Chapter 1

The Driven Damped Linear Oscillator System

Comments and corrections should be sent to pghj@dtu.dk.

1.1 Introduction

A dynamical system with a differential equation given by

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f(t) \tag{1.1}$$

where γ and ω_0 are positive parameters, is called by engineers and physicists the *Driven Damped Harmonic (or Linear) Oscillator*.

The parameter γ is known as the coefficient of damping, and the term $\gamma\dot{x}$ is called the damping term. The parameter ω_0 is called the frequency (or the eigenfrequency) of the oscillator.

Equation (1.1) appears in a large number models of physical systems: for instance, the equation describes both a mass attached to a spring with a linear damper and acted on by an external force, as well as a simple electronic circuit with an inductor, a capacitor and resistor, driven in series by an external voltage. See figure 1.1.

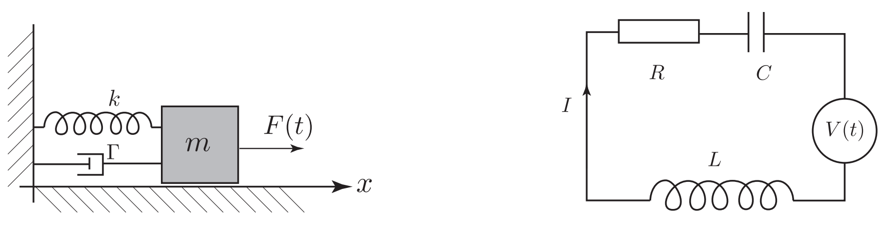


Figure 1.1: Two systems which are both described by driven damped harmonic oscillator equations: The mass-spring-damper system (left) and the inductor-capacitor-resistor system (right). Although the physical interpretations of the dynamical variables ($x(t)$ for the spring, and $I(t)$ for the circuit) are different, the two systems from a mathematical point of view are equivalent.

We shall solve various cases of equation (1.1) in some detail. Equation (1.1) is an inhomogeneous, linear, ordinary differential equation, and we know that the general solution can be written as the complete solution to the homogeneous part (the same left-hand side terms but zero right-hand-side), plus one arbitrary solution to the full inhomogeneous equation. We therefore focus first on finding the complete solution to the homogeneous equation, sometimes called the *unforced* or *free*damped linear oscillator equation.

1.2 The unforced damped linear oscillator

With zero on the right hand side (representing a case with no external forcing), the equation is:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{1.2}$$

We write the equation as a system

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

The eigenvalues of the system matrix are: $\lambda_+ = -\gamma/2 + \omega_0\sqrt{(\gamma/2\omega_0)^2 - 1}$ and $\lambda_- = -\gamma/2 - \omega_0\sqrt{(\gamma/2\omega_0)^2 - 1}$.

Case 1. Weak damping . If $\gamma/2 < \omega_0$ (for many cases $\gamma \ll \omega_0$) the system is called *weakly damped*. In this case the two eigenvalues are complex: $\lambda_\pm = -\gamma/2 \pm i\omega_0\sqrt{1 - (\gamma/2\omega_0)^2}$, and the origo is a stable spiral.

We can in this case write the general solution in x as

$$x(t) = c_1 e^{-\gamma t/2}\cos(\omega_d t) + c_2 e^{-\gamma t/2}\sin(\omega_d t) \tag{1.3}$$

where c_1 and c_2 are constants depending on the initial conditions, and

$$\omega_d = \omega_0\sqrt{1 - (\gamma/2\omega_0)^2} \tag{1.4}$$

In the particular case $\gamma = 0$ (the undamped oscillator), the system is nonhyperbolic, having both eigenvalues on the imaginary axis. The origo in this case becomes a center, and all solutions oscillate with frequancy ω_0 See chapter 6: Hamiltonian systems.

An equivalent way of writing the general solution in x (1.3) is (see Problem 1.1):

$$x(t) = Ce^{-\gamma t/2}\cos(\omega_d t + \varphi) \tag{1.5}$$

where C and φ are constants dependent on the initial conditions.

Note: The damping makes the oscillation frequency ω_d smaller than the corresponding frequency ω_0 for the undamped harmonic oscillations. If $\gamma \ll \omega_0$ we see from (1.4) that $\omega_0 \approx \omega_d$. The case $\gamma \ll \omega_0$ is by far the most important case for harmonic oscillations.

When a hammer hits a church bell in a brief blow, the church bell starts to ring. The surface of the bell performs harmonic oscillations with several frequencies, and, due mainly to a coupling to the surrounding air, the oscillations of the bell are damped. The energy in the oscillations of the bell is slowly transferred to other degrees of freedom, in this case sound waves. The bell emits sound waves corresponding to the dominant (i.e., least damped) frequency of the oscillations. One might say that γ in this case describes the coupling between the oscillations of the bell and the sound field.

Case 2. Strong damping . In the case $\gamma/2 > \omega_0$ the eigenvalues are both real, negative, and the system is hyperbolic. The general solution for $x(t)$ becomes

$$x(t) = c_1\exp(-\lambda_+ t) + c_2\exp(-\lambda_- t) \tag{1.6}$$

and the phase portrait is a node. There is no oscillation in the system, and the amplitude decays expontntially fast to the origin $x = 0$. A system with these parameter values is called *strongly damped* or *overdamped*. The least negative eigenvalue sets the timescale for the damping of the motion.

Case 3. Critical damping . The particular case $\gamma/2 = \omega_0$ is known as *critical damping*. The eigenvalues merge, and the phase portrait is that of a degenerate node. The analytical solution becomes

$$x(t) = c_1\exp(-\gamma t/2) + c_2 t\exp(-\gamma t/2) \tag{1.7}$$

Critical damping is of interest in the design of analog measuring instruments where a mechanical needle indicating a measurement reading has an equation of motion corresponding to that of a damped harmonic oscillator, with the correct reading as its stationary value. If the oscillator was underdamped, the needle would oscillate for a long time before coming to rest at the correct reading. If the oscillator was overdamped, the approach to the correct reading would be monotone but also quite slow. The compromise is the critically damped case which has the fastest approach to the reading, overshooting the reading exactly once.

1.3 The forced weakly damped linear oscillator

We return now to the case of the weakly damped linear oscillator, and consider now the case where an inhomogeneous term (i.e., not containing the dynamical variable x) is present. From a physical point of view such a term is included when the system is acted on by an external driver; this could an external force in the case of the spring system (hence the constant m), or a source of external voltage in the case of the electric circuit.

Of particular interest for applications is the case where the inhomogeneous term is *periodic* in time. Continuing the notation from the spring system, we consequently write the dynamical system as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t) \tag{1.8}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{m}\cos(\omega t) \end{bmatrix}$$

Here we have used $\cos(\omega t)$ to represent the periodic external signal. From Fourier analysis we know that for a general class of periodic signals it is possible to represent the signal as a superposition of 'pure' harmonic terms like $\cos(n\omega t)$ with $n \in \mathbb{N}$. In this sense it is therefore sufficient to solve the problem for a single such term.

From the general structure theorem for linear differential equations we know that the complete solution to the inhomogeneous system consists of the complete solution to the homogenous system plus one arbitrary solution to the inhomogeneous system.

From the previous section we have the complete solution to the homogeneous part of the equation. To find the solution $x_p(t)$ to the inhomogeneous part we make an educated guess: We assume this particular solution $x_p(t)$ to have the form

$$x_p(t) = A\cos(\omega t - \theta)$$

where the constants A and θ must be determined.

To this end, we assume that both the forcing and the particular solution x_p are real parts of complex-valued functions:

$$F/m\cos(\omega t) = \Re(F/m\exp(i\omega t))$$

and

$$A\cos(\omega t - \theta) = \Re(A\exp(i(\omega t - \theta)))$$

Since the equation is linear, the real part of the solution will correspond to the real part of the inhomogenous term.

Substituting the complex-valued functions into equation (1.8), one finds that

$$A\exp(i(\omega t - \theta)) = \frac{F/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)}\exp(i(\omega t))$$

or

$$A(\omega) = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \tag{1.9}$$

$$\tan(\theta(\omega)) = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{1.10}$$

Taking the real part of the complex solution, we have the real particular solution to be

$$x_p(t) = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \theta)$$

and consequently the complete solution is

$$x(t) = c_1 e^{-\gamma t/2} \cos(\omega_d t) + c_2 e^{-\gamma t/2} \sin(\omega_d t) + x_p(t)$$

Notice that the solution part corresponding the homogeneous part of the equation (where c_1 and c_2 can be set to match the initial conditions) will vanish exponentially quickly with time. For this reason, that part of the solution is known as the *transient*. After a time of order

$$t_{\text{transient}} = 2/\gamma$$

only the particular part $x_p(t)$ of the solution remains. For this reason $x_p(t)$ it is also called the *steady-state solution*. The steady state solution, $x(t)$ oscillates with the frequency of the applied force and has a constant amplitude A. With the chosen sign for the phase shift θ in the term $\cos(\omega t - \theta)$, θ gives the lagging of the oscillations of the system behind the phase of the applied driver, $F/m \cos(\omega t)$. A negative value of θ will mean that the oscillations are ahead of the applied force in phase.

1.4 Resonance

It is of some interest to observe how the value of ω, the *driving frequency*, affects the stationary solution to the driven, weakly damped, linear oscillator.

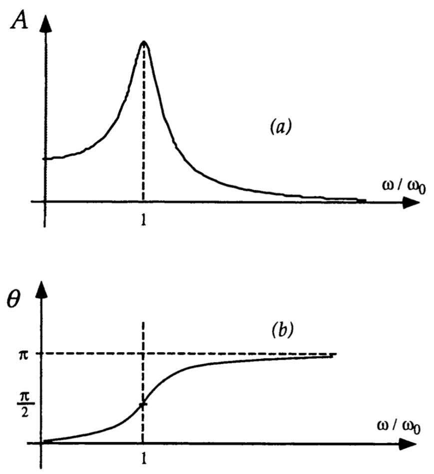


Figure 1.2: Frequency characteristics

We will consider three cases: (1): $\omega \ll \omega_0$, (2): $\omega_0 \ll \omega$, and (3): $\omega \approx \omega_0$.

(1) $\omega \ll \omega_0$. For a very low driving frequency, the stationary solution amplitude A becomes ω-independent :

$$A \to \frac{F/m}{\omega_0^2} \qquad \text{when} \qquad \omega \to 0$$

for the phase shift θ we have

$$\theta \to 0 \qquad \text{when} \qquad \omega \to 0$$

(2) $\omega_0 \ll \omega$. For a very high driving frequency the stationary solution vanished inversely proportional to the square of the driving frequency:

$$A \to \frac{F/m}{\omega^2} \qquad \text{when} \qquad \omega \gg \omega_0$$

for the phase shift θ we have

$$\theta \to \pi \qquad \text{when} \qquad \omega \gg \omega_0$$

The most interesting case is **(3)** $\omega_0 \approx \omega$. When ω is near ω_0 the amplitude reaches its maximum value. This phenomenon is known as **resonance**. Note from (1.10) that the maximum value of A does not occur exactly for $\omega = \omega_0$ but rather for the value

$$\omega_m = \omega_0 \sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}$$

Compare this to the exact oscillation frequency for the undriven (free) damped linear oscillator:

$$\omega_d = \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}$$

Clearly, for a weakly damped linear oscillator, $\omega_m \approx \omega_d \approx \omega_0$.

The obvious physical interpretation of resonance is that for a weakly damped system the largest response (i.e., the largest value of the amplitude) for the driven system occurs when the system is driven near its eigenfrequency. The phenomenon is familiar to anyone who has pushed a swing.

The oscillation is out of phase with the driving near resonance (since $\theta \approx \pi/2$), but the time derivative of the oscillation, the velocity for the case of the spring system, *is* in phase with the driving source at resonance. This brings us to a brief discussion of energy behavior and also a quantification of the 'narrowness' of the resonance peak.

1.5 Energy and Q-values for the weakly damped linear oscillator

The steady state solution balances the work done by the external source with the energy lost to friction in the system.

The time average of the work performed by the external source on a driven oscillator at a given driving frequency ω is

$$P(\omega) = \langle f(t)v(t)\rangle = \langle (F\cos(\omega t))(-A(\omega)\omega\sin(\omega t - \theta(\omega))\rangle \tag{1.11}$$

Here, $\langle y(t)\rangle := \frac{1}{T}\int_T y(t)\mathrm{d}t$ is the time average of any periodic function $y(t)$.

Performing the integral, one obtains

$$P(\omega) = \frac{F^2}{2\gamma m}\frac{(\gamma\omega)^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]} \tag{1.12}$$

Figure (1.3) shows the power absorption as a function of the driving frequency for a weakly damped oscillator.

For $\omega = \omega_0$ the power absorption becomes

$$P = \frac{F^2}{2m\gamma}$$

How quickly away from ω_0 the power absorption drops to half the value its resonnance value can be found by setting in (1.12):

$$\frac{(\gamma\omega)^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} = \frac{1}{2}$$

and using that, near resonance, $\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0(\omega_0 - \omega)$, one finds that for

$$|\omega - \omega_0| = \frac{\gamma}{2}$$

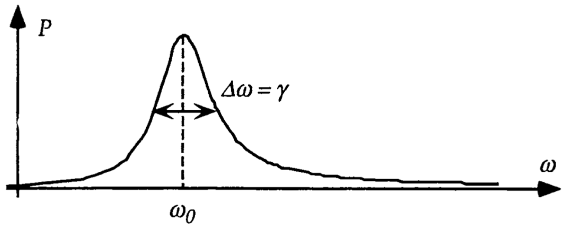


Figure 1.3: The power absorbtion P as a function of ω. At half-maximum, the width of the peak is $\Delta\omega = \gamma$

the power absorption will be at half its maximum value; consequently the full width of the power absorption at half maximum (sometimes referred to as the FWHM) is $\Delta\omega = \gamma$.

The resonance properties of a given weakly damped oscillator are often characterized by the so-called **Q-value**.

Consider first an unforced, weakly damped harmonic oscillator. The energy E dissipated during one oscillation will be small compared to the amount of energy stored in the oscillations. The Q-value for such an oscillator is defined as 2π times the mean energy stored, divided by the energy dissipated, per period. (The factor 2π could have been avoided if the work done per radian had been used instead of the work per period):

$$Q = 2\pi \frac{E}{PT}$$

where $T = 2\pi/\omega_0$ is the approximate period of the wealy damped oscillator; the Q-value is a useful number only if we consider weakly damped oscillations ($\gamma \ll \omega_0$) . We are therefore assuming that

$$\omega_d = \omega_0 \sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2} \approx \omega_0$$

The connection between the dissipated energy $P = \dot{E}$ and the total energy E (which for a spring system is the sum of kinetic and potential energy) is that

$$P = -\gamma E$$

Consequently, Q can be written

$$Q = \frac{\omega_0}{\gamma}$$

A different way of viewing Q is to note that, using the amplitude characteristic $A(\omega)$ for the forced oscillator,

$$Q = \frac{A(\omega_0)}{A(0)}$$

For this reason, Q is often referred to as the *resonance amplification* because it indicated how effective resonant driving is in creating large amplitude oscillations. To have a high resonance amplification the oscillator needs a large value for ω_0 and a small value for γ.

The relative bandwidth W of the amplitude resonance peak is defined to be

$$W = \frac{\Delta\omega}{\omega_0} = \frac{\gamma}{\omega_0}$$

We therefore see that there is a reciprocal relationship between the Q-value and the relative bandwidth W :

$$Q \cdot W = 1$$

This connection is fundamental for all oscillating systems. It may be a mass on a spring or an oscillator in a radio receiver. If we want a system to react in a broad band of frequencies we must pay the price of a decrease in amplification. The Q-value for mechanical oscillators can reach values of 10^3 . This is obtained for, say, a vibrating violin string. A church bell will have a somewhat lower value of Q. For electric oscillators higher values are obtained. For vibrating electrons inside atom, i.e., "antennas" emitting light waves, Q is about 10^7. For nuclei emitting γ-rays, Q-values of about 10^{12} may be reached.

1.6 Exercises

Exercise 1.1 Show that for arbitrary real numbers c_1 and c_2 one can always find real numbers C and φ so that

$$x(t) = c_1 e^{-\gamma t/2}\cos(\omega_d t) + c_2 e^{-\gamma t/2}\sin(\omega_d t) = Ce^{-\gamma t/2}\cos(\omega_d t + \varphi)$$

Exercise 1.2 Show how one gets (1.12) from (1.11).

Exercise 1.3 Consider an undamped harmonic oscillator ($\gamma = 0$) with eigenfrequency ω_0. An external driving force $F\cos(\omega t)$ (where $\omega \neq \omega_0$) is applied. The equation of motion is

$$\ddot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t)$$

(1) Determine the initial conditions such that the undamped oscillator will begin steady state motion immediately (this set of initial conditions is known as the *transient-free initial conditions*).

(2) Determine the value of x_0 and θ in

$$x_p = x_0\cos(\omega t + \theta)$$

for the case described in question (1).

Exercise 1.4

Consider a damped, forced linear oscillator.

(1) Calculate the work performed by the frictional force at resonance, and show that it is equal to the work performed by the impressed force at resonance.

(2) A mass m is suspended from a spring in a homogeneous gravitationl field. The period of oscillations is T = 0.5 s. An impressed force $F\cos(\omega t)$ acts vertically on the body. The amplitude of the force is F = 0.1 N. A frictional force f (proportional to the velocity of the mass) acts on the body also. The amplitude at resonance, A_r, is observed to be 5 cm. Determine the damping constant $b := m\gamma$.

Exercise 1.5

A tuning fork vibrates with the frequency ν = 440 Hz. The tuning fork emits 1/10 of its stored energy in 1 second.

Determine the Q-value of the tuning fork. [Answer: $Q \approx 26240$]

Chapter 2

Lagrangian Dynamics

Comments and corrections should be sent to pghj@dtu.dk.

Put off thy shoes from off thy feet,
for the place whereon thou standest
is holy ground.
Exodus 3:5

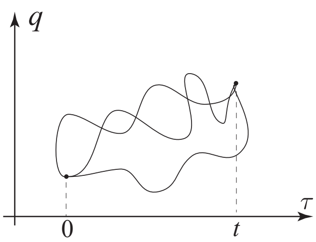
2.1 Introduction

A large class of dynamical systems originate in so-called variational principles.

Among these are mechanical systems where damping is so small that it is ignored[1]. Turning now to such systems, we will revisit familiar systems like the undamped harmonic oscillator and the mathematical pendulum, but also discover how mechanical systems with nontrivial constraints can be dealt with in an elegant fashion using the variational approach.

2.2 Basic Calculus of Variations

A variational formulation converts an initial value problem into a boundary value problem. The variational formulation in this sense takes a global point of view rather than a local, seeking the solution curve as an optimal 'point' in a space consisting of all possible evolution curves $q : \tau \to q(\tau)$ between two points $q(0)$ and $q(t)$.



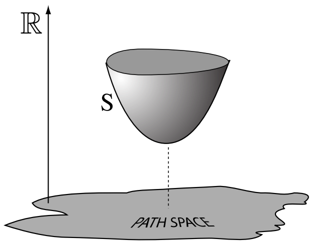


Figure 2.1: The Action S is a real-valued function on the space of all paths through configuration space satisfying the boundary conditions. The solution curve for the dynamical system is the 'point' at which the action is stationary.

A curve $q(\tau)$ is called a motion of the system, and $\dot{q}(\tau)$ is the velocity associated with the the variable q at time τ.

[1]Damping can be incorporated in the formalism which we will descibe here, but for clarity we begin the presentation with undamped systems.

On the space of twice differentiable curves with known boundary values

$$\{q(\tau) \in C^2[0,t] \mid q(0) = q_0, q(t) = q_!\}$$

the functional S is given as an integral:

$$S(\{q\}) = \int_0^t L(q(\tau), \dot{q}(\tau), \tau) \mathrm{d}\tau \tag{2.1}$$

where the function $L = L(q, \dot{q}, t)$ is a differentiable function on $\mathbb{R}^3$ (or more generally $\mathbb{R}^{2n+1}$ where n is the number of configuration variables q). The 'action' S of a curve is how much that curve will give value to the integral (2.1).

The variational principle (sometimes called *principle of least action*) states that we should seek out the curve where the action has a stationary point (has a minimum or a maximum).

Had the functional S been defined on a finite dimensional vector space (e.g., defined on $\mathbb{R}^k$) we would now set the Jacobian of the function equal to zero. For an infinite dimensional vector space we would set the Frechét derivative of S to be zero. We cannot do this here. Instead we resort to a similar but weaker definition of derivative, by demanding that the extremal path $q(\tau)$ is the path where there is no change in the action to lowest order in $\gamma(\tau)$ a small perturbation γ

$$S(q+\gamma) - S(q) = 0 \qquad\qquad \text{to lowest order in } \gamma$$

Here the perturbation path $\gamma(\tau)$ must satisfy $\gamma(0) = \gamma(t) = 0$, so that $(q+\gamma)(\tau)$ is still in the space of paths satisfying the boundary conditions. We will assume that the function L is sufficiently differentiable, so that

$$L(q+\gamma, q\dot{+}\gamma, t) - L(q, \gamma, t) = \frac{\partial L}{\partial q}(q, \gamma, t)\gamma + \frac{\partial L}{\partial \dot{q}}(q, \gamma, t)\dot{\gamma} + \mathcal{O}\left(||\gamma||^2\right)$$

where $||\cdot||$ denotes some norm on the path space. We will also need in the argument below a fairly general

---

Lemma If a continuous function $f(\tau) : [t_0, t_1] \to \mathbb{R}$ satisfies that for any continuous function $\gamma(\tau) : [0, t] \to \mathbb{R}$

$$\int_0^t f(\tau)\gamma(\tau)\mathrm{d}\tau = 0$$

then

$$f(\tau) = 0 \quad \forall\, \tau \in [t_0, t_1]$$

Proof (Exercise 6.3) □

To find the conditon of stationarity of the action integral, we take the lowest order of the variation of the action to be 0:

$$\begin{aligned} 0 = S(q+\gamma) - S(q) &= \int_0^t L(q+\gamma) - L(q)\, \mathrm{d}\tau \\ &= \int_0^t \frac{\partial L}{\partial q}\,\gamma + \frac{\partial L}{\partial \dot{q}}\,\dot{\gamma}\, \mathrm{d}\tau \end{aligned}$$

We then perform an integration by parts on the second term, and use that the boundary terms will vanish since $\gamma(t_0) = \gamma(t_1) = 0$. We thus arrive at

$$0 = \int_0^t \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \gamma(\tau)\, \mathrm{d}\tau$$

Now we invoke the **lemma**: If the above integral is to be zero independent of the choice of $\gamma(\tau)$, then the quantity in the square bracket must be identically zero.

This gives an ordinary differential equation (for $n > 1$ a set of ordinary differential equations $i = 1, \dots n$) for the optimal path:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \dots n. \tag{2.2}$$

This set of equations are called the **Euler-Lagrange equations**. Note that this is a set of (second order) ordinary differential equations (ODE's) with conditions on initial and final values, constituting a boundary value problem.

The variational principle as a means of locating a solution with some kind of extremal property of an integral can be employed in a variety of situations and problems. A famous historical example is the problem of finding the so-called *Brachistochrone* ('shortest-time')-curve along which a mass sliding without friction in a gravitational field between two points would have the least travel-time among all curves connecting the fixed starting and ending points. We are not in this problem finding the details of the motion; we are only finding the curve that satisfies an optimality condition. The curve itself is the solution to the problem.

We turn now to the application of the variational principle in classical mechanics (sometimes called 'analytical mechanics' or 'rational mechanics '). Here we *will* use the principle to study the details of the motion, and we use a special form for the function L, and call it 'the Lagrangian'.

2.3 Classical Mechanics: Configuration space

Given a system of point masses or solid bodies moving in $\mathbb{R}^3$, the first important step in the mathematical analysis is to establish so called 'generalised coordinates', or 'degrees of freedom' for the system.

These are the (smallest number of) coordinates (or geometrical quantities interpreted as coordinates) needed to specify uniquely the positional state of the system. For instance, a pendulum, even though it is moving in a plane, needs only one coordinate (the angle) to specify the configuration. Or: two point masses at each end of a rigid rod (a dumbbell) moving in a plane requires three coordinates to specify its configuration: two numbers to locate one point on the rod, and one number giving the angle of the rod relative to some reference line in the plane.

Although the number n of generalised coordinates is unique for a given system there is often a choice between several kinds of coordinates. For instance, position in a plane may be specified either by cartesian or by polar coordinates. As we shall see later, a good choice of coordinates can sometimes be useful in the further process of solving the equations of motion.

In general we denote the chosen generalised coordinates by $q_1 \dots q_n$. We count the number n of generalised coordinates and say that the mechannical system has n *degrees of freedom*.

For a system of N point masses moving in $\mathbb{R}^3$ and being a system with n degrees of freedom we will have $3N$ (often nonlinear) equations

$$\begin{aligned} x_1 &= x_1(q_1 \dots q_n) \\ &\vdots \\ z_N &= z_N(q_1 \dots q_n) \end{aligned} \tag{2.3}$$

relating the cartesian coordinates $x_1, y_1, z_1 \dots x_N, y_N, z_N$ to the generalised coordinates $q_1 \dots q_n$.

2.4 Classical Mechanics: Kinetic Energy, Potential Energy

Kinetic energy is energy associated with motion. For a single point mass m moving with velocity $\mathbf{v} = \dot{\mathbf{x}}$ in $\mathbb{R}^3$ the kinetic energy is given by $T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.

From the expressions for the cartesian coordinates in terms of the generalised coordinates, one can now by differentiation (remember to use the 'chain rule'!) find expressions for

$$\begin{aligned} \dot{x}_1 &= \dot{x}_1(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}), \\ &\vdots \\ \dot{z}_N &= \dot{z}_N(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}) \end{aligned}$$

so that the kinetic energy T can be written down as a function $T = T(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n})$. The quantities $\dot{q}_1 \dots \dot{q}_N$ are called the *generalised velocities*.

Example The pendulum has $x = \ell \sin\theta$ and $y = \ell\cos\theta$, and consequently $T(\theta, \dot{\theta}) = \frac{1}{2}m(\ell\dot{\theta})^2$. Here, $N = 2$, and $n = 1$

Δ

The physocs of the system becomes more specific when we specify the so-called **potential energy** V as a function of the generalised coordinates (and possibly generalised velocities): $V = V(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n})$.

Well known examples are: potential energy for a point mass near the surface of the Earth ($V = mgz$ where z is the height above the surface of the Earth), potential energy of a spring ($V = \frac{1}{2}kX^2$ where X is the extension of the spring from its equilibrium length), potential energy of two gravitationally interacting masses, m_1 and m_2 ($V = -G\frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$).

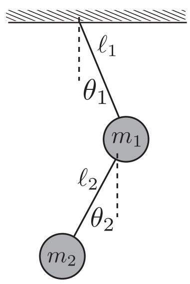
The space in which Lagrangian mechanics can be analysed is the $2n$-dimensional space with coordinates $q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}$. This space is known as the *position-velocity space* (sometimes also called the 'phase space', but we shall reserve that name for a different space, see the section on Hamiltonian mechanics). From a mathematical point of view, the position-velocity space is the tangent bundle of the configuration manifold[2].

2.5 Classical Mechanics: The Lagrangian

The Lagrangian is the function on position-velocity space; which it is the *difference* between the kinetic and the potential energy functions:

$$L = L(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}, t) \equiv T(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}, t) - V(q_1 \dots q_n, \dot{q_1} \dots \dot{q_n}, t)$$

Example For the double pendulum with (point) masses m_1 and m_2 in a gravitational field



the angles θ_1 and θ_2 specify the configuration of the system. Thus, the set (θ_1, θ_2) constitutes our q's. The coordinate transformation from rectangular coordinates are given by (verify this!)

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ y_1 &= l_1(1 - \cos\theta_1) + l_2 \\ x_2 &= l_1\sin\theta_1 + l_2\sin\theta_2 \\ y_2 &= l_1(1-\cos\theta_1) + l_2(1 - \cos\theta_2) \end{aligned}$$

[2]This is not a course in differential geometry, but we will use in a heuristic manner the idea of a manifold as a surface.

Using the chain rule, one can now find the velocities, $\dot{x_1}$, $\dot{y_1}$,$\dot{x_2}$, and $\dot{y_2}$ as functions of $q_1 \dots q_n$ and $\dot{q_1} \dots \dot{q_n}$. Finding this transformation constitutes the necessary *kinematics*. The *physics* is introduced when we identify the kinetic and potential energy functions. In terms of rectangular koordinates, the kinetic energy is here $T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$, and the potential energy is here $V = m_1 g y_1 + m_2 g y_2$ where g is the magnitude of the acceleration of gravity. Δ

2.6 Classical Mechanics: The Euler-Lagrange Equations

We proceed now to take as an axiom of classical mechanics that the evolution of a mechanical system is such that the principle of least action with the Lagrangian as integrand in the action integral, is satisfied. The least-action principle used in this way is sometimes called *Hamilton's Principle*.

In index notation the Euler-Lagrange equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \dots n.$$

Example Single unconstrained particle moving in a potential field in $\mathbb{R}^3$. For this particularly simple case, the Lagrangian is $L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r})$. The Euler-Lagrange equations become $m\ddot{\mathbf{r}} = -\mathbf{grad}V(\mathbf{r})$, which we recognise as Newton's second law for motion in the potential V. We thus see that the Euler-Lagrange equations contain Newton's second law as a special case. When complicated forces act, in particular forces of constraint, the Euler-Lagrange equations still have solutions in accordance with Newton's second law, but setting up the problem via generalised coordinates and proceeding to the Euler-Lagrange equations avoids all mention of forces, in particular internal forces and reaction forces, and may thereby be simpler to write down and (hopefully) solve. Δ

Example Bead on a rotating wire. Consider a bead of mass m forced to move on a straight frictionless wire which is rotating with angular velocity ω about an axis through one endpoint of the wire and perpendicular to the wire.

The kinetic energy of the bead is:

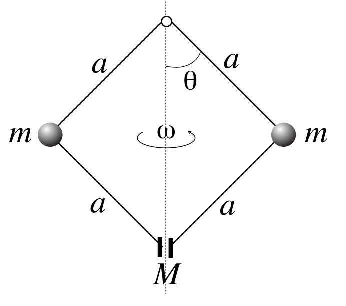
$$T = \frac{m}{2}(\dot{r}^2 + r^2\omega^2)$$

There is no potential energy.

Consequently, the r-Euler-Lagrange equation is : $\ddot{r} = r\omega^2$. In a Newtonian framework one would have to account for the forces constraining the bead to the wire, and then introduce the 'fictitious' centrifugal force experienced in the frame comoving with the wire. Δ

The following example is interesting because of the $\dot{q}$-dependence in the kinetic energy term.

Exapmle The Governor. Consider the system illustrated below (early design for steam engine power control):



The two masses m rotate about the symmetry axis around in a circle perpendicular to the plane of the paper with a constant (impressed angular velocity ω. The arms all have length a, and are fixed at the top to the symmetry axis, at the bottom to a mass M which, as the arms flex, is able to move up or down the symmetry axis.

If we denote the angle between the top arms and the axis by θ, the kinetic and potential energies are given by

$$\begin{aligned} T(\theta,\dot\theta) &= 2\frac{m}{2}\left(a^2\dot\theta^2+\omega^2a^2\sin^2\theta\right)+\frac{M}{4}(4a^2\sin^2(\theta)\dot\theta^2) \\ V(\theta,\dot\theta) &= -2(m+M)ga\cos\theta \end{aligned}$$

and consequently

$$L=\frac{m}{2}\left(a^2\dot\theta^2+\omega^2a^2\sin^2\theta\right)+\frac{M}{4}(4a^2\sin^2(\theta)\dot\theta^2)+2(m+M)ga\cos\theta$$

Δ

2.7 Conservation Laws in Lagrangian Dynamics

Theorem. Suppose the Lagrangian does not depend explicitly on time, $\frac{\partial L}{\partial t}=0$. Then the position-velocity space function

$$E(q,\dot q)\equiv\sum_i\dot q_i\frac{\partial L}{\partial\dot q_i}-L$$

is a constant of the motion: $E(q(t),\dot q(t))=E(q(0),\dot q(0))$ along a solution to the Euler-Lagrange equations.

Proof: Consider the total time derivative of the Lagrangian, and use that along a solution curve we have the Euler-Lagrange equations satisfied:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L &= \sum_i\left(\frac{\partial L}{\partial q_i}\dot q_i+\frac{\partial L}{\partial\dot q_i}\ddot q_i\right) \\ &= \sum_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot q_i}\right)\dot q_i+\frac{\partial L}{\partial\dot q_i}\ddot q_i\right) \\ &= \sum_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot q_i\frac{\partial L}{\partial\dot q_i}\right) \end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i\dot q_i\frac{\partial L}{\partial\dot q_i}-L\right)=0$$

$\square$

Remark. *If* the kinetic energy is a (homogeneous) quadratic function of the generalised velocities, i.e., if $T=\sum_{i,j}a_{i,j}\dot q_i\dot q_j$, then $\sum_i\dot q_i\frac{\partial T}{\partial\dot q_i}=2T$, and thus, *in this case*, we have

$$E=T+V \qquad \text{, i.e., E becomes the total mechanical energy}$$

Question: For *The Governor*, is the total mechanical energy conserved ? [Ans: no].

Definition. A *cyclic* coordinate is a coordinate q_k which does not occur in the Lagrangian, i.e., $\frac{\partial L}{\partial q_k}=0$

Theorem If a coordinate q_k is cyclic, then the quantity $\frac{\partial L}{\partial\dot q_k}$ is a constant of the motion.

Example The Kepler Problem. Consider a point mass moving in R^3 and subject to a potential which depends only on the distance to the center of the coordinate system:

$$V=V(r)=-\frac{\mu}{r}$$

Clearly, spherical polar coordinates are the coordinates most suited to reflect this form of the potential. Using the form of the kinetic energy derived in a previous exercise, we have that

$$L = \frac{m}{2}\left(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right) + \frac{\mu}{r}$$

The angle ϕ is cyclic. Consequently, $r^2\sin\theta^2\dot{\phi}$ is a conserved quantity; let us denote it by M. The Lagrangian therefore has the form

$$L = \frac{m}{2}\left(\dot{r}^2 + (r\dot{\theta})^2 + 2M\dot{\phi}\right) + \frac{\mu}{r}$$

Now we see that the coordinate θ is also 'secretly' cyclic. As a consequence, the quantity $mr^2\dot{\theta}$ is also conserved.

The two conserved numbers are the θ and ϕ components of the vector

$$\underline{\ell} \equiv m\mathbf{r} \times \dot{\mathbf{r}}$$

Since the r-component of this vector will always be zero, we conclude that all three compontnts of the vector $\underline{\ell}$ are constant in time, and that the vector thus is a constant vector.

This vector is called the angular momentum vector, and since its constancy is a consequence only on the geometrical nature of the potential, we note the following theorem:

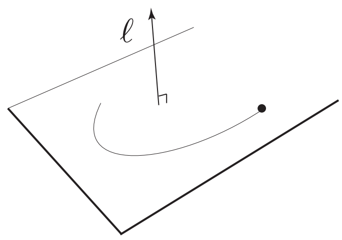
Theorem. For all *central force problems* (i.e., problems where the potential depends only on the radial coordinate r), the angular momentum vector $\underline{\ell} = m\mathbf{r} \times \dot{\mathbf{r}}$ is constant in time.

□

Corollary. For central force problems, the motion takes place in a plane inside R^3.

Proof: If the vector $\mathbf{r}$ were to leave the geometrical plane spanned by the initial vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ it would cause $\underline{\ell}$ to change, in violation of the theorem. □

The corollary therefore tells us that rotational symmetry reduces the number of degrees of freedom to two, and that we should consider the motion to be taking place inside a plane (the plane to which $\underline{\ell}$ is normal). Obviously the plane contains the coordinate origo.



We therefore consider planar polar coordinates (r, θ) in this plane as a new set of coordinates. In these coordinates, (r, θ), the Lagrangian is:

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + (r\dot{\theta})^2) + \frac{\mu}{r}$$

Note, that θ still is cyclic, so that $mr^2\dot{\theta} =$ constant $= \ell$. This is Kepler's Area Law. As a consequence, there is now only one Euler-Lagrange equation:

$$m\ddot{r} - \frac{\ell^2}{mr^3} = -\frac{\mu}{r^2}$$

This looks like 1-D motion in the potential (the 'effective potential')

$$V^{eff} = -\frac{\mu}{r} + \frac{\ell^2}{2mr^2}$$

For the motion the energy $E = \frac{\dot{r}^2}{2} + V^{eff}(r)$ is conserved. Δ

2.8 Exercises

Exercise 6.1 Establish generalised coordinates for

1. A dumbbell in $\mathbb{R}^3$
2. A point mass moving on a helix in $\mathbb{R}^3$.
3. Two masses in $\mathbb{R}^2$ connected with a spring.
4. N point masses moving in $\mathbb{R}^3$.

Exercise 6.2 For the systems in the Exercise 1, and with the generalised coordinates you have chosen, write down the transformations

$$\begin{aligned} x_1 &= x_1(q_1 \dots q_n), \\ y_1 &= y_1(q_1 \dots q_n), \\ z_1 &= z_1(q_1 \dots q_n) \\ &\vdots \\ z_N &= z_N(q_1 \dots q_n) \end{aligned}$$

explicitly.

Exercise 6.3 Prove the Lemma used in the derivation of the Euler-Lagrange equations. Hint: Use contradiction. Prove first that if g is different from 0 (WLOG positive) at some point, then there is a closed interval containing that point in which g is positive. Use this to arrive at a contradiction.

Exercise 6.4 Write the Euler-Lagrange equation for The Governor. Plot (e.g. in PPLANE) the phase plane portrait for various values of the physical parameters.

Exercise 6.5 Prove the theorem that a variable q_i being cyclic variable implies that $\partial L/\partial \dot{q}$ is a constant of the motion. Hint: Use the Euler-Lagrange equations.

Exercise 6.6 For a particle of mass m in a uniform gravitational field in $\mathbb{R}^3$, determine which coordinates are cyclic.

Exercise 6.7 Determine the dynamical equations for a pendulum with a spring (with spring constant k and unloaded length ℓ_0) moving in a plane (x, y) where the gravitational acceleration is $g(e)_y$.