Graphs (7 points)

1. [3 points] Given an adjacency-list representation of a multigraph $G = (V, E)$, describe an $O(V + E)$-time algorithm to compute the adjacency-list representation of the "equivalent" undirected graph $G' = (V, E')$, where E' consists of the edges in E with all multiple edges between two vertices replaced by a single edge and with all self-loops removed.

2. [4 points] The incidence matrix of a directed graph $G = (V, E)$ with no self-loops is a $|V| \times |E|$ matrix $B = (b_{ij})$ such that

$$b_{ij} = \begin{cases} -1 & \text{if edge } j \text{ leaves vertex } i \\ 1 & \text{if edge } j \text{ enters vertex } i \\ 0 & \text{otherwise} \end{cases}$$

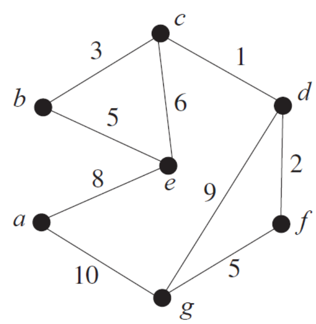
Describe what the entries of the matrix product BB^T represent, where B^T is the transpose of B.

Minimum Spanning Trees (13 points)

3. [2 points] Professor Sabatier conjectures the following converse of Theorem 23.1. Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a safe edge for A crossing $(S, V - S)$. Then, (u, v) is a light edge for the cut. Show that the professor's conjecture is incorrect by giving a counterexample.

4. [4 points] Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

5. [3 points] Consider the following weighted graph:



Use Kruskal's algorithm to find a minimum spanning tree for the graph, and indicate the order in which edges are added to form the tree.

6. [4 points] Let $G = (V, E)$ be any weighted connected graph. If C is any cycle of G, then show (formally) that the heaviest edge of C (i.e., the edge with the largest weight) cannot belong to a minimum spanning tree of G. (Assume that the heaviest edge is unique.) (Hint: use proof by contradiction; also, see the proof of Theorem 23.1).