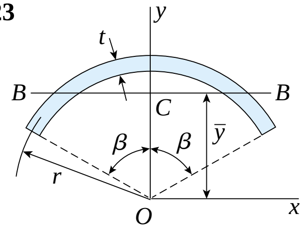

**Thin circular ring** (Origin of axes at center)

Approximate formulas for case when $t$ is small

$$A = 2\pi r t = \pi d t \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$

$$I_{xy} = 0 \qquad I_P = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

**23**




**Thin circular arc** (Origin of axes at center of circle)

Approximate formulas for case when $t$ is small

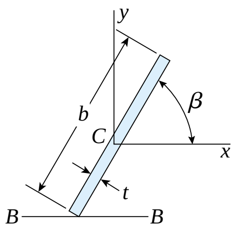
$\beta$ = angle in radians (*Note:* For a semicircular arc, $\beta = \pi/2$.)

$$A = 2\beta r t \qquad \bar{y} = \frac{r \sin\beta}{\beta}$$

$$I_x = r^3 t(\beta + \sin\beta\cos\beta) \qquad I_y = r^3 t(\beta - \sin\beta\cos\beta)$$

$$I_{xy} = 0 \qquad I_{BB} = r^3 t\left(\frac{2\beta + \sin 2\beta}{2} - \frac{1 - \cos 2\beta}{\beta}\right)$$

**24**



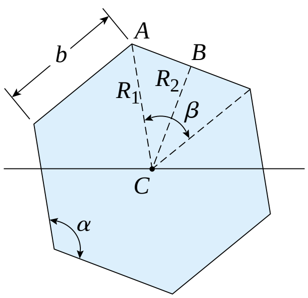

**Thin rectangle** (Origin of axes at centroid)

Approximate formulas for case when $t$ is small

$$A = bt$$

$$I_x = \frac{tb^3}{12}\sin^2\beta \qquad I_y = \frac{tb^3}{12}\cos^2\beta \qquad I_{BB} = \frac{tb^3}{3}\sin^2\beta$$

**25**




**Regular polygon with *n* sides** (Origin of axes at centroid)

$C$ = centroid (at center of polygon)

$n$ = number of sides ($n \geq 3$) $\qquad b$ = length of a side

$\beta$ = central angle for a side $\qquad \alpha$ = interior angle (or vertex angle)

$$\beta = \frac{360°}{n} \qquad \alpha = \left(\frac{n-2}{n}\right)180° \qquad \alpha + \beta = 180°$$

$R_1$ = radius of circumscribed circle (line $CA$) $\qquad R_2$ = radius of inscribed circle (line $CB$)

$$R_1 = \frac{b}{2}\csc\frac{\beta}{2} \qquad R_2 = \frac{b}{2}\cot\frac{\beta}{2} \qquad A = \frac{nb^2}{4}\cot\frac{\beta}{2}$$

$I_c$ = moment of inertia about any axis through $C$ (the centroid $C$ is a principal point and every axis through $C$ is a principal axis)

$$I_c = \frac{nb^4}{192}\left(\cot\frac{\beta}{2}\right)\left(3\cot^2\frac{\beta}{2} + 1\right) \qquad I_P = 2I_c$$