GLOBAL CONTINUUM OF SOLUTIONS FOR SYSTEMS OF ODES WITH PERIODIC BOUNDARY CONDITIONS AND GENERALIZED VARIABLE-EXPONENT OPERATORS

Raúl Manásevich
Departamento de Ingeniería Matemática
and Centro de Modelamiento Matemático (CNRS IRL2807)
FCFM, Universidad de Chile
Casilla 170 Correo 3, Santiago, Chile.

Jorge Novoa*
Departamento de Ingeniería Matemática
FCFM, Universidad de Chile
Santiago, Chile.

November 4, 2025

Abstract. We study nonlinear systems of boundary-value problems in one dimension of the form

$$(\mathcal{S}(t,u'))' = f(t,u,u',\lambda),$$

subject to periodic boundary value conditions. The function $\mathcal{S} : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and satisfies $\mathcal{S}(0,x) = \mathcal{S}(T,x)$ for all $x \in \mathbb{R}^N$, the function f is Carathéodory. Furthermore we assume $\mathcal{S}$ satisfies some monotonicity and coercivity conditions, thereby defining a class of generalized variable-exponent operators that strictly contains the Musielak–Orlicz case. Using an abstract continuation framework and the Leray–Schauder degree, we establish the existence of a global continuum of nontrivial solutions depending on the parameter λ. Our approach builds on a degree-theoretic alternative of Fitzpatrick, Massabò, and Pejsachowicz, reducing the evaluation of the Leray–Schauder degree to a Brouwer degree. This work shows how monotonicity and coercivity conditions in one dimension allow for continuum of solutions in a setting strictly more general than the Musielak–Orlicz framework. Furthermore our results are new even if $\mathcal{S}$ is independent of t.

1. Introduction

The study of differential equations involving nonlinear operators such as the p-Laplacian, the ϕ-Laplacian, operators with variable exponents, and double-phase operators has grown extensively over the last decades. A vast literature has been devoted to these topics; see, for instance, the monographs [19], [3] on variable exponent problems, the works [13], [20] concerning Orlicz–Sobolev frameworks, and the series of papers [22], [23], [24], [25] for results closely related to those presented here.

More recently, attention has been driven toward a very important and broader class of operators generated by Musielak–Orlicz functions, which provide a unified framework of the above-mentioned

Key words and phrases. Nonlinear differential equations, parameter-dependent problems, p-Laplacian, generalized variable exponents, generalized Musielak-Orlicz functions, Leray Schauder degree, solution continuum.

2020 *Mathematics Subject Classification.* 34A34, 34B08, 35J92, 47H11, 47J15.

R. Manásevich and J. Novoa were partially supported by Centro de Modelamiento Matemático (CMM) BASAL fund FB210005 for centers of excellence from ANID-Chile.

*Corresponding author. E-mail address: jnovoa@dim.uchile.cl .

cases. For the theory of Sobolev spaces of Musielak-Orlicz type, we refer to the survey papers [1], [11], and the recent books [17], [2], [9], and the references therein.

In this direction, Wang [21] studied periodic systems of ordinary differential equations in Musielak–Orlicz–Sobolev spaces using variational methods, obtaining existence results by critical point theory. In [7] and [8] systems of boundary value problems containing the operator $(\mathcal{S}(t,u'))'$ were studied with the mapping $\mathcal{S} : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ satisfying continuity, monotonicity, and coercivity assumptions (to be recalled in the next section). In our context we say that the $\mathcal{S}$ function is of Musielak–Orlicz type (MO type) if it has the form $\mathcal{S}(t,x) = \nabla_x \Phi(t,x)$ where $\Phi(t,x)$ is convex in x. It is worth emphasizing that in [7] and [8] and in this paper the structural assumptions on $\mathcal{S}$ (see next section) define a class of generalized variable-exponent operators $(\mathcal{S}(t,u'))'$ which is larger than the Musielak–Orlicz class.

In this paper we deal with some global continuum existence results which were motivated by continuation results of P. Rabinowitz [18], later extended by Fitzpatrick, Massabò, and Pejsachowicz [5], which yield connected components (continua) of solutions for nonlinear operator equations. From [5, Theorem 1.1, 1.2] the following theorem follows.

Theorem 1.1. *Let E be a Banach space, $\Omega \subset E\times\mathbb{R}$ an open set, $T : \overline{\Omega} \to E$ a completely continuous operator, and $F(u,\lambda) = u - T(u,\lambda)$. Then if for some $\lambda^* \in \mathbb{R}$ we have that:*

- (i) *$\Omega_{\lambda^*} \subset E$ is a nonempty open bounded set in E and $F(u,\lambda^*) = 0$ does not have solutions on $\partial\Omega_{\lambda^*}$ where $\Omega_\lambda := \{u \in E : (u,\lambda) \in \Omega\}$.*
- (ii) $d_{LS}[F(\cdot,\lambda^*),\Omega_{\lambda^*},0] \neq 0$.

Then, there exists a connected component ζ of solutions to $F(u,\lambda) = 0$ in $\overline{\Omega}$, such that ζ intersects $\Omega_{\lambda^*}\times\{\lambda^*\}$ and verifies at least one of the following: ζ is unbounded or $\partial\Omega\cap\zeta \neq \emptyset$.

The identification of suitable sets Ω via a priori bounds, and the computation of the Leray–Schauder degree in (ii), are the key steps in applying this theorem to boundary value problems. Theorem 1.1 provides a degree-theoretic approach to continuation which allows the analysis of connected components of solution sets via topological methods. The Leray-Schauder degree d_{LS} is a topological invariant that counts (with multiplicities) the number of solutions to nonlinear operator equations in infinite-dimensional spaces. When $d_{LS}[F(\cdot,\lambda^*),\Omega_{\lambda^*},0] \neq 0$, it guarantees the existence of at least one solution in Ω_{λ^*}, and through continuation arguments yields global information about solution branches. The power of this approach lies in reducing infinite-dimensional problems to finite-dimensional Brouwer degree computations, as we demonstrate in this work. We note that Theorem 1.1 may be viewed as an infinite-dimensional counterpart of a result of Mawhin and Dincă [4].

We apply these ideas to boundary value problems of the form

$$(\mathcal{S}(t,u'))' = f(t,u,u',\lambda), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{1}$$

where the function $\mathcal{S} : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies

- (H_0) The function $\mathcal{S} : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ is continuous, $\mathcal{S}(t,x) = 0$ if and only $x = 0$, and $\mathcal{S}(0,x) = \mathcal{S}(T,x)$ for all $x \in \mathbb{R}^N$.

Condition (H_0) will be assume without mentioning in the rest of this paper. The function $f : I\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}^N$ (here $I = [0,T]$) in (1) is Carathéodory, i.e.,

- (i) for almost every $t \in I$, the map $(x,y,\lambda) \mapsto f(t,x,y,\lambda)$ is continuous on $\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}$;
- (ii) for each fixed $(x,y,\lambda) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}$, the map $t \mapsto f(t,x,y,\lambda)$ is Lebesgue measurable on I;

(iii) for every $\rho > 0$ and every compact $\Lambda \subset \mathbb{R}$, there exists $h_{\rho,\Lambda} \in L^1(I)$ such that

$$|f(t,x,y,\lambda)| \le h_{\rho,\Lambda}(t)$$

for almost every $t \in I$, for all $\lambda \in \Lambda$, and all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \le \rho,\ |y| \le \rho$.

By a *solution* of (1) we understand a function $u : I \to \mathbb{R}^N$ of class C^1 with $\mathcal{S}(t,u'(t))$ absolutely continuous, which satisfies (1) a.e. on $[0,T]$.

We will show that these problems admit a continuum of nontrivial solutions depending on the parameter λ, under suitable assumptions on the functions $\mathcal{S}$ and f.

When λ is absent, existence results for this type of problems were obtained in [7, 8] using continuation theorems and the Leray–Schauder degree, thereby generalizing several results in the literature, for the case when t is not present in the function $\mathcal{S}$ see [14], [15], and [10], [12] for more classical existence results.

The present contribution deals with the existence of a continuum of solutions. We show how continuation methods can be used to establish the degree condition in Theorem 1.1, ultimately reducing the Leray–Schauder degree computation in that theorem to a Brouwer degree computation in finite dimensions; see Theorem 3.1 and Theorem 3.2 in Section 2. We point out that our results are new even if $\mathcal{S}$ is independent of t.

To illustrate our results, let us consider the following boundary value problems that satisfies a strict Hartman condition,

$$(a(t,|u'|)u')' = f(t,u,\lambda), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{2}$$

where $a : [0,T] \times (0,\infty) \mapsto (0,\infty)$ continuous, T-periodic, such that $\lim_{x\to 0} a(t,|x|)x = 0$ and such that $S(t,x) := a(t,|x|)x$ verifies $(H_0) - (H_2)$, and the function $f : [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is continuous. We will show in section 4 that this problem has a continuum of T- periodic solutions generalizing this way results in Xiang-Ling Fang and Xing Fan [6], Knobloch [12], of Mawhin [16] and [7].

This paper is organized as follows. Section 2 is dedicated to set the basic conditions on the function $\mathcal{S}$, study some of its properties, give some examples, and to formulate an abstract formulation of problem (1). In section 3 we establish and prove our main results concerning existence of continuum of periodic solutions for problem (1). In sections 4, 5 we present applications of our main results to systems satisfying respectively a Hartman condition, and a Villari condition, and in section 6 we give an example of the existence of a continuum of periodic solutions which is bounded in the parameter λ.

Throughout the paper, when it is clear from the context, $|\cdot|$ will denote absolute value, and the Euclidean norm on $\mathbb{R}^N$, while the inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$. Also for $N \ge 1$ we will set $C^0 = C(I,\mathbb{R}^N)$, $C^1 = C^1(I,\mathbb{R}^N)$, $C^0_T = \{u \in C^0 \mid u(0) = u(T)\}$, $C^1_T = \{u \in C^1 \mid u(0) = u(T), u'(0) = u'(T)\}$, and $L^p = L^p(I,\mathbb{R}^N)$, $p \ge 1$. The norm in C^0 and C^0_T will be denoted by $||\cdot||_0$, the norm in C^1 and C^1_T by $||\cdot||_1$, and the norm in L^p by $||\cdot||_{L^p}$. Finally $b(0,R)$ denotes the ball of radius R centered at 0 in $\mathbb{R}^N$ while $B(0,R)$ denotes the ball of radius R centered at 0 in C^1_T.

2. Preliminary results and abstract formulation

This section is dedicated to establish the structural hypotheses for the function $\mathcal{S}$, recall from [7, Section 2] some of its properties, show some examples, and to establish the abstract formulation of problem (1).

We assume the following two assumptions on the function $\mathcal{S}$,

(H_1) *For any* $t \in [0,T]$ *and any* $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$,
$$\langle \mathcal{S}(t,x_1) - \mathcal{S}(t,x_2), x_1 - x_2 \rangle > 0,$$

(H_2)
$$\frac{\langle \mathcal{S}(t,x), x\rangle}{|x|} \to \infty \quad \textit{as } |x| \to \infty \textit{ uniformly for } t \in [0,T].$$

It was shown in [7] that these two conditions imply that $\mathcal{S}_t(\cdot) := \mathcal{S}(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ and that $|\mathcal{S}(t,x)| \to \infty$ as $|x| \to +\infty$, uniformly for $t \in [0,T]$. In [7] it was also proved that condition (H_2) is equivalent to the existence of a function $\alpha : [0,+\infty) \to [0,+\infty)$ such that

$$(H_2') \qquad \langle \mathcal{S}(t,x), x\rangle \geq \alpha(|x|)|x|, \quad \text{for all } t \in [0,T],\ x \in \mathbb{R}^N.$$

with $\alpha(s) \to \infty$ as $s \to \infty$. Let us define $\mathcal{S}^{-1}(t,y) = (\mathcal{S}_t)^{-1}(y)$, under the above assumptions, it was proved in [7, Proposition 2.1] the following

Proposition 2.1. (i) *The function* $\mathcal{S}^{-1}$ *is continuous and satisfies* $|\mathcal{S}^{-1}(t,y)| \to \infty$ *as* $|y| \to \infty$, *uniformly for* $t \in [0,T]$.

(ii) *If* $\{\gamma_n\}$ *is a convergent sequence in* C^0, *say* $\gamma_n \to \gamma$ *as* $n \to \infty$, *then* $\mathcal{S}^{-1}(t,\gamma_n(t)) \to \mathcal{S}^{-1}(t,\gamma(t))$, *uniformly in* $[0,T]$, *as* $n \to \infty$.

Next, it will be convenient to define the function
$$G_l(a) = \frac{1}{T}\int_0^T \mathcal{S}^{-1}(t, a + l(t))\,dt, \tag{3}$$
for fixed $l \in C^0$. In this respect the following proposition was proved in [7, Proposition 3.1].

Proposition 2.2. *If* $\mathcal{S}$ *satisfies conditions* (H_1) *and* (H_2), *then the function* G_l *has the following properties:*

(i) *For any fixed* $l \in C^0$, *the equation*
$$G_l(a) = 0, \tag{4}$$
has a unique solution $\widetilde{a}(l)$.

(ii) *The function* $\widetilde{a} : C^0 \to \mathbb{R}^N$, *defined in* (i), *is continuous and sends bounded sets into bounded sets.*

With these results at hand we are ready to establish an abstract formulation of problem (1), following similar steps in [7].

Let us denote by N_f the Nemytskii operator associated to f with parameter λ, which is defined by
$$N_f : C^1([0,T],\mathbb{R}^N) \times \mathbb{R} \to L^1(I,\mathbb{R}^N), \quad N_f(u,\lambda)(t) = f(t,u(t),u'(t),\lambda).$$
Because of the parameter and for the sake of completeness we sketch a proof of the second part of next proposition at the end of the paper

Proposition 2.3. *The map* $N_f : C^1([0,T],\mathbb{R}^N) \times \mathbb{R} \to L^1(I,\mathbb{R}^N)$ *is continuous. Let* $B \subset C^1([0,T],\mathbb{R}^N) \times \mathbb{R}$ *be bounded, i.e.,*
$$\sup_{(u,\lambda)\in B} \|u\|_0 \leq \rho, \qquad \sup_{(u,\lambda)\in B} \|u'\|_0 \leq \rho, \qquad \sup_{(u,\lambda)\in B} |\lambda| \leq M$$
for some $\rho, M > 0$. *Then the set*
$$\mathcal{F} := \{\, N_f(u,\lambda) : \ (u,\lambda) \in B \,\} \subset L^1(I,\mathbb{R}^N)$$
is equi-integrable.

Let us set $\mathcal{G} : C^1_T \times \mathbb{R} \to C^1_T$ the completely continuous operator defined by:

$$\mathcal{G}(u, \lambda) := Pu + QN_f(u, \lambda) + (\mathcal{K} \circ N_f)(u, \lambda) \tag{5}$$

where N_f is the Nemytskii operator associated to f, the linear continuous operators $P : C^1_T \to C^1_T$, $Q : L^1 \to L^1$, are given by

$$P(u) := u(0), \quad Q(h) := \frac{1}{T}\int_0^T h(s)\,ds$$

and $\mathcal{K} : L^1 \to C^1_T$ is defined as follows

$$\mathcal{K}(h) = H\{\mathcal{Z}(a((I-Q)h)) + H((I-Q)h)\},$$

with I the identity in C^1_T, and for $h \in L^1$, $H(h)(t) = \int_0^t h(s)\,ds$, it was proved in [7, Lemma 3.2] that $\mathcal{K}$ is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C^1_T. The function $\mathcal{Z} : C^0 \to C^0$ is defined by $\mathcal{Z}(l)(t) = \mathcal{S}^{-1}(t, l(t))$, $\mathcal{Z}$ is continuous and sends bounded sets into bounded sets, while the function $a : L^1 \to \mathbb{R}^N$ is such that, for a fixed $h \in L^1$, is the unique solution of the equation

$$\frac{1}{T}\int_0^T \mathcal{S}^{-1}(t, a + H(h)(t))\,dt = 0. \tag{6}$$

It is important to point out that this function verifies $a(0) = 0$.

Then, as in [7], it holds that (1) is equivalent to the fixed point problem in C^1_T

$$u = \mathcal{G}(u, \lambda),$$

Finally the following proposition, proved in [7, Proposition 2.2], will be useful in the application of our results.

Proposition 2.4. *For $i = 1, \cdots k$, let $N_i \in \mathbb{N}$ and $\mathcal{S}_i : [0,T] \times \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ be a function $\mathcal{S}_i : [0,T] \times \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ is continuous such that $\mathcal{S}_i(0, x^i) = \mathcal{S}_i(T, x^i)$, for all $x^i \in \mathbb{R}^{N_i}$, $\mathcal{S}_i(t, x^i) = 0$ if and only $x^i = 0$, which satisfies the following conditions.*

(i) *$\langle \mathcal{S}_i(t,z) - \mathcal{S}_i(t,y), z-y\rangle_i \geq 0$, (with $\langle \cdot,\cdot\rangle_i$ denoting the inner product in $\mathbb{R}^{N_i}$), for any $t \in [0,T]$ and for any $z, y \in \mathbb{R}^{N_i}$, with equality holding true if and only if $z = y$;*

(ii)

$$\frac{\langle \mathcal{S}_i(t,z), z\rangle_i}{|z|} \to \infty \quad \text{as } |z| \to \infty \text{ uniformly for } t \in [0,T].$$

Then the function $\mathcal{S} : [0,T] \times \prod_{i=1}^k \mathbb{R}^{N_i} \to \prod_{i=1}^k \mathbb{R}^{N_i}$, defined by

$$\mathcal{S}(t,x) = (\mathcal{S}_1(t,x^1), \cdots, \mathcal{S}_k(t,x^k)), \text{ for } t \in [0,T] \text{ and } x = (x^1, x^2, \ldots, x^k) \in \prod_{i=1}^k \mathbb{R}^{N_i},$$

satisfies conditions (H_1) and (H_2) with $N = \sum_{i=1}^k N_i$.

From this proposition it follows immediately that the following $\mathcal{S}$ functions satisfy conditions (H_1) and (H_2).

(1) In our first example we take $\mathcal{S}(t,x) = (|y|^{p(t)-2}y, |z|^{q(t)-2}z)$, where $x = (y,z)$, $y \in \mathbb{R}^{N_1}$, $z \in \mathbb{R}^{N_2}$, with $N_1 + N_2 = N$ and $p, q : [0,T] \mapsto (1,\infty)$ are continuous T-periodic functions. If

$\Phi(t,(y,z)) = \frac{1}{p(t)}|y|^{p(t)} + \frac{1}{q(t)}|z|^{q(t)}$, then $\nabla_{(y,z)}\Phi(t,(y,z)) = \mathcal{S}(t,(y,z))$ and thus $\mathcal{S}(t,(y,z))$ is a Musielak-Orlicz function.

(2) In our second example we consider the so called double phase operator, $\mathcal{S}(t,x) = |x|^{p(t)-2}x + a(t)|x|^{q(t)-2}x$, where $x \in \mathbb{R}^N$, $p, q : [0,T] \mapsto (1,\infty)$ are continuous T-periodic functions and $a : [0,T] \mapsto (0,\infty)$ is also continuous and T-periodic. In a similar form if $\Phi(t,x) = \frac{1}{p(t)}|x|^{p(t)} + \frac{a(t)}{q(t)}|x|^{q(t)}$, then $\mathcal{S}(t,x) = \nabla_x \Phi(t,x)$, and the $\mathcal{S}$ function is Musielak-Orlicz.

The next example provides an $\mathcal{S}$ function which is not Musielak-Orlicz and which generalizes example 1.

(3) Let $\mathcal{S}(t,x) = (|y|^{p(t)-2}y, \phi(z))$, where $x = (y,z)$, $y \in \mathbb{R}^{N_1}$, $z \in \mathbb{R}^{N_2}$, with $N_1 + N_2 = N$, $p : [0,T] \mapsto (1,\infty)$ is a continuous T-periodic function, and ϕ is a homeomorphism of $\mathbb{R}^{N_2}$ with $\phi(0) = 0$, that satisfies

(i) For any $z_1, z_2 \in \mathbb{R}^{N_2}$, $z_1 \neq z_2$, $\langle \phi(z_1) - \phi(z_2), z_1 - z_2 \rangle > 0$,

(ii) $\frac{\langle \phi(z), z \rangle}{|z|} \to \infty \quad \text{as } |z| \to \infty$.

The $\mathcal{S}$ function in this example is not Musielak-Orlicz in general, since ϕ is not necessarily the derivative of a convex function.

Finally we show a way to easily construct $\mathcal{S}$ functions which are of not of Musielak-Orlicz type starting from any $\mathcal{S}$ function that satisfies conditions (H_1) and (H_2). Let us consider

$$\tilde{\mathcal{S}}(t,x) = \mathcal{S}(t,x) + B(t)x,$$

where $B(t)$ is a skew symmetric matrix for each $t \in [0,T]$, with $B(0) = B(T)$, and which is continuous. Then if $\mathcal{S}(t,x)$ satisfies conditions (H_1) and (H_2) so does $\tilde{\mathcal{S}}(t,x)$, but it is not of Musielak–Orlicz type even though the function $\mathcal{S}(t,x)$ could possible be. In this form starting from Examples 1,2 above we can generate functions $\tilde{\mathcal{S}}$ which are not of Musielak–Orlicz type.

For further properties of the functions $\mathcal{S}$ and related operators, see [7, Section 2, 3].

3. Existence of a Continuum of Periodic Solutions: A General Result

Let us set $E = C^1_T$, $\lambda^* = 0$, and for a set $\Omega_0 \subset C^1_T$, define

$$\Omega = (\Omega_0 \times \{0\}) \cup (C^1_T \times \mathbb{R} \setminus \{0\}). \tag{7}$$

Set also

$$T = \mathcal{G}(u,\lambda), \quad F(u,\lambda) = u - \mathcal{G}(u,\lambda).$$

In view of these definitions, in particular that of the set Ω, we next recast Theorem 1.1 in a more convenient form for our purposes.

Theorem 3.1. *Let $T : C^1_T \times \mathbb{R} \to C^1_T$ a completely continuous operator and define $F(u,\lambda) = u - T(u,\lambda)$, and let $\Omega_0 \subset C^1_T$ be a nonempty open bounded subset such that:*

(1) *the equation $F(u,0) = 0$ does not have solutions on $\partial\Omega_0$,*

(2) *$d_{LS}(F(\cdot,0),\Omega_0,0) \neq 0$,*

then, there exists a connected component ζ of solutions to $F(u,\lambda) = 0$ which intersects $\Omega_0 \times \{0\}$ and at least one of the following must hold: ζ is unbounded or intersects $(C^1_T \setminus \Omega_0) \times \{0\}$.

From Theorem 3.1 it is clear that we have to study the equation $F(u,0) = u - \mathcal{G}(u,0) = 0$, problem which turns out to be equivalent to the problem

$$(\mathcal{S}(t,u'))' = f(t,u,u',0), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{8}$$

The following theorem is a consequence of [7, Theorem 4.1] and Theorem 3.1 above.

Theorem 3.2. *Consider the problem:*

$$(\mathcal{S}(t,u'))' = \mu f(t,u,u',0), \quad u(0)=u(T), \quad u'(0)=u'(T), \tag{9}$$

where $\mathcal{S}:[0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ verifies $(H_0)-(H_2)$ and $f:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Additionally, suppose that there exists a nonempty open bounded set $\Omega_0 \subset C^1_T$ such that the following conditions are satisfied;

(i) *for $\mu \in (0,1]$ there are no solutions of the problem (9) on $\partial\Omega_0$,*

(ii) *The equation*

$$F(a) = \frac{1}{T}\int_0^T f(t,a,0,0)\,dt = 0$$

has no solutions in $\partial\Omega_0 \cap \mathbb{R}^N$,

(iii) $d_B(F,\Omega_0\cap\mathbb{R}^N,0)\neq 0$.

Then,

$$d_{LS}[I-\mathcal{G}(\cdot,0),\Omega_0,0]\neq 0,$$

and there exists a connected component of solutions ζ of (1) such that ζ intersects $\Omega_0\times\{0\}$ and at least one of the following must hold: ζ is unbounded or intersects $(C^1_T\setminus\Omega_0)\times\{0\}$.

Proof. That

$$d_{LS}[I-\mathcal{G}(\cdot,0),\Omega_0,0]\neq 0,$$

follows in the same way as in the proof of [7, Theorem 4.1], indeed repeating step by step the arguments in that proof our result follows. With this at hand the existence of a connected component of solutions of $F(u,\lambda)=0$, and therefore the existence of a connected component of solutions of problem (1) follows directly from Theorem 3.1. □

Remark 1. *Under the hypotheses of Theorem 3.2, if there are not solutions of (9) in $C^1_T\setminus\Omega_0$, then the connected component of solutions given by Theorem 3.2 is unbounded.*

Theorem 3.2 shows an interesting connection between the continuation Theorem [7, Theorem 4.1] and Theorem 1.1 in order to obtain existence of connected components of solutions for problem (1) .

4. A continuum of periodic solutions under a Hartman condition

In this section we will deal with existence of a continuum of periodic solutions under a Hartman condition generalizing results in [12], [16],[6] and [7].

We begin with the following definition. Let $b(0,R)$ be the ball radius R centered at 0 in $\mathbb{R}^N$, the retraction function $\rho_R:\mathbb{R}^N\to\overline{b(0,R)}\subset\mathbb{R}^N$ is defined by

$$\rho_R(x)=\begin{cases} x \text{ if } |x|\le R\\ \frac{Rx}{|x|} \text{ if } |x|>R\end{cases}$$

then $|\rho_R(x)|=R$ if $|x|\ge R$. We next prove

Theorem 4.1. *Let $\mathcal{S}:[0,T]\times\mathbb{R}^N\to\mathbb{R}^N$ be such that it verifies $(H_0)-(H_2)$, and let $f:[0,T]\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$ be a Carathéodory function such that the following Hartman type of condition holds*

$$\langle f(t,x,0),x\rangle > 0 \quad \textit{for all} \;\; t\in[0,T] \;\; \textit{and for all} \;\; x\in\mathbb{R}^N \textit{with } |x|=R, \tag{10}$$

for some fixed $R>0$, then the periodic problem

$$(\mathcal{S}(t,u'))' = f(t,\rho_R(u),\lambda), \;\; u(0)=u(T), \;\; u'(0)=u'(T) \tag{11}$$

has an unbounded connected component of solutions emanating from some point in $B(0, R_4) \times \{0\}$ *for a certain* $R_4 > 0$.

Proof. The proof consists in showing that the conditions of Theorem 3.2 are satisfied with $\Omega_0 = B(0, R_4) \subset C^1_T$ for a suitable $R_4 > 0$. Problem (9) in Theorem 3.2 takes the form

$$(\mathcal{S}(t, u'))' = \mu f(t, \rho_R(u), 0), \quad u(0) = u(T), \; u'(0) = u'(T), \tag{12}$$

and let u be a possible solution to this problem for a certain $\mu \in (0, 1]$. Taking the inner product of both members of the differential system with u, integrating over $[0, T]$ and using integration by parts and the periodic boundary conditions, we obtain

$$\begin{aligned} -\int_0^T \langle \mathcal{S}(t, u'(t)), u'(t)\rangle \, dt &= \mu \int_0^T \langle f(t, \rho_R(u(t)), 0), u(t)\rangle \, dt \\ &= \mu \int_{|u(t)|\le R} \langle f(t, u(t), 0), u(t)\rangle \, dt \\ &+ \mu \int_{|u(t)|>R} \langle f(t, Ru(t)/|u(t)|, 0), u(t)\rangle \, dt \\ &\ge \mu \int_{|u(t)|\le R} \langle f(t, u(t), 0), u(t)\rangle \, dt, \end{aligned} \tag{13}$$

since f is Carathéodory, we can find some $\alpha_{R,\{0\}} \in L^1(0, T)$ such that $|f(t, x, 0)| \le \alpha_{R,\{0\}}(t)$ for a.e. t and $|x| \le R$, and thus:

$$\int_{|u(t)|\le R} |\langle f(t, u(t), 0), u(t)\rangle| \, dt \le R\|\alpha_{R,\{0\}}\|_{L^1}$$

and hence, using (H_2') and defining $K := R\|\alpha_{R,\{0\}}\|_{L^1}/T$ we get:

$$\begin{aligned} \int_0^T \alpha(|u'(t)|)|u'(t)| \, dt &\le \int_0^T \langle \mathcal{S}(t, u'(t)), u'(t)\rangle \, dt \\ &\le \int_{|u(t)|\le R} |\langle f(t, u(t), 0), u(t)\rangle| \, dt \le TK. \end{aligned} \tag{14}$$

Since $\alpha(s) \to \infty$ as $s \to \infty$, it follows that there exists $r > 0$ such that $\alpha(|x|) \ge 1$ for all $|x| \ge r$, so that inequality (14) implies that

$$\int_{|u'(t)|\ge r} |u'(t)| \, dt \le TK,$$

and hence

$$\int_0^T |u'(t)| \, dt \le T(K + r). \tag{15}$$

Consequently, for all t and τ in $[0, T]$, we obtain

$$\begin{aligned} |u(t) - u(\tau)| &= \left| \int_\tau^t u'(s) \, ds \right| \le \left| \int_\tau^t |u'(s)| \, ds \right| \\ &\le \int_0^T |u'(s)| \, ds \le T(K + r). \end{aligned} \tag{16}$$

Now, if $|u(t)| \geq R$ for all $t \in [0,T]$, then the first equality in (13) implies that

$$0 \geq -\int_0^T \langle \mathcal{S}(t,u'(t)), u'(t)\rangle\, dt = \mu \int_0^T \langle f(t, Ru(t)/|u(t)|, 0), u(t)\rangle\, dt > 0,$$

a contradiction, and hence there exists $\tau \in [0,T]$ such that $|u(\tau)| < R$. Combined with (16), this implies that

$$|u(t)| \leq |u(t) - u(\tau)| + |u(\tau)| < T(K+r) + R =: R_0. \tag{17}$$

Let $R_1 = \max\{R_0/T, R_0\}$ (notice that this implies $TR_1 \geq R_0$), then from equation (15) and the definition of R_1, there is a $\tau \in [0,T]$, such that $|u'(\tau)| \leq R_1$, then integrating both members of (12), we obtain

$$(\mathcal{S}(t,u'(t))) = (\mathcal{S}(\tau,u'(\tau))) + \mu \int_\tau^t f(s, \rho_R(u(s)), 0))ds,$$

and hence using (17), we find

$$\begin{aligned} |(\mathcal{S}(t,u'(t)))| &\leq |(\mathcal{S}(\tau,u'(\tau)))| + \left|\int_\tau^t f(s,\rho_R(u(s)),0))ds\right| \\ &\leq \max_{[0,T]\times\overline{b(0,R_1)}} |\mathcal{S}(\cdot,\cdot)| + T\Big(\max_{[0,T]\times\overline{b(0,R)}} |f(\cdot,\cdot,0)|\Big) := R_2 \end{aligned} \tag{18}$$

for all possible solutions of (12) with $\mu \in (0,1]$ and all $t \in [0,T]$.

Consequently, for any $t \in [0,T]$, and thanks to the fact that $x = \mathcal{S}^{-1}(t, \mathcal{S}(t,x))$, we obtain that:

$$|u'(t)| = \left|\mathcal{S}^{-1}\big(t, \mathcal{S}(t,u'(t))\big)\right| \leq \max_{[0,T]\times\overline{b(0,R_2)}} \left|\mathcal{S}^{-1}(t,x)\right| =: R_3.$$

We now fix our set Ω_0, we first set $R_4 := \max\{R_0, R_1, R_2, R_3\} > R$ and then take $\Omega_0 = \{u \in C^1_T \mid ||u||_1 \leq R_4\}$. We next define $F : \mathbb{R}^N \to \mathbb{R}^N$ by

$$F(a) := (1/T)\int_0^T f(t, \rho_R(a), 0)\, dt.$$

Thus, by condition (10), one has

$$\langle F(a), a\rangle > 0$$

for all $a \in \mathbb{R}^N$ with $|a| \geq R$, and in particular for $|a| = R_4$, then it is immediate to see that the Brouwer degree, $d_B[F, B(0,R_4) \cap \mathbb{R}^N, 0] = 1$. Then, by Theorem 3.2, we have that

$$d_{LS}[I - \mathcal{G}(\cdot,0), B(0,R_4), 0] \neq 0,$$

and that there exists a connected component of solutions ζ of problem (11) such that ζ intersects $B(0,R_4) \times \{0\}$ and is unbounded, since it cannot intersect $(C^1_T \setminus B(0,R_4)) \times \{0\}$ for what we have just shown.

□

Next we assume the function $\mathcal{S}$ has the particular form $\mathcal{S}(t,x) = a(t,|x|)x$ where $a : [0,T] \times (0,\infty) \mapsto (0,\infty)$ is continuous, T-periodic, $\lim_{x\to 0} a(t,|x|)x = 0$ and such that $\mathcal{S}$ defined above verifies $(H_0) - (H_2)$. Let us consider the problem

$$(a(t,|u'|)u')' = f(t,u,\lambda), \;\; u(0) = u(T), \; u'(0) = u'(T), \tag{19}$$

where the function $f : [0,T]\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$ is continuous and satisfies a strict Hartman condition, namely, there is $R>0$ such that, for all $t\in[0,T]$ and $x\in\mathbb{R}^N$ with $|x|=R$, it holds that

$$\langle f(t,x,\lambda),x\rangle > 0 \quad \text{for all} \quad \lambda\in\mathbb{R}. \tag{20}$$

Remark 2. *The similar condition on f in the previous theorem corresponds to $\lambda=0$ in* (20).

We now show that, if we additionally assume that f is continuous and satisfies (20), then we can characterize the unboundedness of the connected set of solutions (λ,u) obtained in the previous theorem, in the sense that we can identify along which component the unboundedness occurs, or how it occurs.

Theorem 4.2. *Consider the problem* (19) *where $a(\cdot,\cdot)$ is such that $\mathcal{S}(t,x):=a(t,|x|)x$ verifies $(H_0)-(H_2)$, and let $f:[0,T]\times\mathbb{R}^N\to\mathbb{R}^N$ is continuous such that it verifies* (20)*, then the periodic boundary value problem* (19) *has a connected component of solutions whose projection on the λ axis is unbounded.*

Proof. It is clear that $\mathcal{S}(t,x)=a(t,|x|)x$ satisfies conditions (H_1) and (H_2). By the previous theorem, the problem

$$(a(t,|u'|)u')' = f(t,\rho_R(u),\lambda),\ \ u(0)=u(T),\ u'(0)=u'(T) \tag{21}$$

has a connected component ζ of solutions that is unbounded. We prove next that if (u,λ) belongs to ζ then it satisfies (19) and hence the full ζ is a connected component of problem (19).

So let $(u,\lambda)\in\zeta$, we claim that $u(t)\in\overline{b(0,R)}$ for all $t\in[0,T]$. Indeed if by contradiction, $|u(t_0)|>R$, for some $t_0\in[0,T]$, then for $\sigma\in[0,T)$ (we can take $\sigma=t_0$ without loss of generality) such that

$$\max_{t\in[0,T]}\frac{|u(t)|^2}{2}=\frac{|u(\sigma)|^2}{2}, \tag{22}$$

then by an argument in the proof of [7, Lemma 6.2], one has at the same time that

$$a(\sigma,|u'(\sigma)|)\langle u'(t),u(\sigma)\rangle = 0, \quad (a(t,|u'(t)|))\langle u'(t),u(t)\rangle > 0,$$

for all t in a small neighborhood of σ, which is clearly a contradiction to (22). Thus $|u(t)|\le R$ for all $t\in[0,T]$.

Notice here that we have proved that if $(u,\lambda)\in\zeta$, then $u(t)\in\overline{b(0,R)}$ for all $t\in[0,T]$ and hence the full connected component ζ of periodic solutions of (21) is also a connected component of periodic solutions of (19).

It remains to prove that the projection of this connected component over the λ axis is unbounded. For this since we already have $(u,\lambda)\in\zeta$ implies $||u||_0\le R$, then we only have to find a convenient bound for u'.

Integrating the equation in (19) from 0 to T and using the boundary conditions, as in the derivation of (14) together with condition (H_2'), we get

$$\int_0^T \alpha(|u'(t)|)|u'(t)|\,dt \le \int_0^T a(t,|u'(t)|)\langle u'(t),u'(t)\rangle\,dt$$

$$\le \int_{|u(t)|\le R}|\langle f(t,u(t),\lambda),u(t)\rangle|\,dt \le K(\lambda)RT,$$

where

$$K(\lambda):=\max_J|f(t,x,\lambda)|<\infty \qquad \text{with} \quad J:=[0,T]\times\overline{b(0,R)}.$$

Clearly the function K is continuous for $\lambda \in (-\infty, \infty)$ (notice that to conclude this we need f to be continuous). Thus as in (15), we obtain

$$\int_0^T |u'(t)|\,dt \leq T(RK(\lambda) + r).$$

From here there must exist a $\tau \in [0,T]$ such that $|u'(\tau)| \leq RK(\lambda)+r := K_1(\lambda)$. Then integrating (19), from τ to t, we find

$$a(t, |u'(t)|)u'(t) = a(\tau, |u'(\tau)|)u'(\tau) + \int_\tau^t f(s, u(s), \lambda)\,ds,$$

and thus

$$|\mathcal{S}(t, u'(t))| \leq |\mathcal{S}(\tau, u'(\tau))| + \int_0^T |f(s, u(s), \lambda)|\,ds,$$

where $\mathcal{S}(t, u'(t)) = a(t, |u'(t)|)u'(t)$. Then

$$|\mathcal{S}(t, u'(t))| \leq \max_{[0,T]\times\overline{b(0,K_1(\lambda))}} |\mathcal{S}(\cdot,\cdot)| + K(\lambda)T := K_2(\lambda), \qquad \text{for all} \quad t \in [0,T],$$

where again $K_2(\lambda)$ is finite and continuous on $\lambda \in \mathbb{R}$. Next, by using the identity $x = \mathcal{S}^{-1}(t, \mathcal{S}(t,x))$, we obtain

$$|u'(t)|^2 = |\langle \mathcal{S}^{-1}(t, \mathcal{S}(t, u'(t))), u'(t)\rangle| \leq |\mathcal{S}^{-1}(t, \mathcal{S}(t, u'(t)))||u'(t)|$$

and then

$$|u'(t)| \leq \max_{\tilde{J}} |\mathcal{S}^{-1}(t,y)| =: K_3(\lambda) \qquad \text{where} \quad \tilde{J} = [0,T] \times \overline{b(0, K_2(\lambda))}$$

where, once again, $K_3(\lambda)$ is finite and continuous with $\lambda \in \mathbb{R}$.

We finally have that $(u, \lambda) \in \zeta$ then

$$||u||_1 = ||u||_0 + ||u'||_0 \leq R + K_3(\lambda).$$

This implies $(u, \lambda) \in \zeta$ with λ in a compact interval I_λ of $\mathbb{R}$ implies $||u||_1$ is bounded by a positive constant depending on I_λ. This argument implies the projection of ζ on the λ axis is not bounded. $\square$

We give next a simple example of this situation. Consider the following system of differential equations:

$$(\mathcal{S}(t, u'))' = \Gamma(t, u) + \lambda^2 g(t, u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{23}$$

where $\mathcal{S} = a(t, |x|)x$ is the $\mathcal{S}$ function of the previous theorem and Γ is another $\mathcal{S}$ that satisfies conditions (H_1) and (H_2), the function $g : [0,T] \times \mathbb{R}^N$ is a continuous function satisfying that there exists $R_0 > 0$ such that for all $t \in [0,T], x \in \mathbb{R}^N$ with $|x| = R_0$ it holds that $\langle g(t,x), x\rangle > 0$.

In this case we have that $f(t, x, \lambda) = \Gamma(t, x) + \lambda^2 g(t, x)$, and hence

$$\langle f(t, x, \lambda), x\rangle = \langle \Gamma(t, x), x\rangle + \lambda^2 \langle g(t, x), x\rangle > 0$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| = R_0$. Then all the hypotheses of the last theorem are satisfied and we conclude that there exists a connected component of solutions $\zeta \subseteq C^1_T \times \mathbb{R}$ whose projection on the λ axis is unbounded, and since λ^2 appears in the equation this projection is $\mathbb{R}$.

5. A continuum of periodic solutions under a Villari condition

In our second application we consider problem (1) under a condition of the Villari type. Let $g = (g_1, \dots, g_N) : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, we will say that g satisfies a *generalized Villari condition* if there is a $\rho_0 > 0$ such that for all $u \in C^1_T$, $u = (u_1, \dots, u_N)$ with $\min_{t\in[0,T]} |u_j(t)| > \rho_0$ for some $j \in \{1, \dots, N\}$ it holds that:

$$\int_0^T g_i(t, u(t), u'(t))dt \neq 0,$$

for some $i \in \{1, \dots, N\}$.

Theorem 5.1. *Consider problem* (1)*, where* $\mathcal{S}$ *satisfies* $(H_0)-(H_2)$ *and* $f : [0,T]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ *is Carathéodory. Additionally suppose that the following conditions hold:*

(1) *There exists* $\mathcal{N} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and* $h \in L^1([0,T], \mathbb{R}_+)$ *such that*

$$\langle \mathcal{S}(t,y), \mathcal{N}'(x)y\rangle \geq 0,$$

and

$$|f(t,x,y,0)| \leq \langle f(t,x,y,0), \mathcal{N}(x)\rangle + h(t),$$

for all $x, y \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$.

(2) $f(\cdot,\cdot,\cdot,0)$ *satisfies a generalized Villari condition.*

(3) *There exists* $R_0 > 0$ *such that all the posible solutions to the equation*

$$F(a) := \frac{1}{T}\int_0^T f(t,a,0,0)dt = 0, \quad a \in \mathbb{R}^N,$$

belong to $b(0,R_0)$.

(4) *The Brouwer degree* $d_B[F, b(0,R_0), 0] \neq 0$.

Then there exists some $R > 0$ *such that*

$$d_{LS}[I - \mathcal{G}(\cdot,0), B(0,R), 0] \neq 0,$$

and there exists a connected component of solutions ζ *of* (1) *such that* ζ *intersects* $B(0,R)\times\{0\}$ *and is unbounded.*

Proof. The proof consists in showing that the hypotheses of Theorem 3.2 are satisfied. In the case of this example we notice that by conditions (1) to (4) there exists an $R > 0$ such that for any solution $u \in C^1_T$ of (9), satisfies $\|u\|_1 < R$. To see this we just follow the proof of [7, Theorem 5.1].

Since, without loss of generality, we can take $R > R_0$, and by using the excision property of the Brouwer degree, we obtain that

$$d_B[F, B(0,R)\cap\mathbb{R}^N, 0] = d_B[F, b(0,R_0), 0] \neq 0.$$

Thus, by Theorem 3.2, by taking $\Omega_0 = B(0,R)$, we first have

$$d_{LS}[I - \mathcal{G}(\cdot,0), B(0,R), 0] \neq 0,$$

and then that there is a connected component of solutions that is unbounded, by Theorem 3.1. □

6. A Continuum of Periodic Solutions Unbounded in C^1_T

In this section we consider the problem

$$(\mathcal{S}(t,u'))' = g(t,u) + \lambda h(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{24}$$

We have

Proposition 6.1. *Let $\mathcal{S} : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be such that it verifies $(H_0) - (H_2)$ and let $f : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ defined as $f(t,x,y) = g(t,x) + \lambda h(t,x,y)$ where:*

1. *$g : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and satisfies $\langle g(t,x), x\rangle > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$, $t \in [0,T]$.*
2. *h is Carathéodory and there exists $i \in \{1,\ldots,N\}$ such that $h_i(t,u,u') = h_i(t)$ with $\int_0^T h_i(t)\,dt \neq 0$, and there is a constant $M > 0$ such that $g_i(t,x) \le M$, for all $t \in [0,T]$ and for all $x \in \mathbb{R}^N$.*

Then, there exists an unbounded connected component of solutions ζ in $C^1_T \times \mathbb{R}$ of problem (24) whose projection on the λ axis is bounded, in other words for any $(u,\lambda) \in \zeta$, one has $|\lambda| \le C$, where C is a constant. In particular the connected component is unbounded in the C^1_T direction.

Proof. Problem (24) is equivalent to the fixed point problem

$$u = \mathcal{G}(u,\lambda), \tag{25}$$

with $\mathcal{G} : C^1_T \times \mathbb{R} \to C^1_T$ the completely continuous operator given in (5), and where N_f in (5) is the Nemytskii operator associated to f given by $f(t,u,\lambda) = g(t,u) + \lambda h(t,u,u')$. Let (u,λ) be a solution of (24), we want to show first that λ must be bounded. By integrating this equation from 0 to T, and by periodicity, we obtain:

$$0 = \int_0^T (\mathcal{S}(t,u'))'\,dt = \int_0^T g(t,u)\,dt + \lambda \int_0^T h(t,u,u')\,dt.$$

From here at the i-th coordinate component,

$$\lambda \int_0^T h_i(t)\,dt = -\int_0^T g_i(t,u)\,dt,$$

which implies that:

$$\lambda = \frac{-\int_0^T g_i(t,u)\,dt}{\int_0^T h_i(t)\,dt}.$$

Therefore:

$$|\lambda| \le \frac{\int_0^T |g_i(t,u(t))|\,dt}{\left|\int_0^T h_i(t)\,dt\right|} \le \frac{TM}{\left|\int_0^T h_i(t)\,dt\right|} := C.$$

Now, we notice that from hypothesis (1), it follows that $g(t,0) = 0$, for all $t \in [0,T]$.

To get our results we want to apply Theorem 3.2, and thus we consider the problem

$$(\mathcal{S}(t,u'))' = \mu g(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{26}$$

with $\mu \in (0,1]$.

Clearly $u = 0$ is a solution to problem (26), we want to prove next that is the only solution. Thus let $u \in C^1_T$ be a solution of this problem,. Multiplying by u the equation in (26), and then by integrating from 0 to T, we obtain

$$\int_0^T \langle (\mathcal{S}(t,u'))', u\rangle = \mu \int_0^T \langle g(t,u), u\rangle,$$

which implies

$$\int_0^T \langle \mathcal{S}(t,u'),u'\rangle + \mu \int_0^T \langle g(t,u),u\rangle = 0.$$

Now, by hypothesis (H_1) and by the definition of g, it follows that the integrands of the two integrals in this expression are non-negative implying that $\langle g(t,u(t)),u(t)\rangle = 0$ almost everywhere in $[0,T]$, by the continuity of u this implies that $u(t)=0$ for all $t\in[0,T]$, which is what we wanted to prove.

Next let for $\varepsilon > 0$ and small, let $B(0,\varepsilon)$ the ball with radius ε centered at 0 in C^1_T. The ball $B(0,\varepsilon)$ takes the place of Ω_0 in Theorem 3.2. Clearly there are no solutions of problem (26) in $\partial B(0,\varepsilon)$.

Next, let us consider the equation:

$$G(a) = \frac{1}{T}\int_0^T g(t,a)\,dt = 0. \tag{27}$$

We claim this equation does not have solutions in $\partial B(0,\varepsilon)\cap\mathbb{R}^N$. Indeed this follows immediately from

$$\langle G(a),a\rangle = \frac{1}{T}\int_0^T \langle g(t,a),a\rangle\,dt > 0, \quad a\neq 0.$$

from which we conclude that $G(a)\neq 0$ for $a\in\partial B(0,\varepsilon)\cap\mathbb{R}^N$.

From here, $d_B[G,B(0,\varepsilon]\cap\mathbb{R}^N,0)$ is well defined; that it is different from zero follows from a simple argument which is left to the reader.

By Theorem 3.2, this implies

$$d_{LS}[I-\mathcal{G}(\cdot,0),B(0,\varepsilon),0]\neq 0,$$

and that there exists a connected component of solutions ζ of problem (1) which emanates from $(0,0)$ and either is unbounded or intersects $(C^1_T\setminus B(0,\varepsilon))\times\{0\}$. The fact that this connected component emanates from $(0,0)$ comes from the fact that $u=0$ is the only solution for $\lambda=0$, and for this same argument, the connected component ζ cannot intersect $(C^1_T\setminus B(0,\varepsilon))\times\{0\}$, thus the connected component is unbounded and emanates from $(0,0)$ in $C^1_T\times\mathbb{R}$, and since we know that if $(u,\lambda)\in\zeta$ then $|\lambda|\le C$, we conclude that the set $\{||u||_1 : (u,\lambda)\in\zeta\}$ is unbounded.

$\square$

As an example, let $\mathcal{S}:[0,T]\times\mathbb{R}^2\mapsto\mathbb{R}^2$ satisfy conditions (H_1) and (H_2), with $\mathcal{S}=(\mathcal{S}_1(t,x),\mathcal{S}_2(t,x))$, and let us consider the following system:

$$\begin{cases} (\mathcal{S}_1(t,u'))' = \tanh(u_1)+\lambda e^t, \\ (\mathcal{S}_2(t,u'))' = |u_2|^{p-2}u_2 + \lambda|u_1|^{q-2}u_1, \\ u(0)=u(T), \quad u'(0)=u'(T), \end{cases} \tag{28}$$

where $p,q>1$. Denoting

$$h(t,(x_1,x_2)) := (e^t, |x_1|^{q-2}x_1) \quad , \quad g(t,(x_1,x_2)) := (\tanh(x_1), |x_2|^{p-2}x_2),$$

we notice that

(1) $|g_1(t,x)|\le 1$, since $\tanh(\cdot)$ is bounded by 1.
(2) $\int_0^T h_1(t)dt = \int_0^T e^t dt \neq 0$, since $e^t>0$ for all $t\in\mathbb{R}$.

Also for $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, we have that

$$\langle g(t,x), x\rangle = \tanh(x_1)x_1 + |x_2|^p > 0.$$

Thus the conditions of Proposition (6.1) are verified, and hence we can conclude that there exists a connected component of solutions of (28) that is unbounded in the C^1_T direction and which emanates from $(0,0)$ in $C^1_T \times \mathbb{R}$.

For a couple of examples of functions $\mathcal{S}$ for problem (28) we consider: (1) $\mathcal{S}(t,x) = (|y|^{p(t)-2}y, |z|^{q(t)-2}z)$, where $x = (y,z) \in \mathbb{R} \times \mathbb{R}$, and $p, q : [0,T] \mapsto (1,\infty)$ are continuous T-periodic functions. (2) The double phase $\mathcal{S}$ function given by $\mathcal{S}(t,x) = |x|^{p(t)-2}x + a(t)|x|^{q(t)-2}x$, where $x \in \mathbb{R}^2$, $p, q : [0,T] \mapsto (1,\infty)$ are continuous T-periodic functions and $a : [0,T] \mapsto (0,\infty)$ is also continuous and T-periodic.

7. Sketch of the proof of Proposition (2.3)

Let the function

$$f : I \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$$

satisfy the *Carathéodory conditions* in section 2. We give here the proof of the following

Proposition 7.1 (Images of bounded sets are equi-integrable)**.** *Let $B \subset C^1([0,T], \mathbb{R}^N) \times \mathbb{R}$ be bounded, i.e.,*

$$\sup_{(u,\lambda)\in B} \|u\|_0 \le \rho, \qquad \sup_{(u,\lambda)\in B} \|u'\|_0 \le \rho, \qquad \sup_{(u,\lambda)\in B} |\lambda| \le M$$

for some $\rho, M > 0$. Then the family

$$\mathcal{F} := \{\, N_f(u,\lambda) : \ (u,\lambda) \in B \,\} \subset L^1(I; \mathbb{R}^N)$$

is equi-integrable.

Proof. Let $\Lambda := [-M, M]$, which is compact. By (iii) there exists $\alpha_{\rho,\Lambda} \in L^1(I)$ such that for all $(u,\lambda) \in B$ and a.e. $t \in I$,

$$|N_f(u,\lambda)(t)| = \big|f(t, u(t), u'(t), \lambda)\big| \le \alpha_{\rho,\Lambda}(t).$$

Hence for any measurable $E \subset I$,

$$\sup_{(u,\lambda)\in B} \int_E |N_f(u,\lambda)(t)|\, dt \le \int_E \alpha_{\rho,\Lambda}(t)\, dt \xrightarrow[|E|\to 0]{} 0,$$

since $\alpha_{\rho,\Lambda} \in L^1(I)$. This is exactly uniform (equi-)integrability. ☐

References

[1] I. Chlebicka, A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces, *Nonlinear Analysis TMA,* **175** (2018), 1–27.

[2] I. Chlebicka, P. Gwiazda, A. Swiereczewska-Gwiazda and A. Wroblewska-Kamisnska, *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics, eBook, (2021).

[3] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, LNM vol. 2017, Springer, Berlin, Heidelberg (2011).

[4] G. Dinca and J. L. Mawhin, Brouwer Degree: The Core of Nonlinear Analysis, Birkhäuser, Cham, 2021.

[5] P. M. Fitzpatrick, I. Massabó, and J. Pejsachowicz, On the covering dimension of the set of solutions of some nonlinear equations, *Trans. Amer. Math. Soc.* **296** (1986), 777–798.

[6] X. L. Fan and X. Fan, A Knobloch-type result for $p(t)$-Laplacian systems, *J. Math. Anal. Appl.* **282** (2003), 453–464.

[7] M. García-Huidobro . R. Manásevich, J. Mawhin and S. Tanaka, Periodic solutions for nonlinear systems of Ode's with generalized variable exponents operators, *J. Differential Equations* **388** (2024), 34–58.

[8] M. García-Huidobro . R. Manásevich, J. Mawhin and S. Tanaka,Two point boundary value problems for ordinary differential systems with generalized variable exponents operators *Nonlinear analysis: real world applications*, **81** (2025-02) 104–196.
[9] P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces,* Lecture Notes in Mathematics, LNM vol. 2236, Springer, Cham (2019).
[10] Ph. Hartman, On boundary value problems for systems of ordinary nonlinear, second order differential equations, Trans. Amer. Math. Soc. **96** (1960), 493–509.
[11] D. Huet, Non-linear problems in PDE's, a survey, hal-03951366, 2023.
[12] H.W. Knobloch, On the existence of periodic solutions for second order vector differential equations, *J. Differential Equations,* **9** (1971), 67–85.
[13] M. Krasnosel'skii and J. Rutick'ii, *Convex Functions and Orlicz Spaces,* Noordhoff, Groningen, (1961).
[14] R. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations **145** (1998), 367–393.
[15] R. Manásevich and J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators. J. Korean Math. Soc. **37** (2000), no. 5, 665–685.
[16] J. Mawhin, Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian, *Nonlinear Anal. TMA,* **40** (2000), 497–503.
[17] O. Méndez and J. Lang, *Analysis on Function Spaces of Musielak–Orlicz Type*, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, FL (2019).
[18] P. H. Rabinowitz, *Analyse numérique: Théorie du degré topologique et applications*, Gauthier-Villars, Paris, 1970.
[19] V. Rădulescu and D. Repovš, *Partial Differential Equations with Variables Exponents*, CRC Press, Taylor & Francis Group, Boca Raton , FL, (2015).
[20] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, (1991).
[21] Y. Wang, Existence of periodic solutions for ordinary differential systems in Musielak-Orlicz-Sobolev spaces, JMAA **488** (2020), 124070.
[22] Q. Zhang, Existence of solutions for weighted $p(r)$-Laplacian system boundary value problems, *J. Math. Anal. Applic.,* **327** (2007), 121–141.
[23] Q. Zhang, X. Liu and Z. Qiu, Existence of solutions and multiple solutions for a class of weighted $p(r)$-Laplacian system, *J. Math. Anal. Applic.,* **355** (2009), 620–633.
[24] Q. Zhang, Z. Qiu and X. Liu, Existence of multiple solutions for weighted $p(r)$-Laplacian equation Dirichlet problems, *Nonlinear Analysis TMA,* **70** (2009), 3721–3729.
[25] Q. Zhang, Z. Qiu and X. Liu, Existence of solutions for prescribed variable exponent mean curvature system boundary value problem, *Nonlinear Anal. TMA,* **71** (2009), 2964–2975.