Scientific Computing for Differential Equations

Lecture 02B - Simple Methods and Linear Systems

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02686 Scientific Computing for Differential Equations

Ordinary Differential Equations

Simple methods

$$\dot{x}(t) = f(x(t)) \qquad x(t_0) = x_0$$
$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))d\tau$$

- Explicit Euler method

$$x_{k+1} = x_k + f(x_k)\Delta t$$

- Implicit Euler method

$$x_{k+1} = x_k + f(x_{k+1})\Delta t$$
$$R(x_{k+1}) = x_{k+1} - f(x_{k+1})\Delta t - x_k = 0$$

- Trapezoidal method

$$x_{k+1} = x_k + \left(\frac{1}{2}f(x_k) + \frac{1}{2}f(x_{k+1})\right)\Delta t$$
$$R(x_{k+1}) = x_{k+1} - \frac{1}{2}f(x_{k+1})\Delta t - x_k - \frac{1}{2}f(x_k)\Delta t = 0$$

Simple methods for linear systems - $x \in \mathbb{R}$ (or $x \in \mathbb{C}$)

$$\dot{x}(t) = \lambda x(t) \qquad x(0) = x_0 = 1$$
$$x(t) = \exp(\lambda t) x_0 = \exp(\lambda t)$$

- Explicit Euler method

$$x_{k+1} = x_k + \lambda x_k \Delta t = (1 + \lambda \Delta t) x_k$$

- Implicit Euler method

$$x_{k+1} = x_k + \lambda x_{k+1} \Delta t$$
$$x_{k+1} = \frac{1}{1 - \lambda \Delta t} x_k$$

- Trapezoidal method

$$x_{k+1} = x_k + \left(\frac{1}{2} \lambda x_k + \frac{1}{2} \lambda x_{k+1} \right) \Delta t$$
$$x_{k+1} = \frac{1 + \frac{1}{2} \lambda \Delta t}{1 - \frac{1}{2} \lambda \Delta t} x_k$$

Simple methods for linear systems - $x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$$
$$x(t) = \exp(At)x_0$$

- Explicit Euler method

$$x_{k+1} = x_k + Ax_k\Delta t = (I + A\Delta t)x_k$$

- Implicit Euler method

$$x_{k+1} = x_k + Ax_{k+1}\Delta t$$
$$x_{k+1} = (I - A\Delta t)^{-1} x_k$$

- Trapezoidal method

$$x_{k+1} = x_k + \left(\frac{1}{2}Ax_k + \frac{1}{2}Ax_{k+1}\right)\Delta t$$
$$x_{k+1} = \left(I - \frac{1}{2}A\Delta t\right)^{-1}\left(I + \frac{1}{2}A\Delta t\right)x_k$$

Simple methods for linear systems - $X \in \mathbb{R}^{n \times n}$

$$\dot{X}(t) = AX(t) \qquad X(0) = X_0 = I$$
$$X(t) = \exp(At)X_0 = \exp(At)$$

- Explicit Euler method

$$X_{k+1} = X_k + AX_k\Delta t = (I + A\Delta t)X_k$$

- Implicit Euler method

$$X_{k+1} = X_k + AX_{k+1}\Delta t$$
$$X_{k+1} = (I - A\Delta t)^{-1} X_k$$

- Trapezoidal method

$$X_{k+1} = X_k + \left(\frac{1}{2}AX_k + \frac{1}{2}AX_{k+1}\right)\Delta t$$
$$X_{k+1} = \left(I - \frac{1}{2}A\Delta t\right)^{-1}\left(I + \frac{1}{2}A\Delta t\right)X_k$$

Exercises

- Test the methods on the linear systems from Lecture 01.
- Implement your own algorithm for computing the matrix exponential: $\exp(A)$. Compare it to Matlabs algorithm `expm`