PHYSICAL REVIEW D VOLUME 22, NUMBER 11 1 DECEMBER 1980

Matter effects on three-neutrino oscillations

V. Barger and K. Whisnant
Physics Department, University of Wisconsin, Madison, Wisconsin 53706

S. Pakvasa
Physics Department, University of Hawaii at Manoa, Honolulu, Hawaii 96822

R. J. N. Phillips
Rutherford Laboratory, Chilton, Didcot, Oxon, England
(Received 4 August 1980)

We evaluate the influence of coherent forward scattering in matter upon neutrino oscillations in the three-neutrino picture. We write down the exact solution and also approximate first-order solutions that exhibit general features more transparently. Oscillation characteristics in matter that could be observed in deep-mine experiments are discussed and illustrated using an oscillation solution suggested by solar and reactor data.

I. INTRODUCTION

Interest in neutrino oscillations[1] has been heightened recently by indications from beam-dump experiments,[2-4] from reanalysis[5] of old reactor data,[6] and from a new reactor experiment with reduced sensitivity to spectrum uncertainties.[7] If such oscillations can be clearly resolved, they will not only show that neutrinos are massive but also provide information about their mass differences and mixing matrix.

Deep-mine experiments that measure high-energy events from atmospheric neutrinos offer unique opportunities to probe oscillations in the range $L/E \sim 1-10^5$ m/MeV (where L is the path length and E the energy) that is sensitive to mass-squared differences $\delta m^2 \gtrsim 10^{-5}$ eV^2. Wolfenstein[8] has pointed out, however, that the standard vacuum oscillations can be significantly modified by coherent forward scattering from electrons in matter (that selectively affect ν_e and $\bar{\nu}_e$ components) when the path integral of electron number density N_e is of order $\int N_e dL \sim 10^9 N_A$ cm^{-2} where $N_A = 6\times 10^{23}$. Thus with electron densities in typical terrestrial matter of order $N_e \sim 2N_A$ cm^{-3}, matter effects can occur over the distance of the earth's radius, i.e., in deep-mine events produced by upward neutrinos.

Wolfenstein has given a complete analytic solution for matter corrections to oscillations of two neutrinos.[8] We examine the properties of this solution in detail in Sec. II. We then derive a general solution for matter oscillations with any number of neutrinos, but its implications are not immediately transparent. We therefore also write down first-order approximate solutions that exhibit general properties rather simply. For the case of three neutrinos, we give an exact solution in closed form. In Sec. III we illustrate the properties of the three-neutrino oscillations in matter based on a vacuum-oscillation solution suggested by solar and reactor data.

II. OSCILLATIONS IN MATTER

A. General equations

Consider a set of neutrino charged-current eigenstates ν_α ($\alpha = e, \mu, \tau, \ldots$) and mass eigenstates ν_i ($i = 1, 2, 3, \ldots$) at time $t = 0$, distinguished by their suffixes and related by a unitary transformation,

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle . \tag{1}$$

Then, for a relativistic neutrino beam energy E, we recall the standard amplitude A and probability P for $\nu_\alpha \to \nu_\beta$ transitions after a time t *in vacuo*,

$$\begin{aligned} A(\nu_\alpha \to \nu_\beta) &= \sum_i U_{\alpha i} \exp(-\tfrac{1}{2} i m_i^2 t/E) U^\dagger_{i\beta} , \\ P(\nu_\alpha \to \nu_\beta) &= |A(\nu_\alpha \to \nu_\beta)|^2 , \end{aligned} \tag{2}$$

where m_i are the mass eigenvalues and the interference terms in $|A|^2$ are oscillatory. We usually write L/E in place of t/E in Eq. (2), where L is the length of the flight path (units $\hbar = c = 1$).

To treat neutrino evolution in matter, we consider an arbitrary state vector in neutrino-flavor space,

$$|\psi(t)\rangle = \sum_i \psi_i(t) |\nu_i\rangle . \tag{3}$$

For an initial state ν_α at time $t = 0$, $\psi_i(0) = U_{\alpha i}$, and the $\nu_\alpha \to \nu_\beta$ transition amplitude is

$$A(\nu_\alpha \to \nu_\beta) = \sum_i U^\dagger_{i\beta} \psi_i(t) . \tag{4}$$

The time evolution is controlled by the equation

$$id\psi_j(t)/dt = m_j^2/(2E)\psi_j(t) - \sum_k \sqrt{2}\, GN_e U_{ej} U^\dagger_{ke}\psi_k(t) \equiv H_{kj}\psi_k(t)\,, \tag{5}$$

where N_e is the number of electrons per unit volume and G is the weak coupling constant. In Eq. (5) we have dropped a common overall phase in the ψ_i which produces no observable effects; this includes the phase shift from neutral-current scattering which is the same for all neutrino flavors. The coefficient $\sqrt{2}GN_e$ in Eq. (5) differs from the value GN_e given in Ref. 8; this correction is confirmed by Ref. 9. This equation applies strictly for *neutrinos*; for *antineutrinos*, change the sign of the $\sqrt{2}GN_e$ term and substitute U^* for U.

We assume that the vacuum oscillations are already prescribed so that U and the m_i are known. The problem of propagation is therefore to diagonalize the matrix H defined in Eq. (5). Let us suppose that it can be diagonalized by proceeding to a new set of basis state $|\nu'_i\rangle = V^\dagger_{i\alpha}|\nu_\alpha\rangle = V^\dagger_{i\alpha}U_{\alpha j}|\nu_j\rangle$, where V is a unitary matrix. Then if the eigenvalues of H corresponding to these states $|\nu'_i\rangle$ are written as $M_i^2/2E$, the solution for a uniform medium is given in principle by Eq. (2) with m_i^2 and U replaced by M_i^2 and V, respectively.

There is another way to solve the propagation problem without explicitly constructing the matrix V. For n neutrinos, Eq. (5) has n independent solutions for the row vectors $\psi_j(t)$. We choose the set of solutions $\psi_j^{(i)}$ $(i=1,\ldots,n)$ that are pure mass eigenstates at time $t=0$:

$$\psi_j^{(i)}(t=0) = \delta_{ij}\,. \tag{6}$$

If these row vectors are assembled into an $n\times n$ matrix X according to

$$X_{ij}(t) = \psi_j^{(i)}(t)\,, \tag{7}$$

then X satisfies the matrix equation

$$idX/dt = XH \tag{8}$$

with the boundary condition $X(t=0)=1$. An analytical solution to Eq. (8) is possible for constant N_e,

$$X(t) = \exp(-iHt)\,. \tag{9}$$

Row i of X describes the state that started as the mass eigenstate $|\nu_i\rangle$ at $t=0$; column j describes the amplitude that it has evolved at time t into mass eigenstate $|\nu_j\rangle$. In the presence of matter the transition amplitude of Eq. (2) is modified to

$$A(\nu_\alpha \to \nu_\beta) = \sum_{ij} U_{\alpha i} X_{ij} U^\dagger_{j\beta}\,. \tag{10}$$

The resulting transition probability requires only knowledge of the original mixing matrix U and the mass-squared differences $\delta M^2_{ij} = M_i^2 - M_j^2$, $\delta m^2_{ij} = m_i^2 - m_j^2$ and $M_i^2 - m_j^2$. For computation, the matrix X can be rewritten via Lagrange's formula as

$$X = \sum_k \left[\prod_{j\neq k} \frac{(2EH - M_j^2 1)}{\delta M_{kj}^2}\right] \exp\left(-i\frac{M_k^2 L}{2E}\right). \tag{11}$$

B. Two-neutrino case

We take the charged-current (CC) eigenstates to be ν_e, ν_μ and parametrize the 2×2 matrices U, V that relate them to the mass eigenstates ν_i and matter eigenstates ν'_i by

$$\begin{aligned} U_{e1} &= U_{\mu 2} = \cos\alpha, \quad U_{e2} = -U_{\mu 1} = \sin\alpha\,, \\ V_{e1} &= V_{\mu 2} = \cos\alpha', \quad V_{e2} = -V_{\mu 1} = \sin\alpha'\,. \end{aligned} \tag{12}$$

There is just one mass-squared difference $\delta m^2 = m_1^2 - m_2^2$. Vacuum oscillations of two neutrinos determine only $|\delta m^2|$ and $\sin^2 2\alpha$, leaving the sign of δm^2 and the quadrant of 2α unresolved. By convention we take $\alpha < 45°$. The ambiguity in the sign of δm^2 is resolved by the effects of matter.

It is convenient to define the oscillation length in vacuum l_V and a characteristic length l_M for matter effects by

$$l_V = 4\pi E/\delta m^2, \quad l_M = 2\pi/(\sqrt{2}\, GN_e)\,. \tag{13}$$

For *antineutrinos*, the sign of l_M is reversed. We note that l_V or l_M can have either sign. With E in MeV, δm^2 in eV^2, and N_e in cm^{-3}, the oscillation lengths in meters are given by

$$l_V = 2.48E/\delta m^2, \quad l_M = 1.77\times 10^7 N_A/N_e\,. \tag{14}$$

For terrestrial matter[10] $N_e \sim 2N_A$ cm^{-3} in the mantle ($3.5\times 10^6 < r < 6.4\times 10^7$ m) and $N_e \sim 5N_A$ cm^{-3} in the core ($r < 3.5\times 10^6$ m).

Following Ref. 8, the solution in the *neutrino case* is given by

$$\tan 2\alpha' = \sin 2\alpha/(\cos 2\alpha - l_V/l_M) \tag{15}$$

for the oscillation angle in matter, and

$$\delta M^2 = \delta m^2[1 - 2(l_V/l_M)\cos 2\alpha + (l_V/l_M)^2]^{1/2} \tag{16}$$

for the matter eigenmass-squared difference. The associated oscillation length in matter is

$$l = l_V[1 - 2(l_V/l_M)\cos 2\alpha + (l_V/l_M)^2]^{-1/2}\,. \tag{17}$$

The physical consequences are clear.

(i) If $|l_V| \ll |l_M|$, i.e., the vacuum-oscillation length is very short on the matter scale, then $\alpha' \simeq \alpha$, $\delta M^2 \simeq \delta m^2$, and matter corrections are negligible.

(ii) If $|l_V| \gg |l_M|$, then $\alpha' \simeq 0$ and the matter corrections damp out all oscillation effects.

(iii) At intermediate values $|l_V| \sim |l_M|$ the matter corrections are very significant and differ between neutrinos and antineutrinos. Moreover, matter effects resolve the vacuum-oscillation ambiguity in the sign of δm^2.

(iv) For matter corrections to be observable, the distance traversed in matter must also be an appreciable fraction of l_M. Hence matter corrections are very small in all terrestrial contexts, except when neutrinos traverse a substantial fraction of the earth's diameter and have energies

$$E\ (\mathrm{MeV}) \gtrsim 10^5\,|\delta m^2\ (\mathrm{eV}^2)|\,. \tag{18}$$

(v) For given δm^2 the vacuum-oscillation length l_V depends on E, whereas l_M does not; hence, there is always some energy range where matter effects are important.

(vi) There is always some energy where $l_V/l_M = \cos 2\alpha$ and hence $\alpha' = 45°$ for either ν or $\bar{\nu}$ depending on the sign of δm^2. Hence, there is always some energy where ν or $\bar{\nu}$ matter mixing is maximal. At this energy, the diagonal transition probability vanishes at a distance

$$L = \frac{l_M}{2}\cot 2\alpha\,. \tag{19}$$

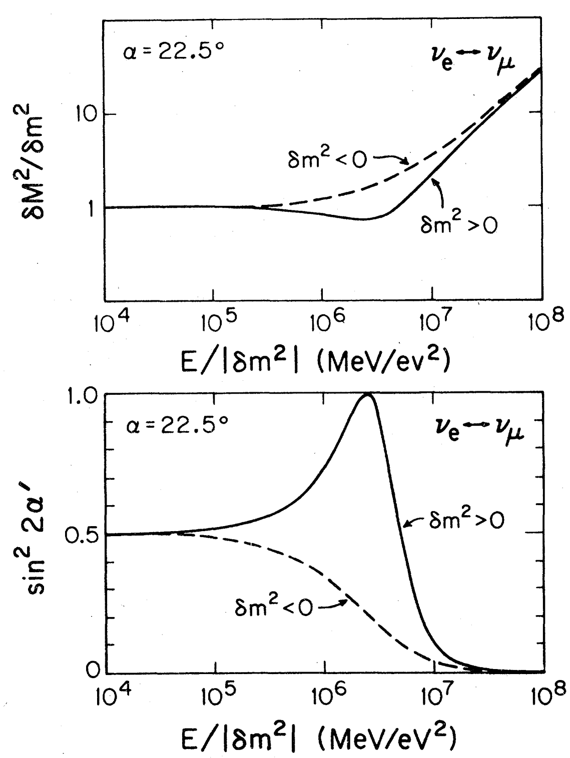

FIG. 1. Matter-to-vacuum eigenmass-squared difference ratio and matter amplitude $\sin^2 2\alpha'$ for oscillations of two neutrinos with vacuum amplitude $\sin^2 2\alpha = 0.5$ ($\alpha = 22.5°$).

With $\alpha = 22.5°$ and $N_e = 2N_A\ \mathrm{cm}^{-3}$, this distance is $L \simeq 5\times 10^6$ m, which would correspond to deep-mine events about 10° below the horizontal direction.

Some of these results are illustrated in Fig. 1, showing the ratio $\delta M^2/\delta m^2$ (describing the correction to the oscillation wavelength) and $\sin^2 2\alpha'$ (describing the oscillation amplitude) versus $E/|\delta m^2|$. This illustration is based on $\alpha = 22.5°$ and $N_e = 2N_A$. As expected, there is little matter correction for E (MeV) $< 10^5\,|\delta m^2\ (\mathrm{eV}^2)|$. The mixing becomes maximal in one channel (ν or $\bar{\nu}$) at one energy. At sufficiently large energy where $l_V \gg l_M$ the mixing is damped out. When δm^2 changes sign, ν and $\bar{\nu}$ exchange roles. The transition probabilities are given simply by

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= P(\nu_\mu \to \nu_e) = 1 - P(\nu_e \to \nu_e) \\ &= 1 - P(\nu_\mu \to \nu_\mu) = \sin^2(2\alpha')\sin^2(\tfrac{1}{4}\delta M^2 L/E)\,. \end{aligned} \tag{20}$$

Figure 2 compares vacuum- and matter-oscillation results for $P(e \to e)$ in the two-neutrino case at a fixed distance $L = 5\times 10^6$ m.

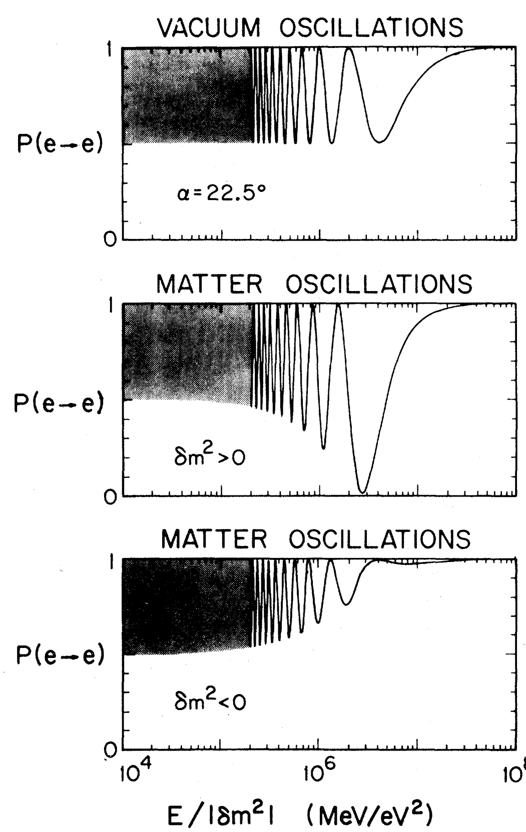

FIG. 2. Comparison of vacuum and matter transition probability $P(e \to e)$ for two neutrinos at fixed $L = 5\times 10^6$ m for density $N_e = 2N_A$.

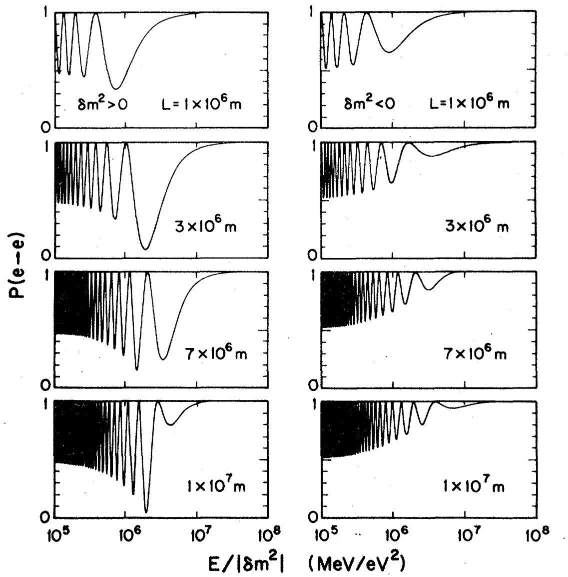

FIG. 3. Dependence of $P(e\to e)$ on distance L with $N_e=2N_A$ for two neutrinos.

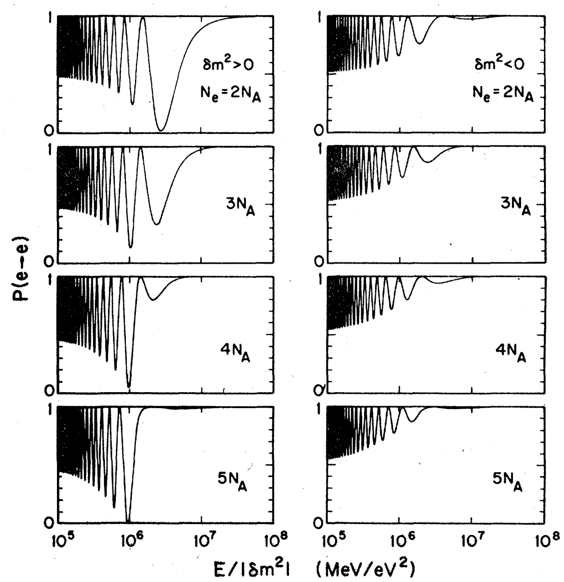

FIG. 4. Dependence of $P(e\to e)$ on N_e at distance $L=5\times10^6$ m for two neutrinos.

Figure 3 compares matter results at distances up to the earth's diameter. As L increases, the oscillation pattern moves to higher E, as might be expected, and the oscillation amplitude also changes. Figure 4 shows results with different electron densities N_e at fixed $L=5\times10^6$ m. Matter damping [point (ii) above] is clearly seen at the higher energies in Figs. 3 and 4 when either the distance L or the density N_e is sufficiently large; in general, significant damping of vacuum oscillations occurs for energies at which $L > l_V > l_M$.

C. Three-neutrino case

For constant density N_e, we have algebraically performed the diagonalization of Eq. (5) in the three-neutrino case. After tedious but straightforward algebra, the "modified mass" parameters M_i^2 which are the eigenvalues of the matrix H_{ij} are found to be

$$M_i^2 = -\tfrac{2}{3}(\alpha^2-3\beta)^{1/2}\cos\left[\tfrac{1}{3}\arccos\left(\frac{2\alpha^3-9\alpha\beta+27\gamma}{2(\alpha^2-3\beta)^{3/2}}\right)\right] + m_1^2 - \alpha/3\,, \tag{21}$$

with

$$\begin{aligned}\alpha &= 2\sqrt{2}EGN_e + \delta m^2_{12} + \delta m^2_{13}\,,\\ \beta &= \delta m^2_{12}\delta m^2_{13} + 2\sqrt{2}EGN_e[\delta m^2_{12}(1-|U_{e2}|^2) + \delta m^2_{13}(1-|U_{e3}|^2)]\,,\\ \gamma &= 2\sqrt{2}EGN_e\delta m^2_{12}\delta m^2_{13}|U_{e1}|^2\,.\end{aligned} \tag{22}$$

Here $\delta m^2_{ij} = m_i^2 - m_j^2$ and the three separate M_i^2 values ($i=1,2,3$) are provided by the three distinct roots of $\cos(\frac{1}{3}\arccos)$. Notice that only mass difference δm^2_{ij} enter the relations determining $\delta M^2_{ij} = M_i^2 - M_j^2$; m_1^2 itself does not.

Equations (10), (11), (21), and (22) define the complete three-neutrino oscillation solution in a uniform medium. Unfortunately, the algebra is rather impenetrable. It is therefore interesting to construct also the approximate solutions to first order in GN_e or first order in δm^2_{ij}.

D. First-order solution in GN_e, any number of neutrinos

The approximate solution of Eq. (5) to first order in GN_e has the form, for any number of neutrinos,

$$\psi_i(t) = \exp(-\tfrac{1}{2}im_i^2t/E)(1+i\sqrt{2}GN_et|U_{ei}|^2)\psi_i(0) + \sum_{j\neq i}\exp[-\tfrac{1}{4}i(m_i^2+m_j^2)t/E]\frac{\sin(\frac{1}{4}\delta m^2_{ij}t/E)}{(\frac{1}{4}\delta m^2_{ij}/E)}\, i\sqrt{2}GN_eU_{ei}U^\dagger_{je}\psi_j(0)\,. \tag{23}$$

This implies the propagation amplitude

$$A(\nu_\alpha \to \nu_\beta) = \sum_i \exp(-\tfrac{1}{2} i m_i^2 L/E)(1 + i\sqrt{2} G N_e L \,|U_{ei}|^2) U_{\alpha i} U^\dagger_{i\beta} + \sum_{j\neq i} \exp[-\tfrac{1}{4} i (m_i^2 + m_j^2) L/E] \frac{\sin(\frac{1}{4}\delta m^2_{ij} L/E)}{(\frac{1}{4}\delta m^2_{ij}/E)} i\sqrt{2} G N_e U_{ei} U^\dagger_{i\beta} U_{\alpha j} U^\dagger_{je} \,. \tag{24}$$

For antineutrinos, reverse the sign of G and substitute U^* for U.

The following remarks should be made:

(i) The matter correction is of order $GN_e t = GN_e L$; the perturbative solutions, Eqs. (23) and (24), are only valid when this is small compared to unity.

(ii) If $\delta m^2_{ij} L/E$ is of order unity or less, the off-diagonal ($j \neq i$) terms in the summation in Eq. (24) are of full strength $\sqrt{2} G N_e L$. If $\delta m^2_{ij} L/E$ is large, however, the $j \neq i$ terms are only of order $\sqrt{2} G N_e E/\delta m^2_{ij}$; i.e., they are suppressed by a factor of order $E/(\delta m^2_{ij} L)$. Depending on the numerical values involved, these off-diagonal terms may then be negligible.

(iii) If all δm^2_{ij} are large enough for all off-diagonal terms to be ignored, Eq. (5) can be solved *to all orders* in GN_e to give (for $t \ll E/\delta m^2_{ij}$)

$$\psi_i(t) = \exp(-\tfrac{1}{2} i m_i^2 t/E + i\sqrt{2} G N_e |U_{ei}|^2 t)\psi_i(0)\,, \tag{25}$$

$$A(\nu_\alpha \to \nu_\beta) = \sum_i \exp(-\tfrac{1}{2} i m_i^2 L/E + i\sqrt{2} G N_e L \,|U_{ei}|^2) U_{\alpha i} U^\dagger_{i\beta}\,. \tag{26}$$

In other words, when the vacuum oscillations are sufficiently fast, they suppress off-diagonal matter corrections, in the mass-eigenstate basis, and the effect is simply to introduce effective mass parameters,

$$\begin{aligned} m_i^2 &\to m_i^2 - \sqrt{2} G N_e E\,|U_{ei}|^2\,, \\ \delta m^2_{ij} &\to \delta m^2_{ij} - 2\sqrt{2} G N_e E(|U_{ei}|^2 - |U_{ej}|^2)\,. \end{aligned} \tag{27}$$

However, the off-diagonal terms that we have assumed negligible are themselves of order $GN_e E/\delta m^2_{ij}$; the corrections to δm^2_{ij} are of the same relative order and hence are equally negligible.

(iv) To summarize, the solution to first order in GN_e shows that the matter correction is potentially of order $\sqrt{2} G N_e L$. However, if the vacuum oscillations have sufficiently short wavelength (such that $\delta m^2_{ij} \gg 2\sqrt{2} G N_e E$ for all i,j) the matter corrections are always negligible. These general properties have already been seen in the two-neutrino case above.[8]

E. First-order solution in δm^2_{ij}, any number of neutrinos

If $\delta m^2_{ij} \ll 2\sqrt{2} G N_e E$, the region $L \gtrsim l_v$ where vacuum oscillations develop is beyond the range of applicability of Eq. (23). However, on general physical grounds we can surmise that amplitudes relating ν_e to other CC states will not build up, because the matter effect gives ν_e a relatively fast-changing phase. In this regime, the matter correction simply decouples ν_e while the other CC eigenstates have normal vacuum oscillations to one another. This can be seen explicitly by constructing a solution in which the matter effect is treated exactly and the mass-squared differences are treated perturbatively instead. Writing an arbitrary neutrino state vector in terms of the charge-current eigenstates as

$$|\psi(t)\rangle = \sum_\alpha \psi_\alpha(t)\,|\nu_\alpha\rangle\,, \tag{28}$$

the propagation equation can be cast in the form

$$i d\psi_\alpha(t)/dt = m_\alpha^2/(2E)\psi_\alpha(t) + \sum_{i\beta} U^\dagger_{i\alpha} U_{\beta i}\left(\frac{m_i^2 - m_1^2}{2E}\right)\psi_\beta(t)\,. \tag{29}$$

Here we have adopted the convenient shorthand $m_\alpha^2 = m_1^2$ ($\alpha \neq e$), $m_e^2 = m_1^2 - 2\sqrt{2} G N_e E$. The solution to first order in $\delta m^2_{i1} = m_i^2 - m_1^2$ is very similar to Eq. (23):

$$\psi_\alpha(t) = \exp(-\tfrac{1}{2} i m_\alpha^2 t/E)\left(1 - it\sum_i |U_{\alpha i}|^2 \frac{\delta m^2_{i1}}{2E}\right)\psi_\alpha(0) - \sum_{\beta\neq\alpha}\sum_i \exp[-\tfrac{1}{4} i (m_\alpha^2 + m_\beta^2) t/E] \frac{\sin(\frac{1}{4}\delta m^2_{\alpha\beta} t/E)}{(\frac{1}{4}\delta m^2_{\alpha\beta}/E)} i U^\dagger_{i\alpha} U_{\beta i}\left(\frac{\delta m^2_{i1}}{2E}\right)\psi_\beta(0)\,. \tag{30}$$

For all $\alpha \neq e$, $\beta \neq e$ this simply gives the lowest-order terms in the normal vacuum-oscillation amplitude Eq. (2). For $\alpha = e$, however, the off-diagonal couplings are strongly suppressed by the relative factor $\sin(\frac{1}{4}\delta m^2_{\alpha\beta} t/E)/(\frac{1}{4}\delta m^2_{\alpha\beta} t/E)$ and the ν_e mode is essentially decoupled.

Thus our perturbative solutions show how the general properties of oscillations in matter extrapolate from two neutrinos to the n-neutrino case.

III. ILLUSTRATIONS OF MATTER EFFECTS

We illustrate matter effects in the three-neutrino system using a representative vacuum-oscillation solution from Ref. 11 (solution A) and taking $N_e = 2N_A$ cm^{-3} as before. The leading mass-square difference $\delta m^2_{31} \simeq 0.9$ eV^2 was chosen to correspond to indications from reactors.[5,6] The mixing matrix U was chosen in accordance with accelerator limits on $P(\nu_\mu \to \nu_e)$ and the reactor data on $P(\bar{\nu}_e \to \bar{\nu}_e)$ and to give an average value $\langle P(\nu_e \to \nu_e)\rangle \simeq \frac{1}{3}$ for solar neutrinos.[12] The signs of the $\delta m^2_{ij} = m_i^2 - m_j^2$ were essentially undetermined; hence, there is a fourfold sign ambiguity. In the notation of Refs. 5 and 11, the parameters for the solution that we illustrate are

θ_1 (deg)	θ_2 (deg)	θ_3 (deg)	$\lvert\delta m^2_{31}\rvert$	$\lvert\delta m^2_{21}\rvert$
50	20	30	0.9	0.05 or 10^{-4}.

(31)

Here the δm^2_{ij} are in eV^2. The CP-violating phase δ is set equal to zero for simplicity. In Ref. 11 the secondary mass difference of solution A was chosen to be $|\delta m^2_{21}| = 0.05$ eV^2. Since matter effects are most sensitive to the smallest δm^2_{ij}, we consider the alternative possibilities in Eq. (31).

Figure 5 shows the results for δM^2_{ij} versus E; the four possible choices of sign for δm^2_{ij} and the two choices of magnitude for $|\delta m^2_{21}|$ are distinguished. At low E, $\delta M^2_{ij} = \delta m^2_{ij}$; as E increases, a given δM^2_{ij} begins to change when E (MeV) $> 10^5\ |\delta m^2_{ij}$ (eV^2)$|$. Results for antineutrinos are the same as those for neutrinos with the signs of the δm^2_{ij} reversed.

We do not attempt to illustrate the behavior of all the individual mixing matrix elements, but concentrate instead on the final observable result, namely, the transition probabilities for $\nu_\mu \to \nu_\mu$, $\nu_e \to \nu_\mu$, $\nu_e \to \nu_e$, $\nu_e \to \nu_\tau$, and $\nu_\mu \to \nu_\tau$. Figure 6 shows transition probabilities at the distance $L = 5 \times 10^6$ m (approximately one earth's radius) for $N_e = 2N_A$ cm^{-3}. For other L values, the calculations cannot be simply scaled with L/E since the matter-oscillation length is energy-independent. Quantitative comparisons with deep-mine data will require a numerical solution of Eq. (5) at each L, taking into account the variation of N_e with distance.

We make the following observations about the results in Fig. 6:

(i) The onset of matter effects occurs in general in two different E regions, in association with the two vacuum mass-squared differences [see, e.g., $P(e \to e)$ in Fig. 6(c)].

(ii) The secondary mass difference δm^2_{21} controls the energy above which matter effects are present. For E (MeV) $\lesssim 10^5\,|\delta m^2_{21}$ (eV^2)$|$ the probabilities are governed by vacuum oscillations.

(iii) The leading mass difference δm^2_{31} controls the energy above which all oscillation effects become small. For E (MeV) $\gtrsim 5 \times 10^7\,|\delta m^2_{31}$ (eV^2)$|$, $P(e \to e)$ and $P(\mu \to \mu)$ approach unity and all off-diagonal transitions go to zero, as also is the case for vacuum oscillations.

(iv) In channels that are effectively decoupled from the leading mass-squared difference δm^2_{31} by the choice of vacuum mixing matrix, there is only the E region and the probabilities diagonalize for E (MeV) $\gtrsim 5 \times 10^7\,|\delta m^2_{21}$ (eV^2)$|$. In the illustration this decoupling occurs for the μ channels.

(v) For the intermediate energy range $10^5\,|\delta m^2_{21}$ (eV^2)$| < E$ (MeV) $< 5 \times 10^7\,|\delta m^2_{31}(eV^2)|$, oscillations exist with mixing angles and transition probabilities which may differ from the vacuum val-

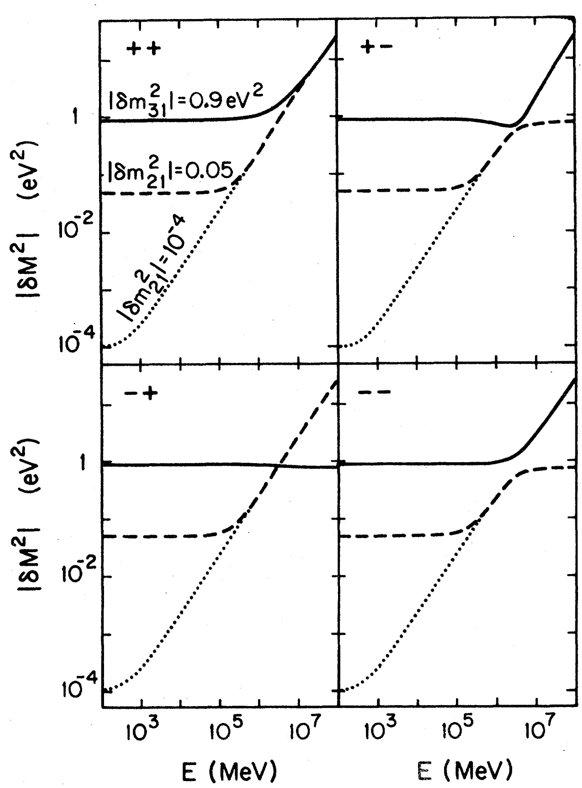

FIG. 5. Eigenmass-squared differences δM^2 in matter for the three-neutrino vacuum-oscillation parameters of Eq. (31). The signs of $(\delta m^2_{21}, \delta m^2_{31})$ label the fourfold ambiguity of solutions.

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(vi) Diagonal probabilities may approach zero and off-diagonal probabilities unity at certain energies. See $P(e\to e)$ in Fig. 6(c) and $P(e\to\tau)$ in Fig. 6(d). This is the analog to the maximal matter mixing of the two-neutrino case in Fig. 1. The energies of these extrema are correlated with the leading mass scale, E_{extrema} (MeV) $\simeq 3\text{–}4\times 10^6 |\delta m^2_{31}\ (\text{eV}^2)|$.

Qualitatively, three-neutrino matter oscillations display all the features of the two-neutrino case with the added complication of two δm^2 scales. For particular energy and length values, dramatic matter corrections can be expected.

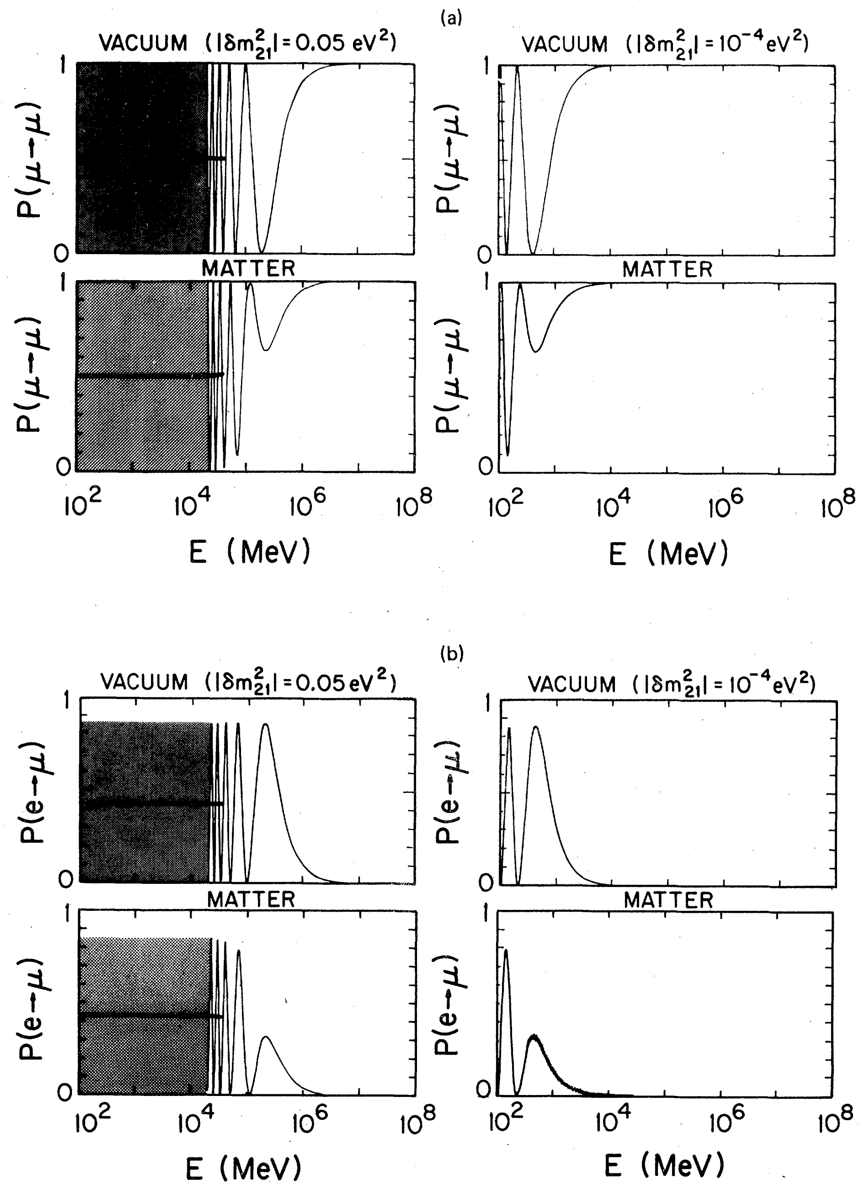

FIG. 6. Transition probabilities in vacuum and matter for the parameters of Eq. (31) at $L=5\times10^6$ m for $N_e=2N_A$ cm^{-3}. (a) $P(\mu\to\mu)$; (b) $P(e\to\mu)$; (c) $P(e\to e)$; (d) $P(e\to\tau)$; (e) $P(\mu\to\tau)$. The signs denote $(\delta m^2_{21}, \delta m^2_{31})$; unlabeled cases are essentially independent of the signs of the δm^2. The shaded areas are the regions of rapid oscillations; the central solid lines are the averaged values.

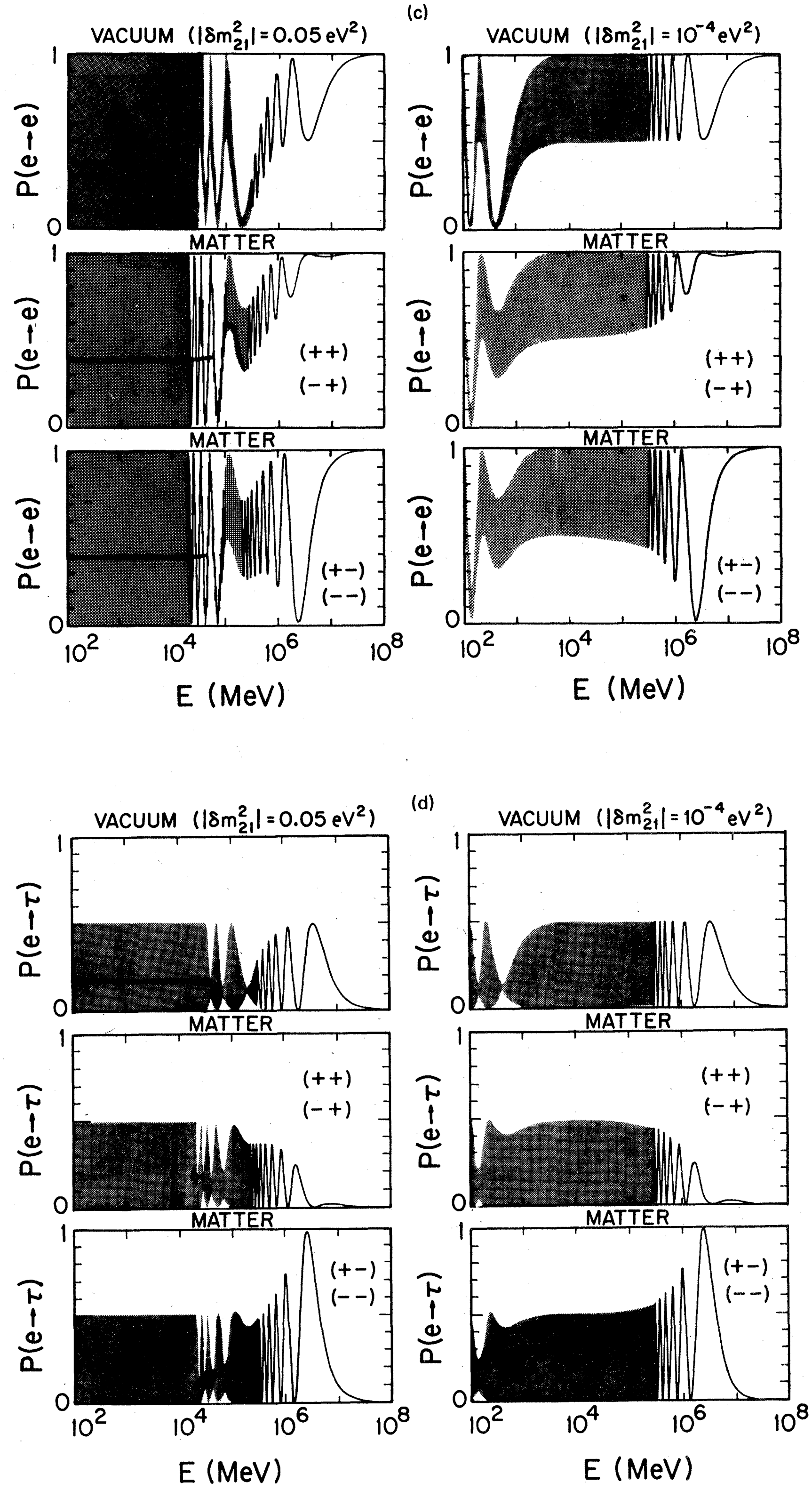

FIG. 6 (*Continued*.)

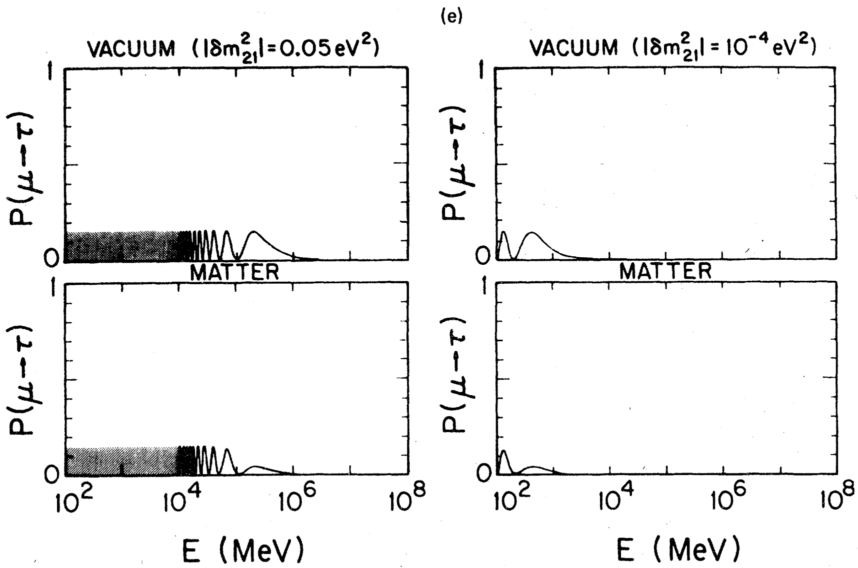

FIG. 6 (*Continued.*)

ACKNOWLEDGMENTS

We thank D. Cline for discussions. This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contract No. DE-AC02 76 ER00881-152.

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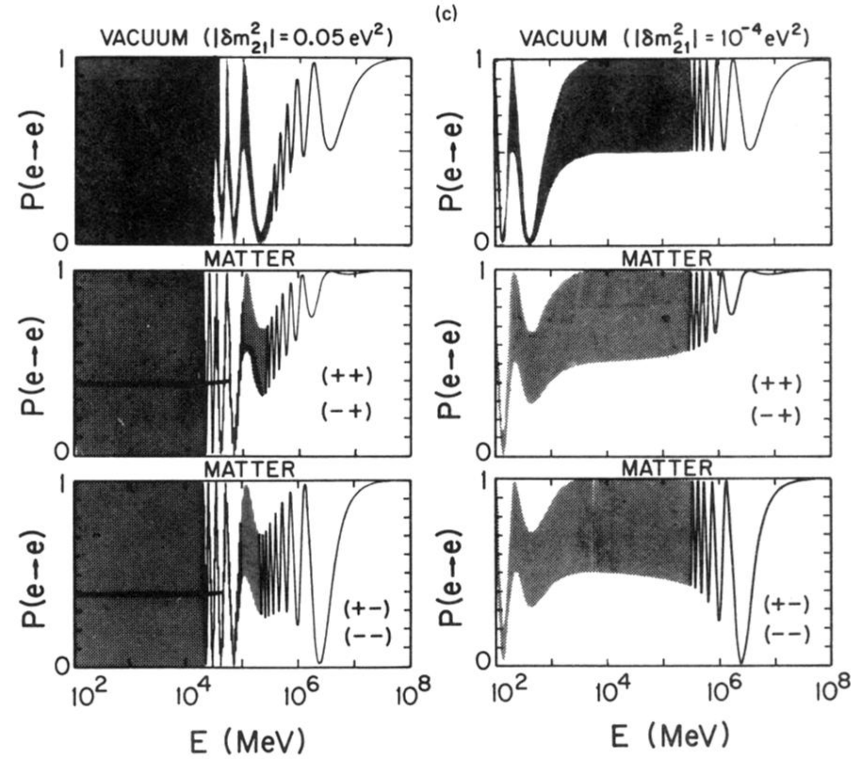

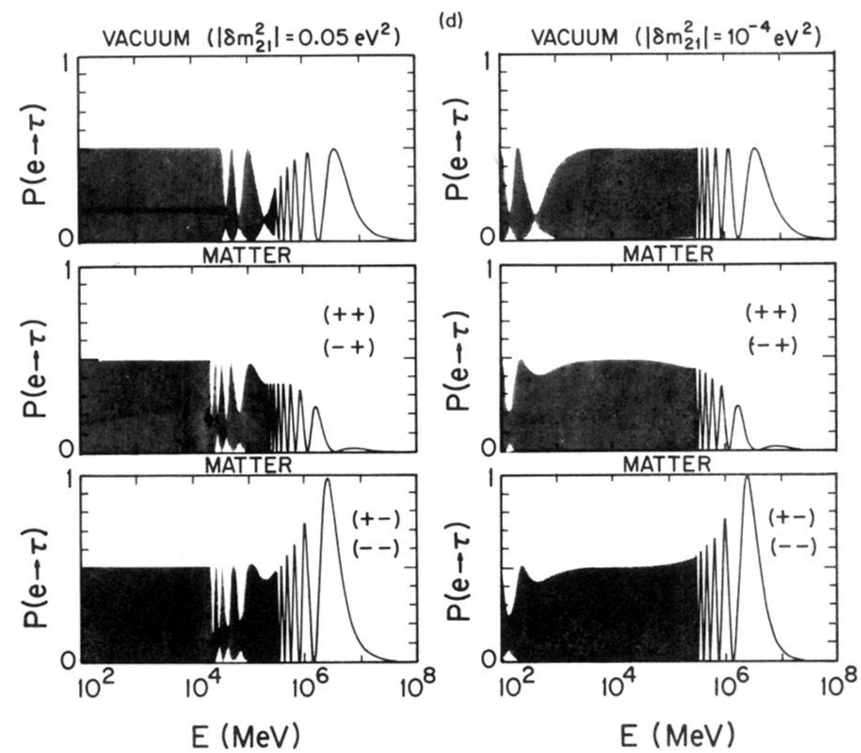

FIG. 6 (*Continued*.)

(e)

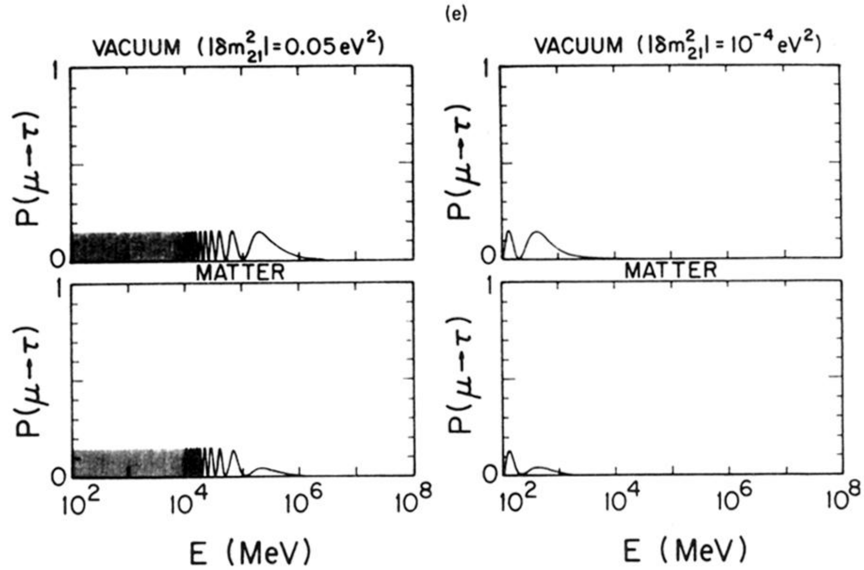

FIG. 6 (*Continued.*)

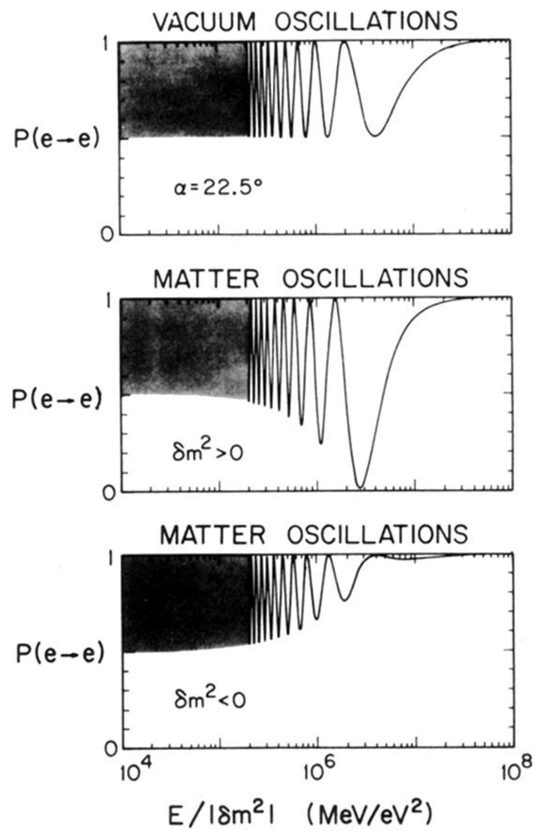

FIG. 2. Comparison of vacuum and matter transition probability $P(e \to e)$ for two neutrinos at fixed $L = 5 \times 10^6$ m for density $N_e = 2N_A$.

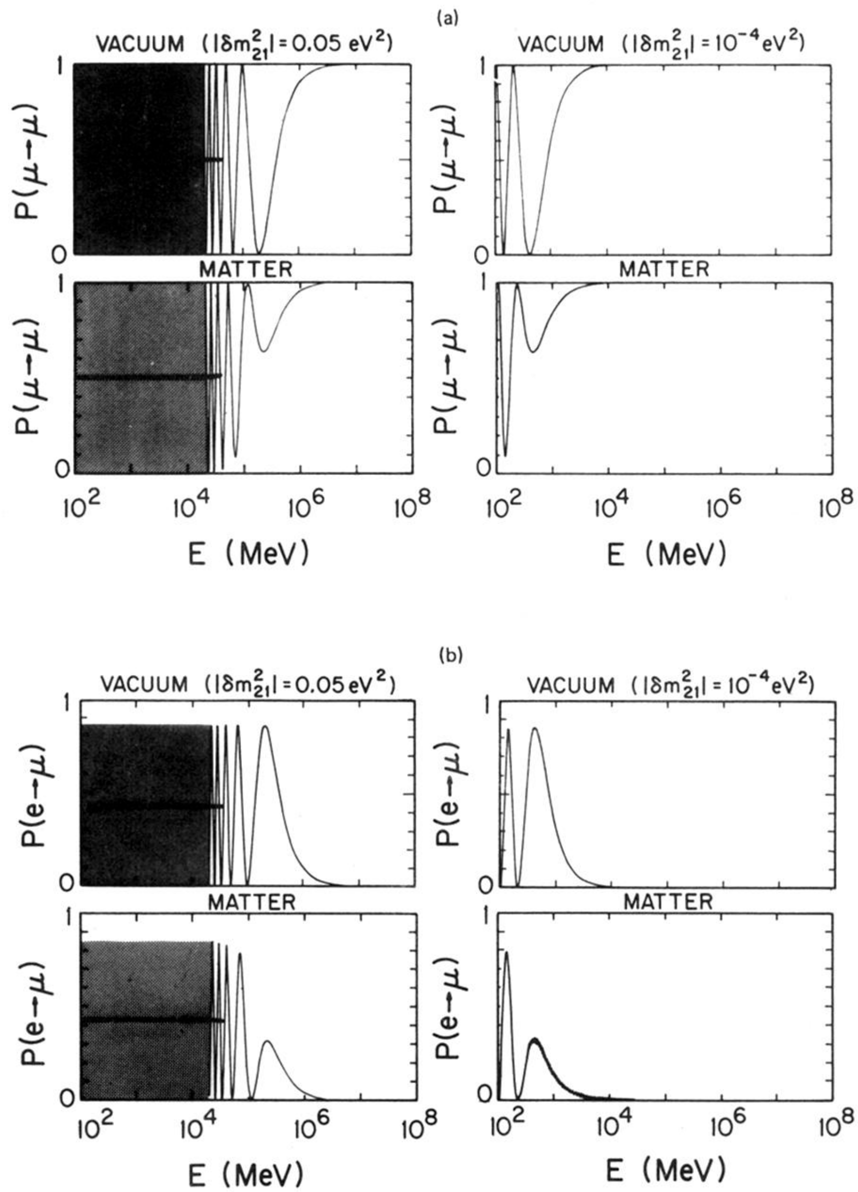

FIG. 6. Transition probabilities in vacuum and matter for the parameters of Eq. (31) at $L = 5 \times 10^6$ m for $N_e = 2N_A$ cm^{-3}. (a) $P(\mu \to \mu)$; (b) $P(e \to \mu)$; (c) $P(e \to e)$; (d) $P(e \to \tau)$; (e) $P(\mu \to \tau)$. The signs denote $(\delta m^2{}_{21}, \delta m^2{}_{31})$; unlabeled cases are essentially independent of the signs of the δm^2. The shaded areas are the regions of rapid oscillations; the central solid lines are the averaged values.