Norwegian University of Science
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Department for Computer Science

Methods in Artificial Inteligence TDT4171 Spring 2021

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Exercise 2

1 **a)** The Bayesian network for this problem is shown in figure Figure 1.

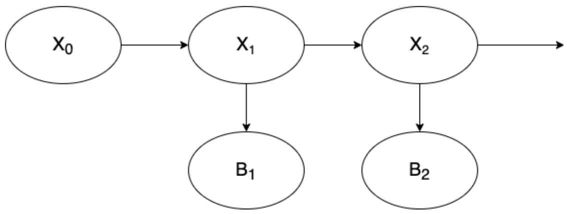


Figure 1: Bayesian network for the problem.

Here X_t is the state distribution on time t and B_t is the evidence distribution at time t.
The probability distributions are be given by tables 1, 2 and 3.

	$P(X_t)$	
X_{t-1}	t	f
t	0.8	0.2
f	0.3	0.7

Table 1: Probability distribution for $P(X_t|X_{t-1})$.

	$P(B_t)$	
X_t	t	f
t	0.75	0.25
f	0.2	0.8

Table 2: Probability distribution for $P(B_t|X_t)$.

$P(X_0)$	
t	f
0.5	0.5

Table 3: Probability distribution for $P(X_0)$.

Formulation as a hidden Markov model:

$$\mathbf{T} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\mathbf{P}(X_0) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$\mathbf{T}$ is the transition model and $\mathbf{P}(X_0)$ is the initial distribution.
The sensor model $\mathbf{O}_t$ depends on the evidence given:

$$\mathbf{e}_t = BirdsNearby \implies \mathbf{O}_t = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\mathbf{e}_t = NoBirdsNearby \implies \mathbf{O}_t = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.8 \end{bmatrix}$$

b) This kind of operation is filtering, where we calculate the current belief state given all the evidence to date. The results are given in Figure 2. This operation gives us the probability distribution of the current state given all the evidence to date.

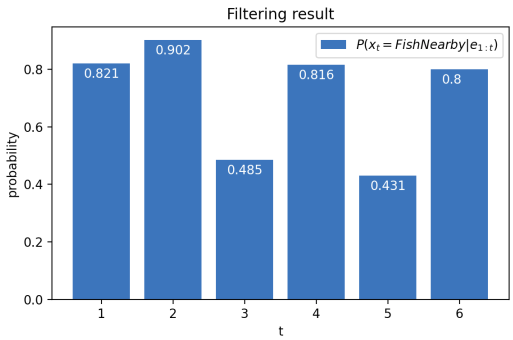


Figure 2: Filtering for $t = 1, ..., 6$.

c) This kind of operation is prediction, where we calculate the distribution of a future state, based on the previous evidence, but no new addition evidence are given. This is the same as filtering without adding new evidence. The results are given in Figure 3.

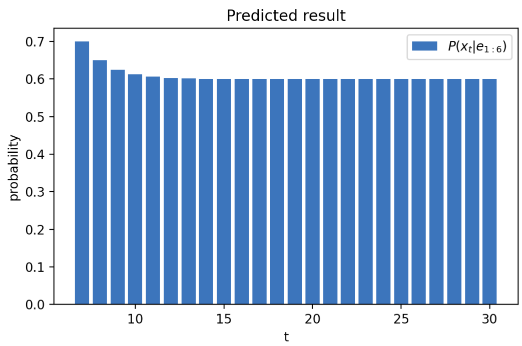


Figure 3: Prediction for $t = 7, ..., 30$.

As we can see, the belief state converges towards $\begin{bmatrix}0.6 & 0.4\end{bmatrix}^\top$ as t increases. This is because $\begin{bmatrix}0.6 & 0.4\end{bmatrix}^\top$ is the eigenvector for the eigenvalue 1 in the matrix $T^\top$. This will be explained further in Task 2b.

d) This kind of operation is smoothing, where we calculate the distribution of a past state, given all the evidence up to the present. The results are given in Figure 4.

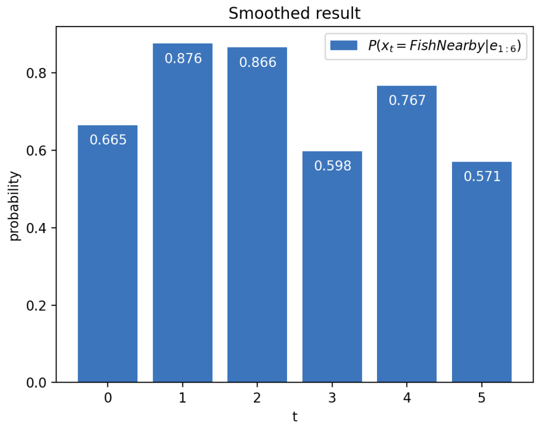


Figure 4: Smoothing for $t = 0, ..., 5$.

e) This kind of operation is most likely sequence, where we calculate the most like sequence of states given all the evidence up to the present. The results where the sequence given in Figure 5.

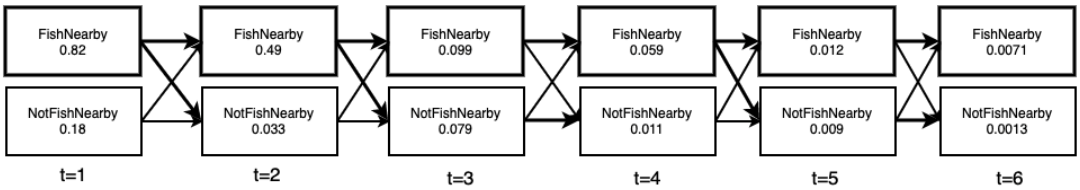
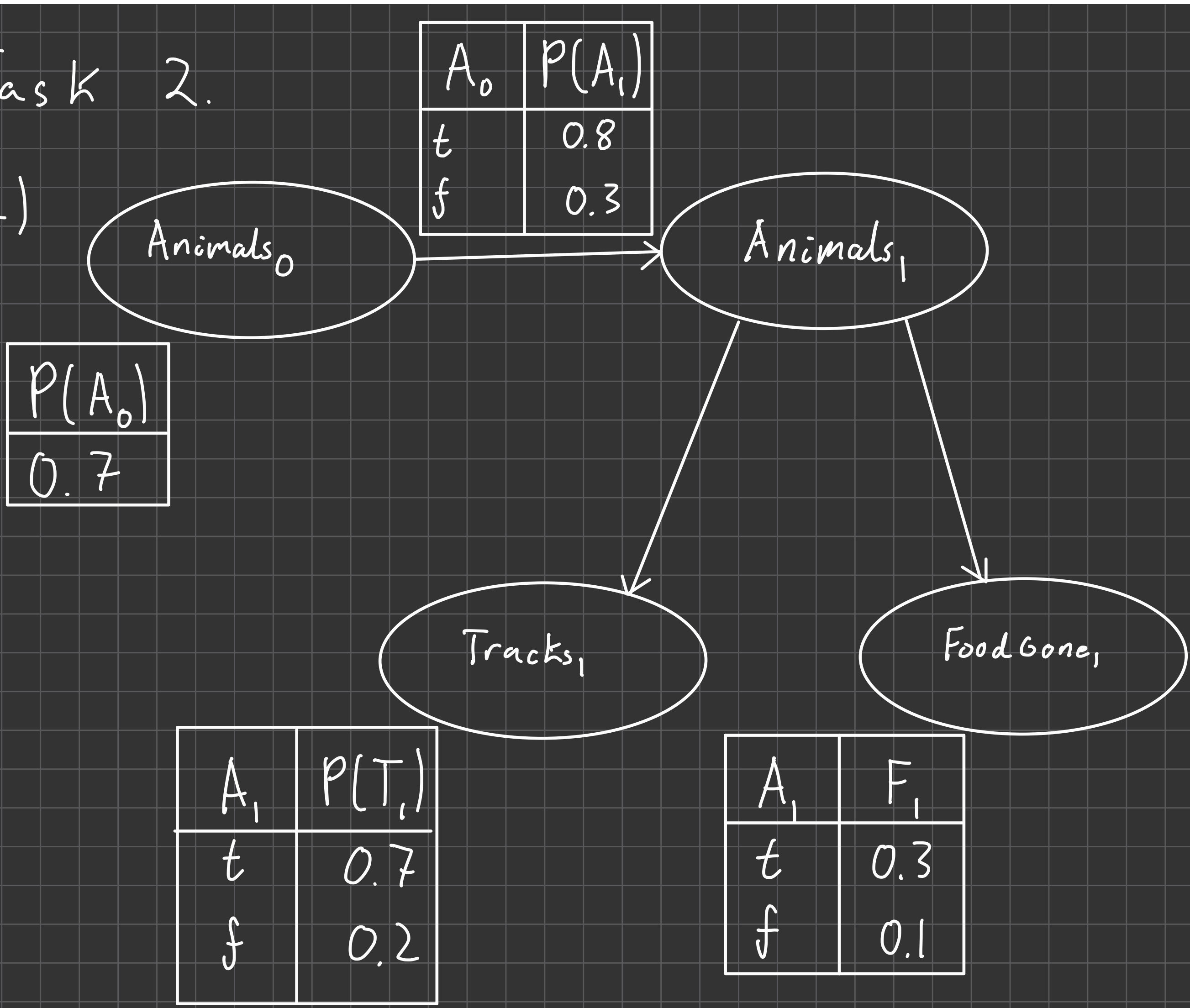


Figure 5: Viterbi sequence for $t = 1, ..., 6$.

Task 2.

a)



$P(A_0)$
0.7

A_0	$P(A_1)$
t	0.8
f	0.3

A_1	$P(T_1)$
t	0.7
f	0.2

A_1	F_1
t	0.3
f	0.1

A = Animals, T = Tracks, F = Food Gone.

b) From equation 15.5 in the book we have that:

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha\, P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})$$

In matrix-form we have:

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha\, O_{t+1} T^T P(X_t \mid e_{1:t})$$

Which is a recursive equation with the base case:

$$P(X_0 | e_{1:0}) = P(X_0) = P(A_0) = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

T is the transition model defined as:

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

O_t is the sensor-model.

$$O_t = \begin{bmatrix} P(t_t \cap f_t | a_t) & 0 \\ 0 & P(t_t \cap f_t | \neg a_t) \end{bmatrix}$$

Since T_t and F_t is conditionally independent given A_t we have:

$$O_t = \begin{bmatrix} P(t_t | a_t) P(f_t | a_t) & 0 \\ 0 & P(t_t | \neg a_t) P(f_t | \neg a_t) \end{bmatrix}$$

Now we can start calculating:

$t = 1$:

$$P(X_1 | e_{1:1}) = \alpha \, O_1 T^T P(X_0)$$

$$= \alpha \begin{bmatrix} 0.7 \cdot 0.3 & 0 \\ 0 & 0.2 \cdot 0.1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.21 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.1365 \\ 0.007 \end{bmatrix} = \begin{bmatrix} 39/41 \\ 2/41 \end{bmatrix} = f_{1:1}$$

$t=2$:

$$P(X_2 \mid e_{1:2}) = \alpha O_2 T^T P(X_1 \mid e_{1:1})$$

$$= \alpha \begin{bmatrix} 0.3 \cdot 0.3 & 0 \\ 0 & 0.8 \cdot 0.1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 39/41 \\ 2/41 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.09 & 0 \\ 0 & 0.08 \end{bmatrix} \begin{bmatrix} 159/205 \\ 46/205 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1431/20500 \\ 92/5125 \end{bmatrix} \approx \begin{bmatrix} 0.79544 \\ 0.20456 \end{bmatrix} = f_{1:2}$$

$t=3$:

$$P(X_3 \mid e_{1:t}) = \alpha O_3 T^T P(X_2 \mid e_{1:2})$$

$$= \alpha \begin{bmatrix} 0.3 \cdot 0.7 & 0 \\ 0 & 0.8 \cdot 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.79544 \\ 0.20456 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.21 & 0 \\ 0 & 0.72 \end{bmatrix} \begin{bmatrix} 0.69772 \\ 0.30228 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.14652 \\ 0.21764 \end{bmatrix} \approx \begin{bmatrix} 0.40235 \\ 0.59765 \end{bmatrix} = f_{1:3}$$

$t=4$:

$$P(X_4 | e_{1:4}) = \alpha \, O_4 T^T P(X_3 | e_{1:3})$$

$$= \alpha \begin{bmatrix} 0.7 \cdot 0.7 & 0 \\ 0 & 0.2 \cdot 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.40235 \\ 0.59765 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.49 & 0 \\ 0 & 0.18 \end{bmatrix} \begin{bmatrix} 0.501175 \\ 0.498825 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.2458575 \\ 0.0897885 \end{bmatrix} \approx \begin{bmatrix} 0.732 \\ 0.268 \end{bmatrix} = f_{1:4}$$

c) Prediction is the same as filtering without adding new evidence:

$$P(X_{t+k+1} | e_{1:t}) = \sum_{x_{t+k}} P(X_{t+k+1} | x_{t+k}) P(x_{t+k} | e_{1:t})$$

We want to calculate this with $t=4$ and $k=0, 1, 2, 3$.

We can rewrite this using matrices:

$$P(X_{t+k+1} \mid e_{1:t}) = T^T f_{1:t+k} = f_{1:t+k+1}$$

$k=0$:

gotten from previous task

$$P(X_5 \mid e_{1:4}) = T^T f_{1:4}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.732266 \\ 0.267733 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.66613 \\ 0.33387 \end{bmatrix} = f_{1:5}$$

$k=1$:

$$P(X_6 \mid e_{1:4}) = T^T f_{1:5}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.666133 \\ 0.333867 \end{bmatrix} \approx \begin{bmatrix} 0.63307 \\ 0.36693 \end{bmatrix} = f_{1:6}$$

$k=2$:

$$P(X_7 \mid e_{1:4}) = T^T f_{1:6}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.633066 \\ 0.366934 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.61653 \\ 0.38347 \end{bmatrix} = f_{1:7}$$

$k=3$:

$$P(X_8 \mid e_{1:4}) = T^T f_{1:7}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.616533 \\ 0.383467 \end{bmatrix} \approx \begin{bmatrix} 0.60827 \\ 0.39173 \end{bmatrix} = f_{1:8}$$

d) $$\lim_{t \to \infty} P(X_t \mid e_{1:4}) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$P(X_t \mid e_{1:4}) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\Rightarrow P(X_{t+1} \mid e_{1:4}) = T^T P(X_t \mid e_{1:4})$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Thus $P(X_t \mid e_{1:4})$ has converged to $[0.6 \;\; 0.4]^T$.

Next I will demonstrate why it converges to $[0.6 \;\; 0.4]^T$

We have that T^T is a regular stochastic matrix. Then T^T has a unique equilibrium vector q. For every vector x_0 where the Markov-process starts, the Markovchain $\{x_t\}$ will converge towards q when $t \to \infty$.

This q can be found by finding the eigenvector corresponding to the eigenvalue 1:

$$T^T - I_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} q = 0$$

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \sim \begin{bmatrix} -0.2 & 0.3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -0.2q_1 + 0.3q_2 = 0$$

$$q_1 = 1.5 q_2$$

$$\Rightarrow \quad q = t \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

We want q to be a probability vector:

$$\Rightarrow \quad q = \frac{1}{1+1.5} \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

e) $P(X_t \mid e_{1:4}) = \alpha P(X_t \mid e_{1:t}) P(e_{t+1:4} \mid X_t)$

$= \alpha f_{1:t} \times b_{t+1:4}$ (pointwise multiplication)

$$b_{k+1:t} = P(e_{k+1:t} \mid X_k)$$

$$= \sum_{x_{k+1}} \underbrace{P(e_{k+1} \mid x_{k+1})}_{\text{sensor model}} \underbrace{P(e_{k+2:t} \mid x_{k+1})}_{b_{k+2:t}} \underbrace{P(x_{k+1} \mid X_k)}_{\text{transition model}}$$

$$= T O_{k+1} b_{k+2:t}$$

The base case for this recursive process is

$$b_{t+1:t} = P(e_{t+1:t} \mid X_t) = P(\mid X_t) 1 = [1 \quad 1]^T$$

$t = 3$:

have from previous task

$$P(X_3 \mid e_{1:4}) = \alpha f_{1:3} \times b_{4:4}$$

$$b_{4:4} = T O_4 b_{5:4}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.49 & 0 \\ 0 & 0.18 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 \\ 0.72 \end{bmatrix} = \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$\Rightarrow \quad P(X_3 | e_{1:4}) = \alpha \begin{bmatrix} 0.40235191 \\ 0.59764809 \end{bmatrix} \times \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.1722066 \\ 0.1631579 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.513491 \\ 0.486509 \end{bmatrix}$$

$t = 2$:

$$P(X_2 | e_{1:4}) = \alpha f_{1:2} \times b_{3:4}$$

$$b_{3:4} = T O_3 b_{4:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 & 0 \\ 0 & 0.72 \end{bmatrix} \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.08988 \\ 0.19656 \end{bmatrix}$$

$$= \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$\Rightarrow \quad P(X_2 | e_{1:4}) = \alpha \begin{bmatrix} 0.7954419 \\ 0.2045581 \end{bmatrix} \times \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0884659 \\ 0.0336613 \end{bmatrix} \approx \begin{bmatrix} 0.724375 \\ 0.275625 \end{bmatrix}$$

$t = 1$:

$$P(X_1 | e_{1:4}) = \alpha f_{1:1} \times b_{2:4}$$

$$b_{2:4} = TO_2 b_{3:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.09 & 0 \\ 0 & 0.08 \end{bmatrix} \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.01000944 \\ 0.01316448 \end{bmatrix} = \begin{bmatrix} 0.01064045 \\ 0.01221797 \end{bmatrix}$$

$$\Rightarrow \quad P(X_1 | e_{1:4}) = \alpha \begin{bmatrix} 0.95121951 \\ 0.04878049 \end{bmatrix} \times \begin{bmatrix} 0.01064045 \\ 0.01221797 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0101214 \\ 0.0005960 \end{bmatrix} \simeq \begin{bmatrix} 0.94439 \\ 0.05561 \end{bmatrix}$$

$t = 0$:

$$P(X_0 | e_{1:4}) = \alpha f_{1:0} \times b_{1:4}$$

$$b_{1:4} = TO_1 b_{2:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.0106405 \\ 0.0122180 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.0022345 \\ 0.00024436 \end{bmatrix} = \begin{bmatrix} 0.001836472 \\ 0.000841402 \end{bmatrix}$$

$$\Rightarrow \quad P(X_0 | e_{1:4}) = \alpha \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \times \begin{bmatrix} 0.001836472 \\ 0.000841402 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.00128553 \\ 0.00025242 \end{bmatrix} \approx \begin{bmatrix} 0.83587 \\ 0.16413 \end{bmatrix}$$