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SHANGHAI JIAO TONG UNIVERSITY

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THESIS OF BACHELOR



论文题目：警匪追逐中破坏数的研究

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警匪追逐中破坏数的研究

摘 要

给定一个图 G ，警察和小偷可以在 G 上进行追逐游戏。在该游戏中，警察会带着特定的目标追捕小偷，比如抓住小偷，从而可以定义 G 的抓捕时间（警察抓到小偷需要经过的最少步数）。同时，警察的目标也可以是限制小偷的可行范围，从而可以定义 G 的破坏数（小偷能够经过不同顶点的最大个数）。在 2019 年，Cox 和 Sanaei 提出两个猜想：“1）对于任一给定的 $r \in (0, 1)$ ，存在一系列图使得图的破坏数和抓捕时间的比率可以趋近 r 。2）对于佩利图 P_n ，当 $n > 9$ 时，其破坏数等于 $\frac{n-1}{2}$ 。”我们证明了猜想 1 并在 $n = 13$ 时验证了猜想 2。我们还引入并讨论了“保证抓捕情况下最小破坏数”的概念。

关键词：警匪追逐，破坏数，抓捕时间

A STUDY OF DAMAGE NUMBER IN COP ROBBER GAME

ABSTRACT

Given a graph G , cops and robbers can play chase on G . In this game, the cops pursuit the robbers with a certain goal, such as capturing the robber, in which we can define capture time (the minimum number of steps the cops need to catch the robbers). Besides, the goal of the robbers can also be to limit the range of movement of the robbers, in which we can define damage number (the maximum number of vertices the robbers can visit). In 2019, Cox and Sanaei made two conjectures: 1) For any given $r \in (0, 1)$, there is a series of graphs such that the ratio of the damage number and the capture time can approach r . 2) For Paley graph \mathcal{P}_n , the damage number equals $\frac{n-1}{2}$ when $n > 9$. We proved Conjecture 1 and we also proved Conjecture 2 in the special case that $n = 13$. We introduced and discussed the concepts of capture-and-smallest-damage number.

Key words: Cop Robber Game, Damage Number, Capture Time



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Chapter 1 Introduction

In the real world, humans will inevitably encounter scenarios of pursuit and evasion. For example, tracking by Sensor Actuator Network (SAN) can be modeled as a pursuit-evasion game, where the object being tracked is trying to evade by moving in a way that prevents the SAN from observing it. Other scenarios include surveillance, search-and-rescue, and quality inspection. There are two advantages to formulate these tasks as a pursuit-evasion game. The game-theoretical formulation naturally captures the adversarial nature of these tasks^[1]. Thus, it is meaningful to study the pursuit-evasion game. The pursuit-evasion game is a vertex pursuit game, where the environment is represented by a graph.

The *Cop and Robber* game is a specific vertex pursuit game. In this game, a simple, undirected and reflexive graph $G = (V, E)$ is given, player I, known as the cop, wants to catch player II, known as the robber. First, the cop chooses a vertex to start, then the robber chooses a vertex. At each round, the cop moves first. The cop can move from one vertex to the adjacent vertex or pass the round, i.e., stay at the original vertex. Then, the robber moves from one vertex to the adjacent vertex or passes the round. In each round, both the two players are aware of the other's positions and they can design their moving strategies. The rule of the game is that, if the robber is caught after a finite number of moves, the cop wins the game, otherwise, the robber wins. As the *Cop and Robber* is a game of complete information, there is a winning strategy for the cop or the robber. If G is a graph the cop has a winning strategy, G is called a *cop-win* graph, otherwise, G is called a *robber-win* graph.

There are other settings of the *Cop and Robber* game. For example, the game with k cops and l robber, where $k > 1$, where the minimum number of cops who guarantees the cop's winning strategy on a graph G (the cop number of G) is studied^[2]. And the games played on directed graphs^[1] or game with partial information^[3, 4]. For more versions, see the surveys^[5, 6].

Recently, other variations of the *Cop and Robber* game have been studied. For example,^[4] introduce a new graph parameter called *burning number*, which measures the speed of the spread of contagion in a graph. Besides, Clarke et al.^[7] propose a new model on directed acyclic graphs, in which the contamination spreads slowly. Cox and Sanaei^[8] introduces the *damage number* of a graph, which is the minimum number of distinct vertices the robbers can visit without capture. The motivation of the proposed



damage number is that in some situations, the damage by the intruder is severe or costly and the highest priority is to contain the damage instead of capturing the intruder.

In this paper, we will first introduce some remarkable results in *cop-win* graph and *damage number*. Then, we present our proof and analysis for the previous problems about *damage number*. Furthermore, we propose a new meaningful parameter *capture-and-smallest-damage number*, which stands for the minimal capture time in *cop-win* graph with the smallest damage number.

Chapter 2 Definition

We give definitions appearing in this paper to refer as follows.

Definition 2.1 $N(v)$: $\{v' \in V | (v, v') \in E\}$, the neighbourhood of v .

Definition 2.2 $N[v]$: $N(v) \cup \{v\}$, the closed neighbourhood of v .

Definition 2.3 $\deg(v)$: $|N(v)|$, the degree of vertex v .

Definition 2.4 $\text{dis}(u, v)$: the distance of vertex u, v .

Definition 2.5 $\Delta(G)$: $\max\{\deg(v) | v \in V(G)\}$, the maximum degree of vertex in graph G .

Definition 2.6 $\text{ecc}(v)$: $\max\{\text{dis}(u, v) | u \in V(G)\}$, the eccentric distance of vertex v in graph G .

Definition 2.7 $\text{rad}(G)$: $\min\{\text{ecc}(v) | v \in V(G)\}$, the radius of graph G .

Definition 2.8 Corner: a vertex $v \in V(G)$ such that $N[v] \subseteq N[u]$ for some $u \in V(G)$.

Definition 2.9 $\text{capt}(G)$: the minimum number of cop's moves to capture the robber in the graph G , no matter how the robber moves.

Definition 2.10 $\text{dmg}(G)$: the maximum number of vertices of G that the robber can damage, no matter how the cop moves.

Definition 2.11 Dismantlable graph $DG(k)$: a graph with k separate corners v_1, \dots, v_k and $G - \{v_1, \dots, v_k\}$ is also $DG(k)$.

Definition 2.12 Strong regular graph $SRG(n, k, \lambda, \mu)$: a graph with n vertices, each having k neighborhoods. Every two connected vertices have λ common neighbors, two disconnected vertices μ common neighbors.

Chapter 3 Cop-win Graph

In this section, we introduce some results of *cop-win* graph. First, there are some lemmas.

Lemma 3.1 A cop-win graph is a graph with connected component 1.

Lemma 3.2 (Triangle Lemma) If G is a cop-win graph, then each edge $(u, v) \in E(G)$ is either a bridge or u and v have a common neighbor w , which forms a triangle over (u, v) .

Proof If (u, v) is not a bridge, and u and v don't have a common neighbor, then there is a C_4 in G . Besides, no vertex connects 3 vertices of this C_4 . Thus, the robber can move to stay at the diagonal vertex of the cop. \square

There are some remarkable theorems of *cop-win* graph. We present them as follows with brief proof ideas.

Theorem 3.3 (Nowakowski and Winkler^[9]) A retract of a cop-win graph is cop-win.

Proof Assume that G' is the retract of G . The cop just follows the strategy in G . \square

Although the proof of this theorem seems to be simple, it leads to some important corollaries.

Corollary 3.4 If G is cop-win and v is a corner in G , then $G - v$ is cop-win.

Corollary 3.5 If G is cop-win and G' is also cop-win by removing a corner of G , then $\text{capt}(G) \leq \text{capt}(G') + 1$.

Theorem 3.6 (Bonato et al.^[10]) If G is a finite $DG(2)$, $\text{capt}(G) \leq \lfloor \frac{n}{2} \rfloor$.

Proof Induct on n . Each time select two vertices because of 2-dismantable property. Then get a retract of the graph and use inductive assumption. \square

Theorem 3.7 (Gavenčiak^[11]) If $n \geq 7$, $\text{capt}_{\max}(n) = n - 4$, and for $n \leq 7$, $\text{capt}_{\max}(n) = \lfloor \frac{n}{2} \rfloor$.

Proof (1) For $n \leq 7$, enumerate all possible n , we have:



n	1	2	3	4	5	6	7
$\text{capt}_{\max}(n)$	0	1	1	2	2	3	3

(2) For $n \geq 8$, the lower bound is guaranteed by using the corollary 3.5 to induct on n . The upper bound is guaranteed by constructing a family of graphs H_n in Figure 3–1 which has $\text{capt}_{\max}(H_n) = n - 4$. \square

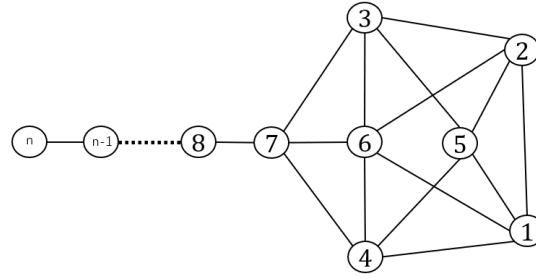


Figure 3–1 Graph H_n .

Note that this theorem improves the results in the previous paper^[10] Theorem 2.

Chapter 4 Damage Number

The following results about *damage number* are all from the previous work^[8]. We indicate briefly ideas of their proofs.

Theorem 4.1 If $n \geq 4$, $\text{dmg}(C_n) = \lfloor \frac{n-1}{2} \rfloor$.

Proof (1) Cop can protect $\lceil \frac{n-1}{2} \rceil$: when the robber passes, passes as the same; when the robber moves, moves at the opposite direction. (2) Robber can damage $\lfloor \frac{n-1}{2} \rfloor$: starts as close to cop as possible, then moves at clockwise. \square

Theorem 4.2 $\lfloor \frac{\text{rad}(G)}{2} \rfloor \leq \text{dmg}(G) \leq n - \Delta(G) - 1$.

Theorem 4.3 If $n \leq 8$, then $\text{dmg}(G) \leq \lfloor \frac{n}{2} \rfloor$.

Proof For $n \leq 8$, enumerate all possible n , we have:

n	1	2	3	4	5	6	7	8
$\text{dmg}(G)$	0	0	0	1	2	2	3	4
Example	-	-	-	C_4	C_5	C_6	C_7	M_8

where M_8 is the Möbius ladder M_8 , as shown in Figure 4–1. \square

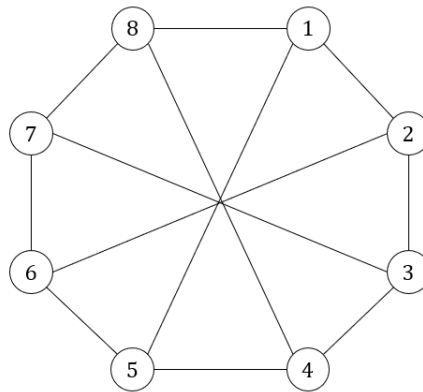


Figure 4–1 Möbius ladder M_8 .

Theorem 4.4 For the Möbius ladder M_8 , $\text{dmg}(M_8) = 4$.

Proof (1) It is clear that the cop can protect 4 vertices by passing all the rounds. (2) The robber can damage three vertices in the first three consecutive moves. It can be proved by enumerating the first three steps. (3) The robber can damage the fourth vertex by discussing case by case. \square

Some graph has *damage number* greater than half of $|V|$.

Theorem 4.5 For the Pappus graph G shown in Figure 4–2, $\text{dmg}(G) \geq 10$.

Proof (1) If the robber is not forced by the cop to complete a cycle, then the robber can damage at least ten vertices. This is because of three facts, the Pappus graph is bipartite, the robber can access three vertices and the cop can only protect one vertex in a round. (2) If the robber is forced by the cop to complete a cycle, we only need to consider 6-cycle and 8-cycle. We can finish the proof by enumerating all possible cases. \square

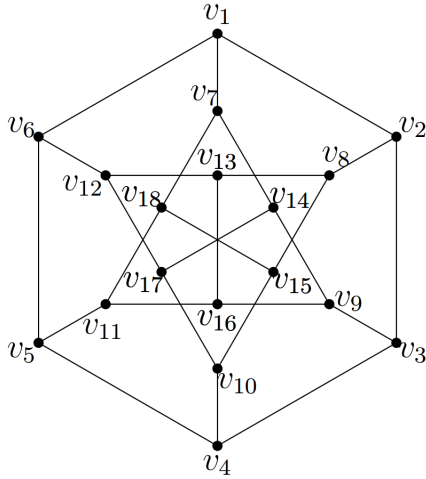


Figure 4–2 The Pappus graph.

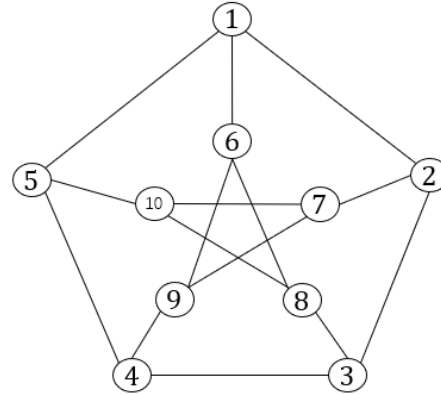


Figure 4–3 The Peterson graph.

Theorem 4.6 For the Peterson graph G shown in Figure 4–3, $\text{dmg}(G) = 5$.

Proof (1) The robber can damage five vertices in the first five consecutive moves. (2) The cop can protect five vertices by force the robber to damage a 5-cycle and then protect the others. \square

Theorem 4.7 For the graph $G = \text{SRG}(n, k, \lambda, \mu)$, $\text{dmg}(G) \geq \min\{k - \lambda, k - \mu + 1\}$.

Proof Thinking backward, assume that the robber is already not able to damage any vertex. \square

Theorem 4.8 For a Paley graph \mathcal{P}_n , $n > 9$, $\frac{n+3}{4} + 1 \leq \text{dmg}(\mathcal{P}_n) \leq \frac{n-1}{2}$.

Proof Utilize that the maximum clique in a Paley graph of order n has \sqrt{n} vertices. \square

Chapter 5 Main Results

5.1 Damage Number of Paley Graph

Problem description: For a Paley graph \mathcal{P}_n , decide whether $\text{dmg}(\mathcal{P}_n) = \frac{n-1}{2}$ for $n > 9$ or not.

In graph theory, Paley graphs are a family of dense undirected graphs. The number of vertices $|V|$ is a prime power and $|V| \equiv 1 \pmod{4}$. Let vertex set $V = \{v_0, v_1, \dots, v_{|V|-1}\}$, the edge set $E = \{(v_i, v_j) | i - j \in (\mathbb{F}_{|V|}^\times)^2\}$. This means that two vertices connect if and only if the difference of their indices is a quadratic residue of $|V|$.

Note that we already have the the results for $n = 2$ and $n = 3$, i.e., $\text{dmg}(\mathcal{P}_5) = 2$ and $\text{dmg}(\mathcal{P}_9) = 3$.

Theorem 5.1 For the Paley graph \mathcal{P}_9 , $\text{dmg}(\mathcal{P}_9) = 3$.

Proof Consider $\mathcal{P}_9 = C_3 \square C_3$ in Figure 5–1, where \square is the Cartesian product of graphs. □

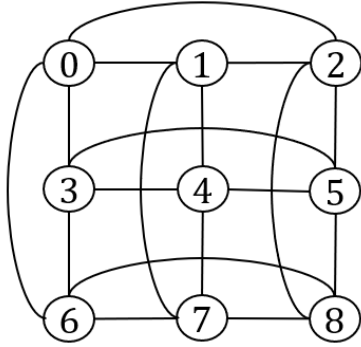


Figure 5–1 Graph \mathcal{P}_9 as $C_3 \square C_3$.

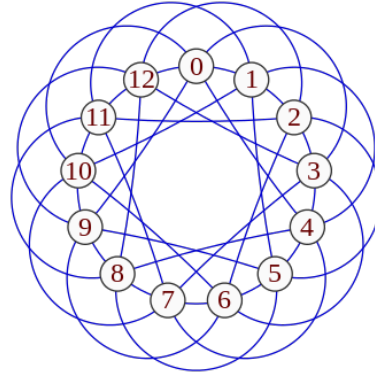


Figure 5–2 Graph \mathcal{P}_{13} . The image is from Wikipedia.

Theorem 5.2 For the Paley graph \mathcal{P}_{13} , $\text{dmg}(\mathcal{P}_{13}) = 6$.

Proof First, notice that it is already proved that the damage number of $\mathcal{P}_{13} \geq 5$ in Theorem 4.8. Thus, we only need to prove that the robber can damage another more vertex. To prove it, we notice the graph \mathcal{P}_{13} in Figure 5–2. We find that for any vertex

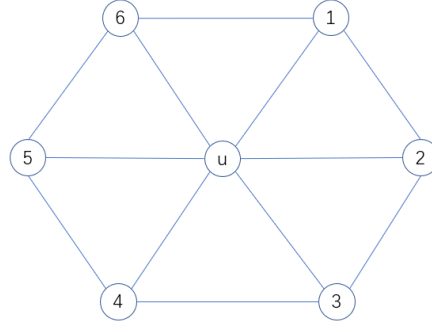


Figure 5-3 Graph A_6 .

u in graph \mathcal{P}_{13} , it has its 6 neighbors like A_6 in Figure 5-3. This result is crucial to complete the proof.

Assume that the robber has already damaged 4 vertices u_i , ($i = 1, 2, 3, 4$), and he is now on the vertex u_5 . It is the robber's move. As the robber has 6 neighbors, he must have two neighbors different from u_i , ($i = 1, 2, 3, 4$), we name these two neighbors as v_1, v_2 . There are two big cases to consider: 1) the cop is not adjacent to the robber; 2) the cop is adjacent to the robber. We discuss these two cases in the following.

Case 1: the cop is not adjacent to the robber. First, it is impossible for u_5 adjacent to all the u_i , ($i = 1, 2, 3$). If u_5 is adjacent to u_i , ($i = 1, 2, 3, 4$), as w and u_5 has 3 common neighbors, we assume that the third neighbor is u_1 . To form the A_6 of u_5 , we assume that $(v_1, u_4), (v_1, v_2), (v_2, u_1)$ are connected. This is shown in Figure 5-4. However, in this case, v_2 can not form A_6 . Thus, it is impossible for w to connect v_1, v_2 .

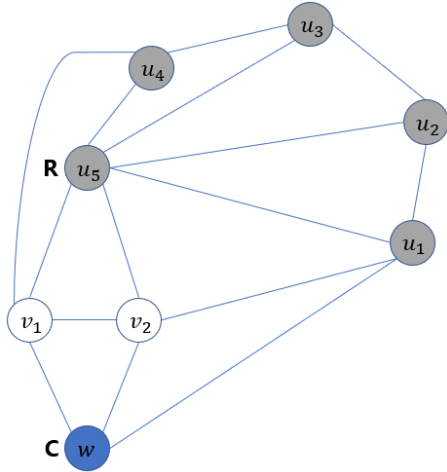


Figure 5-4 Case 1.

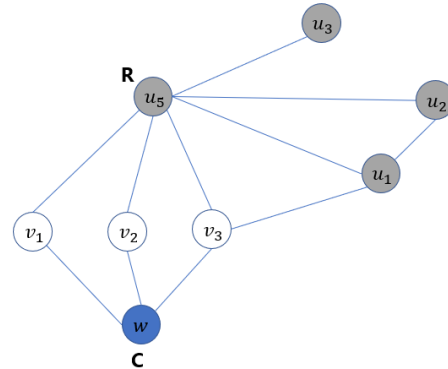


Figure 5-5 Case 1.

Thus, u_5 is not adjacent to all the u_i , ($i = 1, 2, 3$). Then u_5 connects to another undamaged vertex v_3 . We can assume that u_5 is connect to three vertices of u_i , ($i = 1, 2, 3$). Otherwise, u_5 is connected to at least 4 vertices, but w can only protect 3 vertices. We

assume that u_5 connects to u_1, u_2, u_3 . As u_1 should connect one of $v_i, (i = 1, 2, 3)$, we assume u_1 connects to v_3 . And $v_i, (i = 1, 2, 3)$ should have and only have an edge. Otherwise, if more than one edge, one vertex of $v_i, (i = 1, 2, 3)$ can not form A_6 . We get the graph shown in Figure 5–5. As v_1, v_2 is equivalent, we have two cases to consider. *Case 1.1:* v_2 connects to v_3 . In this case, v_2 is not connected to v_1 . Then v_2 connects to u_3 and $(v_1, u_3), (v_1, u_2)$ is connected. If one of u_1, u_2, u_3 can have 3 new neighbors different from u_4 (the damaged vertex). Then the robber can move to that vertex $x \in \{u_1, u_2, u_3\}$. No vertex can protect these three undamaged neighbors of x , otherwise one of the neighbor of x can not form A_6 . Figure 5–6 is an example. In Figure 5–6, $x = u_1$ and x has three new undamaged neighbors k_1, k_2, k_3 . As u_1 forms the A_6 , if one vertex connects to all the k_1, k_2, k_3 to protect them, then k_2 can not form A_6 .

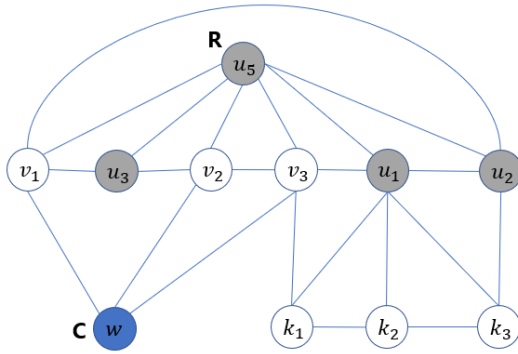


Figure 5–6 Case 1.1.

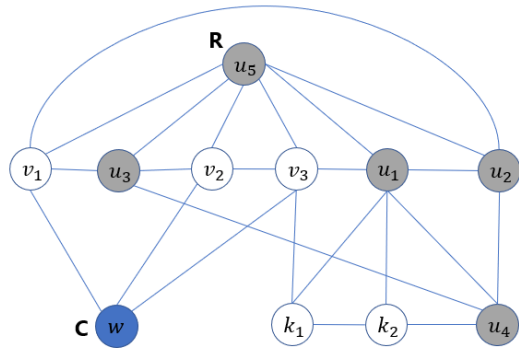


Figure 5–7 Case 1.1.

If u_1, u_2, u_3 all connect to u_4 , we consider u_1 . u_1 can connect to two undamaged vertices k_1, k_2 and form A_6 . This situation is like the above, no vertex can protect both v_3, k_1, k_2 , otherwise k_1 can not form A_6 . This is shown in Figure 5–7.

Case 1.2: v_2 connects to v_1 . This is similar to *Case 1.1*, as shown in Figure 5–8 and Figure 5–9.

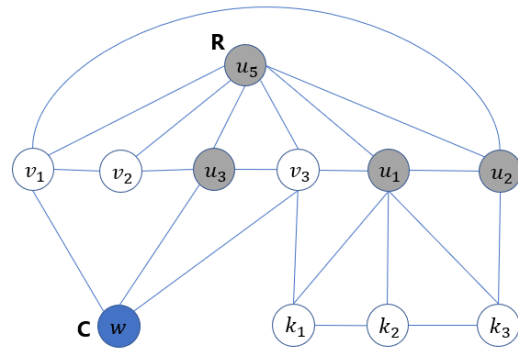


Figure 5–8 Case 1.2.

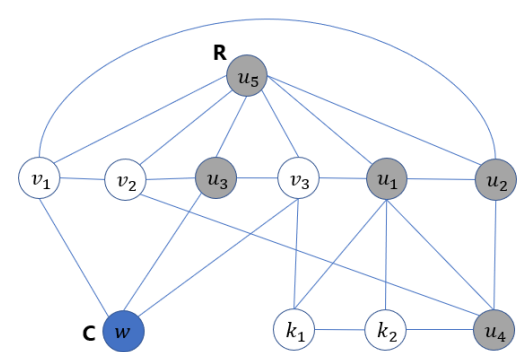


Figure 5–9 Case 1.2.

Case 2: the cop is adjacent to the robber. Considering whether w is damaged, we have

two cases.

Case 2.1: w is not damaged. Assume that u_5 connects to the damaged vertices u_1, u_2, u_3 . If one of u_1, u_2, u_3 can have 3 new neighbors different from u_4 (the damaged vertex). Then we can prove as the above. Assume that u_1, u_2, u_3 all connect to u_4 . For u_1 , it has two undamaged neighbors k_1, k_2 . When u_1 forms A_6 , no vertex can protect both v_2, k_1, k_2 , otherwise k_1 can not form A_6 . This is shown in Figure 5–10.

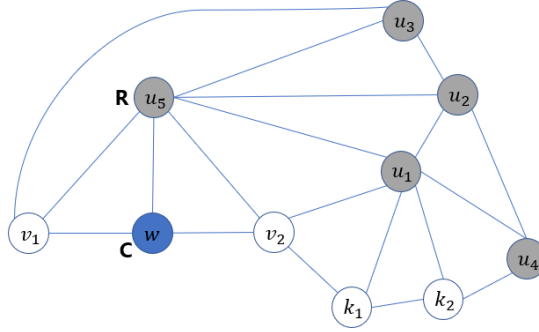


Figure 5–10 Case 2.1.

Case 2.2: w is damaged. Assume that u_5 connects to the damaged vertices u_1, u_2, u_3 . And we can prove like *Case 2.1*.

The core of the above proof is the structure of A_6 in \mathcal{P}_{13} . Therefore, we also present a similar structure in \mathcal{P}_{17} in Figure 5–11, which we define as A_8 . However, we find that A_8 seems to be not enough to decide whether $\text{dmg}(\mathcal{P}_{17}) = 8$ or not.

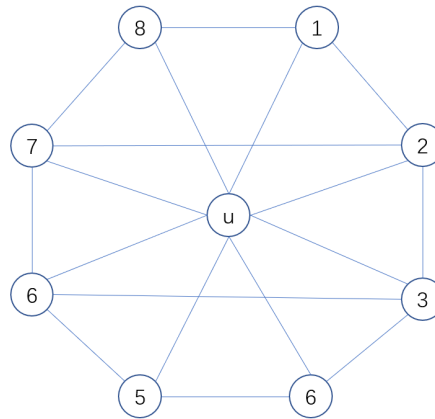


Figure 5–11 Graph A_8 .

5.2 The Ratio of Damage Number and Capture Time

Problem description: $\forall r \in (0, 1)$, decide whether there is a series of graph G_n , such that $\frac{\text{dmg}(G_n)}{\text{capt}(G_n)} \longrightarrow r \quad (n \longrightarrow \infty)$.

From the previous work, we can easily get the results of $r = \frac{1}{2}$ and $r = 1$.

Theorem 5.3 When $r = \frac{1}{2}$, for graph H_n , $\frac{\text{dmg}(H_n)}{\text{capt}(H_n)} \longrightarrow \frac{1}{2} \quad (n \longrightarrow \infty)$.

Proof From the previous paper^[8] Theorem 4.1, we have $\text{dmg}(H_n) = \lfloor \frac{n-1}{2} \rfloor - 2$. From^[11] Lemma 7, we have $\text{capt}(H_n) = n - 4$. Combining these two results, we get $\frac{\text{dmg}(H_n)}{\text{capt}(H_n)} = \frac{\lfloor \frac{n-1}{2} \rfloor - 2}{n-4} \longrightarrow \frac{1}{2} \quad (n \longrightarrow \infty)$. \square

Theorem 5.4 When $r = 1$, for graph P_n , $\frac{\text{dmg}(P_n)}{\text{capt}(P_n)} \longrightarrow 1 \quad (n \longrightarrow \infty)$.

Proof As $\text{capt}(P_n) = \lfloor \frac{n}{2} \rfloor = \text{dmg}(P_n) + 1$, we have $\frac{\text{dmg}(P_n)}{\text{capt}(P_n)} = \frac{\lfloor \frac{n}{2} \rfloor - 1}{n-4} \longrightarrow 1 \quad (n \longrightarrow \infty)$. \square

For $r \in [0, 1)$, we use the graph H_n proposed in the work^[11], as shown in Figure 3–1. We construct the chain of H_n , named as $CH(m_1, k_1, m_2, k_2)$, where m_1, k_1, m_2, k_2 are positive integers, s.t. $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$. A graph $CH(m_1, k_1, m_2, k_2)$ contains m_1 H_7 heading left, m_2 H_7 heading right, and $k_1 \times m_1$ nodes in the left chain and $k_2 \times m_2$ nodes in the right chain. To make it more clear, we present this graph in Figure 5–12.

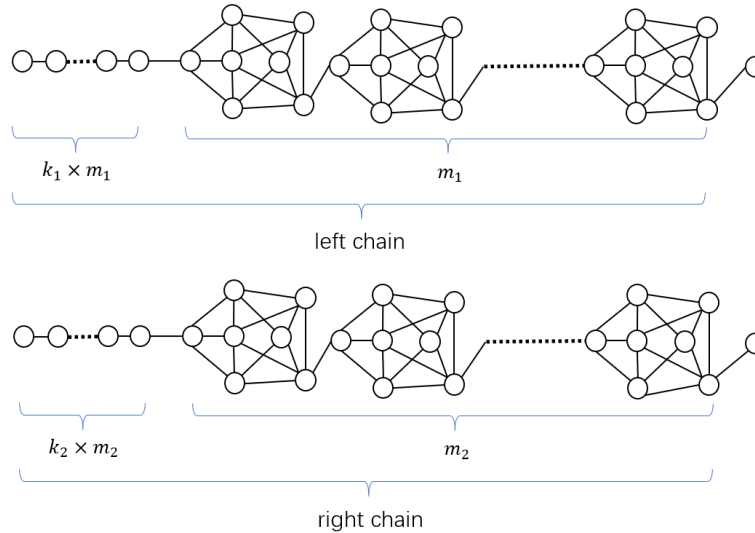


Figure 5–12 Graph chain of H_n , $CH(m_1, k_1, m_2, k_2)$.

Theorem 5.5 For the graph $CH(m_1, k_1, m_2, k_2)$, where m_1, k_1, m_2, k_2 are positive integers, *s.t.* $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\text{dmg}(CH)}{\text{capt}(CH)} = \frac{1}{2}(\frac{3+k_1}{4+k_1} + \frac{3+k_2}{4+k_1}t)$, where $t = \frac{m_2}{m_1}$.

Proof Notice that in H_7 shown in Figure 3–1 where $n = 7$, if the robber is in vertex 1, and the cop in a vertex outside this H_7 and connected to vertex 1, which we set vertex x . And if x is not connect to any vertex in this H_7 and the output vertex in this H_7 is vertex 7. Then, the maximal capture time used in this H_7 is 4. That is $x \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 7$. Besides, the maximal damage number within this H_7 is 3. That is $1 \rightarrow 6 \rightarrow 7$.

Thus, we have:

$$\text{capt}(CH) = 4m_1 + k_1m_1 \quad (5-1)$$

$$\text{dmg}(CH) = \frac{1}{2}(3m_1 + k_1m_1 + 3m_2 + k_2m_2) \quad (5-2)$$

As $t = \frac{m_2}{m_1}$, we have:

$$\begin{aligned} \frac{\text{dmg}(CH)}{\text{capt}(CH)} &= \frac{1}{2} \frac{3m_1 + k_1m_1 + 3m_2 + k_2m_2}{4m_1 + k_1m_1} \\ &= \frac{1}{2} \left(\frac{3+k_1}{4+k_1} + \frac{3+k_2}{4+k_1}t \right) \end{aligned} \quad (5-3)$$

This completes the proof. □

Theorem 5.6 For the graph $CH(m_1, k_1, m_2, k_2)$, where m_1, k_1, m_2, k_2 are positive integers, *s.t.* $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\text{dmg}(CH)}{\text{capt}(CH)} \rightarrow r \in (\frac{3}{8}, 1)$ ($n \rightarrow \infty$), where n is the total number of vertices.

Proof Because $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have:

$$0 \leq t \leq \frac{4+k_1}{4+k_2} \quad (5-4)$$

Thus, by Theorem 6.3, we have:

$$\frac{\text{dmg}(CH)}{\text{capt}(CH)} \rightarrow r \in (\frac{3}{8}, 1) \quad (5-5)$$

This completes the proof. □

Similarly, we construct the graph modified chain of H_n , named as $MCH(m_1, k_1, m_2, k_2)$, where m_1, k_1, m_2, k_2 are positive integers, *s.t.* $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$. This graph is shown in Figure 5–13. Note that the vertex connected with bidirectional arrow is the same vertex.

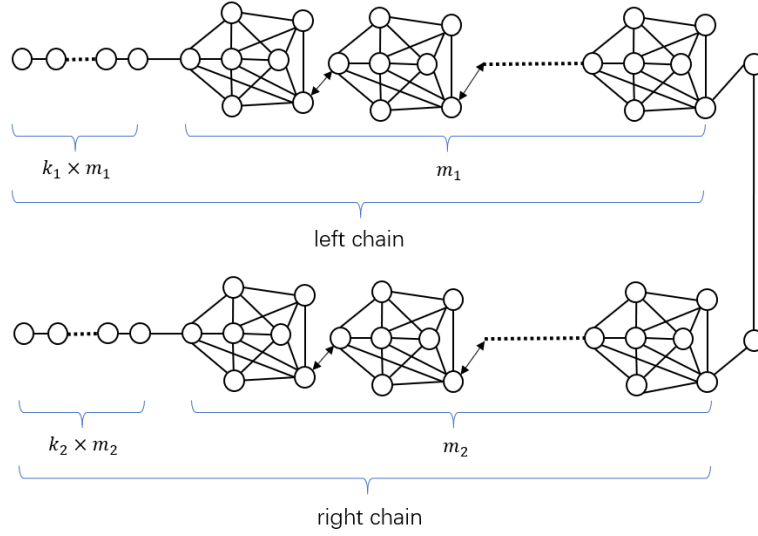


Figure 5-13 Graph modified chain of H_n , $MCH(m_1, k_1, m_2, k_2)$.

Theorem 5.7 For the graph $MCH(m_1, k_1, m_2, k_2)$, where m_1, k_1, m_2, k_2 are positive integers, s.t. $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\text{dmg}(MCH)}{\text{capt}(MCH)} \rightarrow r \in (\frac{1}{8}, 1)$ ($n \rightarrow \infty$), where n is the total number of vertices.

Proof The proof of this theorem is similar to the proofs of Theorem 5.5 and Theorem 5.6. Notice that in each block similar to H_7 , the robber can only damage one vertex instead of three vertices. □

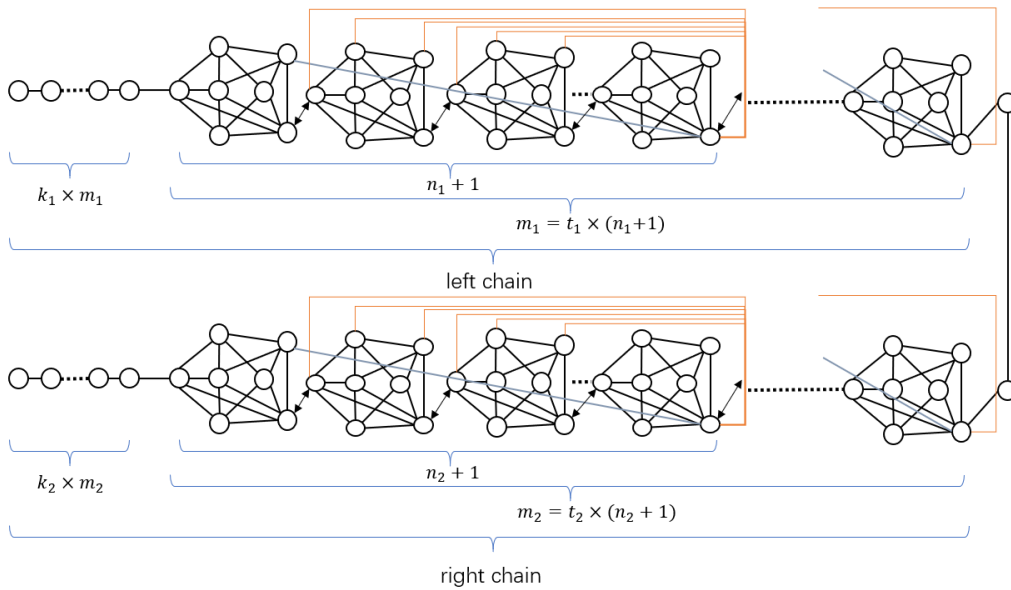


Figure 5-14 Graph modified high-way chain of H_n , $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$.



Finally, we use the “high-way” connection to get the graph modified high-way chain of H_n , i.e $MHCH$ shown in Figure 5–14.

Theorem 5.8 For the graph modified high-way H_n , $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, s.t. $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\text{dmg}(MHCH)}{\text{capt}(MHCH)} = \frac{1}{2} \frac{\frac{3}{n_1} + k_1 + (\frac{3}{n_2} + k_2)t}{4 + k_1}$, where $t = \frac{m_2}{m_1}$.

Proof To prove this theorem, we need to notice that if the robber wants to damage more vertices, he needs to escape by the “high-way”, i.e. the pink line in Figure 5–14. Otherwise, he will be stuck in one of $n_1 + 1$ blocks and can not damage the vertices outside the block. \square

Theorem 5.9 For the graph modified high-way H_n , $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, s.t. $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\text{dmg}(MHCH)}{\text{capt}(MHCH)} \rightarrow r \in (0, 1) \quad (n \rightarrow \infty)$, where n is the total number of vertices.

Proof Utilizing Theorem 5.8, it is easy to prove this theorem. \square

Chapter 6 Capture-and-smallest-damage Number

Motivated by Cox and Sanaei^[8], we introduce a new parameter of the *cop-win* graph related to *damage number*, the *capture-and-smallest-damage number*. We notate the *capture-and-smallest-damage number* of graph G as $\text{dmg}_{\text{capt}}(G)$. In some real-world scenarios, we need to achieve the goal of capturing the robber and limit the damage done by the robber to the minimum. Mathematically, we need to design a graph that is *cop-win* and the damage number is small after capturing the robber.

It is not easy. For example, in H_n , $\text{capt}(H_n) = n - 4$ and $\text{dmg}(H_n) = \lfloor \frac{n-1}{2} \rfloor - 2$, but $\text{dmg}_{\text{capt}}(H_n)$ is not small.

Theorem 6.1 For the graph H_n , $\text{dmg}_{\text{capt}}(H_n) = \text{capt}(H_n) - 1 = n - 5$.

Proof Because the cop needs to capture the robber instead of limit the robber within vertex 5 in Figure 3–1, thus the cop needs to use the strategy starting at vertex 1 or 2. \square

Note that $\text{dmg}_{\text{capt}}(H_n)$ is much greater than $\text{dmg}(H_n)$, because the cop needs to capture the robber in the end. This goal limits the strategy of the cop. Generally we have:

Theorem 6.2 For any graph G , $\text{dmg}(G) \leq \text{dmg}_{\text{capt}}(G) \leq \text{capt}(G) - 1$.

For graph $MHCH$, we have:

Theorem 6.3 For the graph modified high-way H_n , $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, *s.t.* $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\text{dmg}_{\text{capt}}(MHCH) = \text{capt}(MHCH) - 1$.

Proof This theorem can be easily proved because the cop needs to capture the robber in the end. \square

We believe that it is meaningful to find or construct some graph G that $\text{dmg}_{\text{capt}}(G)$ is small. We leave it as future work.

Chapter 7 Conclusion and Open Problems

In this undergraduate thesis, we study some problems of damage number in cop and robber game. The following are two new results we obtained.

Result 1: We constructed a series of graphs that can satisfy an abstract ratio of damage number dividing capture time.

Result 2: We proved that Paley graph \mathcal{P}_{13} has damage number 6.

Finally, there are some open problems to explore in the following.

Problem 1: Determine whether the damage number of Paley graph \mathcal{P}_n is extreme when $n \geq 17$.

Problem 2: Determine whether there are a series of graphs, such that their capture-and-smallest-damage number is much smaller than the capture time.

Problem 3: Determine the capture-and-smallest-damage number for k cops and 1 robber game, where $k > 1$.



Bibliography

- [1] ISLER V, KARNAD N. The role of information in the cop-robber game[J]. Theor. Comput. Sci., 2008, 399(3): 179-190.
- [2] HAHN G, MACGILLIVRAY G. A note on k-cop, l-robber games on graphs[J]. Discrete mathematics, 2006, 306(19-20): 2492-2497.
- [3] CLARKE N E. A game of cops and robber played with partial information[J]. Congressus numerantium, 2004, 166: 145.
- [4] CLARKE N E, CONNON E L. Cops, robber, and alarms[J]. Ars Combinatoria, 2006, 81: 283-296.
- [5] ALSPACH B. Searching and sweeping graphs: a brief survey[J]. Le matematiche, 2004, 59(1, 2): 5-37.
- [6] FOMIN F V, THILIKOS D M. An annotated bibliography on guaranteed graph searching[J]. Theoretical computer science, 2008, 399(3): 236-245.
- [7] CLARKE N E, FINBOW S, FITZPATRICK S L, et al. Seepage in directed acyclic graphs[J]. Australas. J Comb., 2009, 43: 91-102.
- [8] COX D, SANAEI A. The damage number of a graph[J]. Australian Journal of Combinatorics, 2019, 75(1): 1-16.
- [9] NOWAKOWSKI R, WINKLER P. Vertex-to-vertex pursuit in a graph[J]. Discrete Mathematics, 1983, 43(2-3): 235-239.
- [10] BONATO A, GOLOVACH P, HAHN G, et al. The capture time of a graph[J]. Discrete Mathematics, 2009, 309(18): 5588-5595.
- [11] GAVENČIAK T. Cop-win graphs with maximum capture-time[J]. Discrete Mathematics, 2010, 310(10-11): 1557-1563.