The Vasa Ship

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1 Introduction

The Vasa is a Swedish warship that sank in Stockholm harbor during its inaugural voyage on August 10, 1628. Despite a stability test indicating potential instability, King Gustav II Adolf ordered its launch. Within just 20 minutes of sailing, a slight gust of wind caused the ship to capsize and sink. The Vasa ship was eventually raised on April 24, 1961, and is now displayed at the Vasa Museum in Stockholm.

In this project, I will be using a simple ship model to study various factors that influence its dynamics and stability characteristics.

2 Ship Model

I will consider a long, semicircular ship with a radius R, rocking about an axis in the longitudinal direction of the ship. The ship can be compact, with a constant mass density σ, or hollow, with a thin hull and deck with a constant mass density λ. Here, σ and λ represent mass per unit area and mass per unit length, respectively, given in meters along the length of the ship. In this project, I will assume a compact ship, where the center of mass of the ship, C, is located at a distance

$$h = \frac{4R}{3\pi} \approx 0.42R$$

below the midpoint of the deck, M. (For a hollow ship: $\frac{2R}{\pi+2}$.) In equilibrium, the ship displaces a volume of water determined by Archimedes' principle, which can be expressed in terms of the cross-sectional area of the ship below the water surface. This area is given by

$$A_0 = \frac{\sigma \pi R^2}{2\sigma_0}$$

for a compact ship. (For a hollow ship: $\frac{\lambda(\pi+2)R}{\sigma_0}$.) Here, $\sigma_0 = 1000\, kg/m^2$ represents the density of water. The displaced volume of water will vary as the ship rocks. It is convenient to use the sector angle γ to denote the displaced

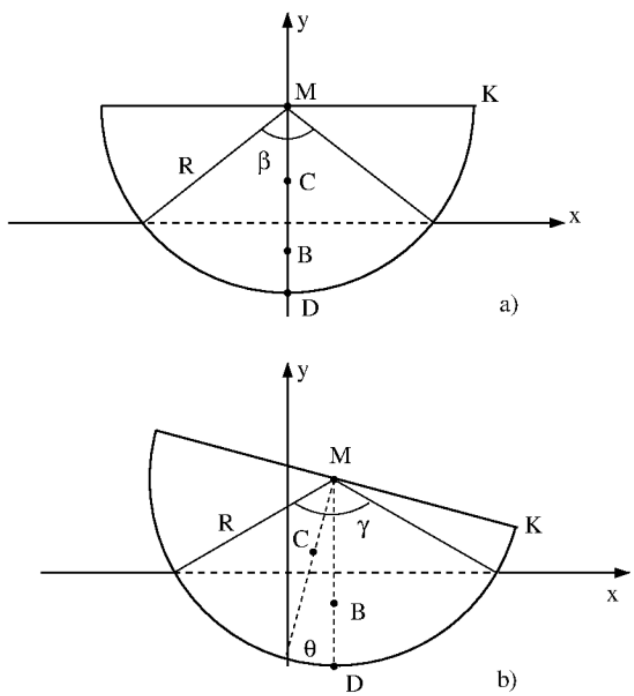

Figure 1: Geometry of the ship model. a) In equilibrium. b) After rotating an angle θ about C and translating C. The rotation angle is chosen positive counterclockwise, so $\theta < 0$ in the figure. M: metacentre (see next chapter) and midpoint of the deck; C: centre of gravity; B: centre of buoyancy; D: deepest point of the ship; K: edge of the ship and lowest point of the deck.

volume of water. The figure above shows the ship in equilibrium (a) and in a state (b) where it has rotated and shifted. In equilibrium, I denote the sector angle as β. The area of the entire sector is then $\frac{\beta R^2}{2}$. The triangle above the waterline has an area

$$R \sin \frac{\beta}{2} \cdot R \cos \frac{\beta}{2} = \frac{1}{2} R^2 \sin \beta$$

so the area of the displaced water becomes

$$A_0 = \frac{1}{2} R^2 (\beta - \sin \beta).$$

Typically, the mass densities σ and σ_0 are given quantities. The sector angle in equilibrium is then determined by solving the equation

$$\beta - \sin \beta = \frac{\pi \sigma}{\sigma_0}$$

using an appropriate method. The center of gravity of the displaced water is denoted by B in the figure. With a sector angle γ, the distance from M to B is given by

$$Y_{MB} = R \cdot \frac{4\sin^3(\gamma/2)}{3(\gamma - \sin\gamma)}$$

During a pure rotation about C, M follows a circular path with a radius of h, with C as the center. If the rotation angle is θ, M is vertically displaced by $\Delta y_M = h(\cos\theta - 1)$ and horizontally displaced by $\Delta x_M = -h\sin\theta$. Here, Δy_M is always negative, while Δx_M is positive when the ship has rotated clockwise.

The positions B and D always lie directly below M and therefore have the same horizontal displacement as M. The distance from M to D is of course always equal to the ship's radius R. The sector angle γ depends on the rotated angle θ and the vertical displacement Δy_C. I consider vertical coordinates for M, C, B, and D while keeping the surface at height $y = 0$.

In equilibrium:

$$y_M^0 = R\cos\frac{\beta}{2}$$

$$y_C^0 = y_M^0 - h = R\cos\frac{\beta}{2} - \frac{4R}{3\pi}$$

$$y_B^0 = y_M^0 - Y_{MB}^0 = R\cos\frac{\beta}{2} - \frac{4R\sin^3(\beta/2)}{3(\beta - \sin\beta)}$$

$$y_D^0 = y_M^0 - R = R\cos\frac{\beta}{2} - R$$

After a rotation θ about C:

$$y_M^\theta = y_M^0 + \Delta y_M = R\cos\frac{\beta}{2} + \left(\frac{4R}{3\pi}\right)(\cos\theta - 1)$$

$$y_C^\theta = y_C^0 = R\cos\frac{\beta}{2} - \frac{4R}{3\pi}$$

$$y_D^\theta = y_D^0 + \Delta y_M = R\cos\frac{\beta}{2} - R + \left(\frac{4R}{3\pi}\right)(\cos\theta - 1)$$

After a rotation θ and a vertical displacement Δy_C:

$\mathrm{y}_M = R\cos\frac{\beta}{2} + \frac{4R}{3\pi}(\cos\theta - 1) + \Delta y_C$

$y_C = R\cos\frac{\beta}{2} - \frac{4R}{3\pi\beta} + \Delta y_C$

$y_D = R\cos\frac{\beta}{2} - R + \frac{4R}{3\pi}(\cos\theta - 1) + \Delta y_C$

The sector angle γ satisfies $\cos\frac{\gamma}{2} = \frac{y_M}{R}$, so

$$\gamma = 2\arccos\left[\cos\frac{\beta}{2} - \frac{4}{3\pi}(1-\cos\theta) + \frac{\Delta y_C}{R}\right].$$

The area of the displaced water is given by

$$A = \frac{1}{2}R^2(\gamma - \sin\gamma),$$

which means that the buoyant force $F_B = A\sigma_0 g$ is not equal to the weight of the ship (and any load on deck) in equilibrium. This leads to a "bobbing" motion up and down. The magnitude of Δy_C is the vertical displacement of the center of gravity relative to the equilibrium, i.e., $\Delta y_C = y_C - y_C^0$.

Since the distance from M to B is Y_{MB}, the vertical coordinate of the buoyancy center B is given by $y_B = y_M - Y_{MB}$. The moment of inertia of the ship about the axis through the center of gravity C is

$$I_C = \frac{1}{2}mR^2\left(1 - \frac{32}{9\pi^2}\right),$$

where m is the mass. It follows that the moment of inertia I_M about an axis through M is half of the moment of inertia for a complete circular compact cylinder with a mass of $2m$, i.e.,

$$I_M = \frac{1}{2}\left(\frac{1}{2}\right)(2mR^2) = \frac{mR^2}{2},$$

and using the parallel axis theorem, we have $I_C = I_M - mh^2$ with $h = \frac{4R}{3\pi}$.

3 External Forces on the Ship

In addition to gravity, there are contact forces between the ship and the water. The buoyant force F_B is the net normal force on the ship from the surrounding water. According to Archimedes, F_B is equal to the weight of the displaced water, i.e., $F_B = \sigma_0 A g$. The buoyant force acts at the buoyancy center B, which corresponds to the center of gravity of the displaced water. With a symmetric ship like this, B will always be directly below the midpoint of the deck M. The intersection point of the plumb line through B and the plumb line through C (and B) in equilibrium is often called the ship's metacenter. For our ship, the midpoint of the deck and the metacenter are the same point M. When the ship is in equilibrium, the buoyant force is naturally equal to the weight of the ship, $F_B = mg = \sigma_0 A_0 g$.

When the ship is in motion, there are frictional forces between the ship and the water. The simplest approach is to completely ignore friction. A more realistic approximation would be to assume a frictional force f proportional to the area of the ship's boundary surface with water (i.e., the arc length $R\gamma$) and the velocity of this boundary surface.

$$f = -k_f R\gamma\omega,$$

where k_f has units of kg/s. Wind and waves can also be included in simple or more complex ways. Here, I keep it as simple as possible and use a horizontally directed harmonic force perpendicular to the longitudinal direction:

$$F_w(t) = F_0 \cos\omega wt,$$

where f acts at the waterline.

4 Harmonic Oscillations

Let's assume that gravity mg and buoyancy F_B are the only external forces acting on the ship. I further assume pure rotation about the center of gravity C and $F_B = mg$. The torque of buoyancy about C is then $\tau_B = -mgh\sin\theta$, and if I consider small oscillations around the equilibrium, I can approximate $\sin\theta \approx \theta$. Newton's second law for rotation about C becomes

$$-mgh\theta = I_C\ddot{\theta},$$

which represents a harmonic oscillator with angular frequency

$$\omega_0 = \sqrt{\frac{mgh}{I_C}}$$

and period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I_C}{mgh}}$. Using the expressions for h and I_C in the equation, the period becomes

$$T = \pi\sqrt{\frac{3\pi R}{2g}\left(1 - \frac{32}{9\pi^2}\right)}$$

The proposed frictional force f gives a contribution $\tau_f = -fY_{CD}$ to the external torque about C, with $Y_{CD} \approx 0.58R$ (this "arm" varies with the angular displacement θ). The ship is then expected to perform damped harmonic oscillations given by

$$\theta(t) = \theta_0 e^{-\delta t}\sin(\omega t)$$

with a time constant $1/\delta \approx \frac{m}{k_f\gamma}$ and angular frequency $\omega = \sqrt{\omega_0^2 - \delta^2}$.

5 Moving Load

A moving load is simplest to model as a point mass m_L initially at rest at position M, i.e., in the middle of the deck. Any static and kinetic friction between the load and the surface can be characterized by the respective friction coefficients μ_s and μ_k. The simplest approach is to neglect friction between the load and the deck. A moving load results in a richer dynamics and can affect the stability characteristics of the ship. It is convenient to use the position of the load relative to the metacentre, s_L, such that when $s_L = R(-R)$, it is positioned as far right (left) on the deck as possible.

Assuming the load slides frictionlessly along the deck, only gravity and the normal force N from the deck act on the load, determining its motion according to Newton's second law. Note that I use bold font for N to emphasize that it has components in both the x- and y-directions. All other forces in the project act in only one direction. From Newton's third law, it also follows that the load exerts an equal and opposite force to N on the ship. This creates a torque from the load on the ship.

6 Equations of Motion

To describe the motion/dynamics of the ship, I use Newton's second law for the motion of the center of mass and rotational motion. The equation describing the motion of the center of mass is given by:

$$\sum \mathbf{F} = m\mathbf{A} = m\frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}$$

where $\mathbf{A}$, $\mathbf{V}$, and $\mathbf{R}$ are the acceleration, velocity, and position of the ship's center of mass, respectively. In component form, I can write $\mathbf{R} = (x_C, y_C)$. The rotational motion is given by:

$$\sum \tau = I_C \frac{d\omega}{dt}$$

where I have dropped the vector notation for the torque, as all torques in the problem act in the z-direction.

As mentioned in the previous section, a moving load on the ship's deck will result in a contact force from the load on the ship, contributing to both $\mathbf{F}$ and τ.

Let's summarize the forces acting on the ship:

$$F_G = -mg \ (y)$$

$$F_B = A\sigma_0 g \ (y)$$

$$f = -k_f R\gamma\omega \ (x)$$

$$F_w = F_0 \cos \omega_w t \ (x)$$

$$F_L^y = -m_L g \cos^2 \theta \ (y)$$

$$F_L^x = m_L g \cos \theta \sin \theta \ (x)$$

The torques acting on the ship will vary both as a function of the magnitude of the force and as a function of the distance between the center of mass and the extension of the force ("arm"). This means, for example, that the torque due to gravity acting on the ship is always zero. For simplicity, I provide all the expressions for the different torques acting on the ship (with respect to the axis through C):

$$\tau_B = -F_B h \sin \theta$$

$$\tau_f = f(y_C - (R(\cos(\gamma/2)) - 1))$$

$$\tau_w = F_w y_C$$

$$\tau_L = -m_L g \cos(\theta) s_L$$

In this project, I will solve the coupled system of differential equations given by the equations in this section. This will be done step by step, gradually adding the different forces as I progress through the tasks.