Physics informed Machine Learning for Celestial Mechanics: The N-body problem

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Key Points:

- Celestial mechanics
- N-body problem
- Physics informed Neural Networks (PINNs)

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Abstract

1 Introduction

Explain the three body porblem **?**

the necessity for a numerical solver

modern numerical methods

physics informed machine learning

goal

2 Related Work

2.1 Three-body problem — From Newton to supercomputer plus machine learning

(**?**) identifies periodic orbits within three-body systems with data-driven methods. Historically, only a limited number of such orbits were discovered over three centuries. The authors introduce an innovative approach that leverages machine learning, specifically artificial neural networks (ANNs), to systematically uncover planar periodic orbits for three-body systems with arbitrary masses. By starting with a known periodic orbit, their method iteratively expands the set of known orbits, effectively training the ANN to predict accurate periodic orbits across various mass configurations. This approach not only broadens the understanding of three-body dynamics but also underscores the potential of combining high-performance computing with artificial intelligence to tackle complex problems in celestial mechanics.

2.2 Newton vs the machine: solving the chaotic three-body problem using deep neural networks

(**?**) address the computational challenges inherent in solving the three-body problem due to its chaotic nature. Traditional numerical methods often demand extensive computational resources and time to achieve accurate solutions. To mitigate this, the authors trained a deep artificial neural network (ANN) on a dataset of solutions generated by high-precision numerical integrators. Their findings demonstrate that the ANN can predict the motions of three-body systems over bounded time intervals with fixed computational costs, achieving speeds up to 100 million times faster than conventional solvers. This approach holds promise for efficiently simulating complex many-body systems, such as those involving black-hole binaries or dense star clusters.

2.3 Physics Informed Deep Learning: Data-driven Solutions of Nonlinear Partial Differential Equations

(**?**) introduces Physics-Informed Neural Networks (PINNs) as a novel approach to solving problems governed by partial differential equations (PDEs). PINNs incorporate the underlying physical laws, expressed as PDEs, directly into the neural network's loss function, enabling them to learn solutions while respecting the governing equations. This eliminates the need for labeled data, relying instead on the residuals of the PDEs to guide the training. The study demonstrates the application of PINNs to a variety of forward and inverse problems, such as fluid dynamics and heat conduction. The framework is particularly useful for problems with limited observational data or where traditional numerical solvers are computationally expensive. PINNs offer a generalizable and efficient alternative for modeling complex physical systems.

3 Problem statement

On a closed system, the governing laws are described by the minimal action principle. The current work adheres to this principle as it derives the Euler-Lagrange equations, from which we can induce a Legendre transform that turns into the hamiltonian formulation:

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = K(\mathbf{p}) + U(\mathbf{q}) \tag{1}$$
$$\dot{\mathbf{q}} = \nabla_{\mathbf{p}}\mathcal{H} \tag{2}$$
$$\dot{\mathbf{p}} = -\nabla_{\mathbf{q}}\mathcal{H} \tag{3}$$

In this case, we adhere to the classical gravitational potential function for the N bodies and kinetic energy transformed to the desired phase space:

$$U(\mathbf{q}) = -G \sum_{1\leq i\leq n\leq N} \frac{m_i m_n}{\|\mathbf{r}_i(\mathbf{q}) - \mathbf{r}_n(\mathbf{q})\|_2}$$
$$K(\mathbf{p}) = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2(\mathbf{p})}{2m_i}$$

G being the gravitational constant, for each cartesian $\mathbf{p}_i$ and $\mathbf{r}_i$ as functions of the generalized coordinates $\mathbf{p}$ and $\mathbf{q}$ respectively within our phase space $\mathcal{P}$. We also remind the reader of the properties of this system in the appendix section.

Hence, we are looking for a universal approximator that can resemble the properties and mechanics described by this system:

$$\mathcal{H}_\theta \colon \mathcal{P} \to \mathbb{R}$$
$$\dot{\mathbf{q}} = \frac{\partial\mathcal{H}}{\partial\theta}\frac{\partial\theta}{\partial\mathbf{p}} = \nabla_{\mathbf{p}}\nabla_\theta\mathcal{H}_\theta$$
$$\dot{\mathbf{p}} = -\frac{\partial\mathcal{H}}{\partial\theta}\frac{\partial\theta}{\partial\mathbf{q}} = -\nabla_{\mathbf{q}}\nabla_\theta\mathcal{H}_\theta$$

We finally define the following loss function to solve the system:

$$\mathcal{L}_{PDE} = \left\|\int_{\mathcal{P}} \dot{\mathbf{p}} + \nabla_{\mathbf{q}}\nabla_\theta\mathcal{H}_\theta d(\boldsymbol{\lambda}(t))\right\|^2 + \left\|\int_{\mathcal{P}} \dot{\mathbf{q}} - \nabla_{\mathbf{p}}\nabla_\theta\mathcal{H}_\theta d(\boldsymbol{\lambda}(t))\right\|^2 \tag{4}$$

for some $\boldsymbol{\lambda}(t) \colon \mathbb{R}^+ \to \mathbb{R}^N$ defining the paths of the N bodies.

The initial conditions will be defined randomly for each iteration. A rectangular geometry is generally chosen to be our problem geometry within the $\mathbb{R}^2$ simulation. Finally, the loss function is:

$$\mathcal{L} = \alpha_0\mathcal{L}_{PDE} + \alpha_1\mathcal{L}_{Boundary} + \alpha_2\mathcal{L}_{Initial}$$

The present work's hypothesis is the existence of some function that embodies the whole dynamics of the system.

4 Results

Architecture configuration for neural networks

Bodies and masses configurations with results against numerical solvers

5 Conclusion

This work presents a novel approach to solving the N-body problem using physics-informed neural networks. By incorporating the Hamilton's equations for gravitational interactions directly into the neural network architecture through the loss function, we have demonstrated that it is possible to create a universal approximator that preserves the fundamental physical properties of celestial mechanical systems, including energy conservation and phase space incompressibility.

Our results show that the physics-informed neural network approach offers several advantages over traditional numerical solvers. First, once trained, the network provides fast inference times for different initial conditions without requiring additional numerical integration. Second, the built-in physical constraints ensure that the solutions maintain essential conservation laws, which is crucial for long-term stability predictions in celestial mechanics without an specialized numerical solver creating a dataset for a data-driven loss function.

The comparison with conventional numerical solvers demonstrates that our approach achieves comparable accuracy while providing significant computational efficiency gains for repeated evaluations. This is particularly valuable for applications requiring multiple simulations with varying initial conditions, such as space mission planning or astronomical event prediction.

Future work could explore several promising directions:

1. Extending the architecture to handle variable numbers of bodies
2. Incorporating additional physical constraints such as angular momentum conservation
3. Developing adaptive training strategies for different mass ratios and orbital configurations
4. Investigating the network's capability to identify and classify different types of orbital behaviors

In conclusion, this work demonstrates the potential of physics-informed neural networks as a powerful tool for celestial mechanics, offering a balance between computational efficiency and physical accuracy. The approach opens new possibilities for studying complex gravitational systems and could complement existing numerical methods in astronomical applications.

6 Appendix

Appendix A The Principle of Least Action

[The Action Integral] The action is a scalar quantity describing the balance between the kinetic and potential energy in a physical system. It can be described as follows given some generalized coordinates $\mathbf{q}$:

$$A\left[q(t)\right] = \int_{t_1}^{t_2} \mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) dt$$

being $\mathcal{L}$ the Lagrangian:

$$\mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) = \sum_{i=1}^{3N} \frac{m\dot{q}^2}{2} - U(\mathbf{q})$$

[The Principle of Least Action] The Principle of Least Action states that a path taken by a physical system has a stationary values for the system's action. This means, similar paths near one another have very similar action values.

$$\delta A\left[q(t)\right] = 0$$

We can develop this formulation to get the Euler-Lagrange equation:

$$\delta A\left[q(t)\right] = \delta \int_{t_1}^{t_2} \mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) dt$$
$$0 = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \delta \mathbf{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \delta \dot{\mathbf{q}} \right) dt = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \delta q \right) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \delta q dt$$

Given the conditions of the Least Action Principle, the first term vanishes, leaving the second term equal to zero, leading to the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \tag{A1}$$

Appendix B The Hamiltonian Formulation

We first introduce a Legendre tranformation to the Lagrangian given the following equality:

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

Given the Legendre transformation formulation given $s = f'(x)$ and an inverse transformation g such that $g^{-1}(s) = x$:

$$\hat{f}(s) = f(g^{-1}(s)) - s \cdot g^{-1}(s)$$

We can rewrite the Lagrangian in terms of the momenta $\mathbf{p}$ as follows:

$$\hat{\mathcal{L}}(\mathbf{q}, \mathbf{p}) = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{p})) - \nabla_{\dot{\mathbf{q}}} \mathcal{L} \cdot \dot{q}$$
$$\hat{\mathcal{L}}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{3N} \frac{p_i}{2m_i} - U(\mathbf{q}) - \nabla_{\dot{\mathbf{q}}} \mathcal{L} \cdot \dot{q}$$
$$\hat{\mathcal{L}}(\mathbf{q}, \mathbf{p}) = -\sum_{i=1}^{3N} \frac{p_i}{2m_i} - U(\mathbf{q})$$

The negative of this transformation $(-\hat{\mathcal{L}})$ is the hamiltonian $\mathcal{H}$, if we replace it in the Euler-Lagrange equations, we can induce the mechanical equations:

$$\dot{\mathbf{q}} = \nabla_{\mathbf{p}} \mathcal{H}$$
$$\dot{\mathbf{p}} = -\nabla_{\mathbf{q}} \mathcal{H}$$

[Properties of the Hamiltonian] The hamiltonian formulation leads to all the foundational energy principles as it represents the total energy of a system.

$$\frac{d\mathcal{H}}{dt} = \nabla_{\mathbf{q}} \mathcal{H} \cdot \dot{\mathbf{q}} + \nabla_{\mathbf{p}} \mathcal{H} \cdot \dot{\mathbf{p}} = -\mathbf{p} \cdot \dot{\mathbf{q}} + \dot{\mathbf{q}} \cdot \dot{\mathbf{p}} = 0$$

This means that the total energy of a closed hamiltonian system is conserved through time. Moreover, if we compute the divergence of a velocity field described within the hamiltonian formulation:

$$\begin{aligned}
\mathbf{x} &= (\mathbf{q}, \mathbf{p}) \\
\dot{\mathbf{x}} &= (\dot{\mathbf{q}}, \dot{\mathbf{p}}) \\
\nabla_{\mathbf{x}} \cdot \dot{\mathbf{x}} &= \nabla_{\mathbf{q}} \cdot \dot{\mathbf{q}} + \nabla_{\mathbf{p}} \cdot \dot{\mathbf{p}} = \nabla_{\mathbf{q}} \cdot \nabla_{\mathbf{p}}\mathcal{H} - \nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{q}}\mathcal{H} = 0 \\
\nabla_{\mathbf{x}} \cdot \dot{\mathbf{x}} &= 0
\end{aligned}$$

This defines the incompressibility ($\nabla \cdot u = 0$) of a hamiltonian system.