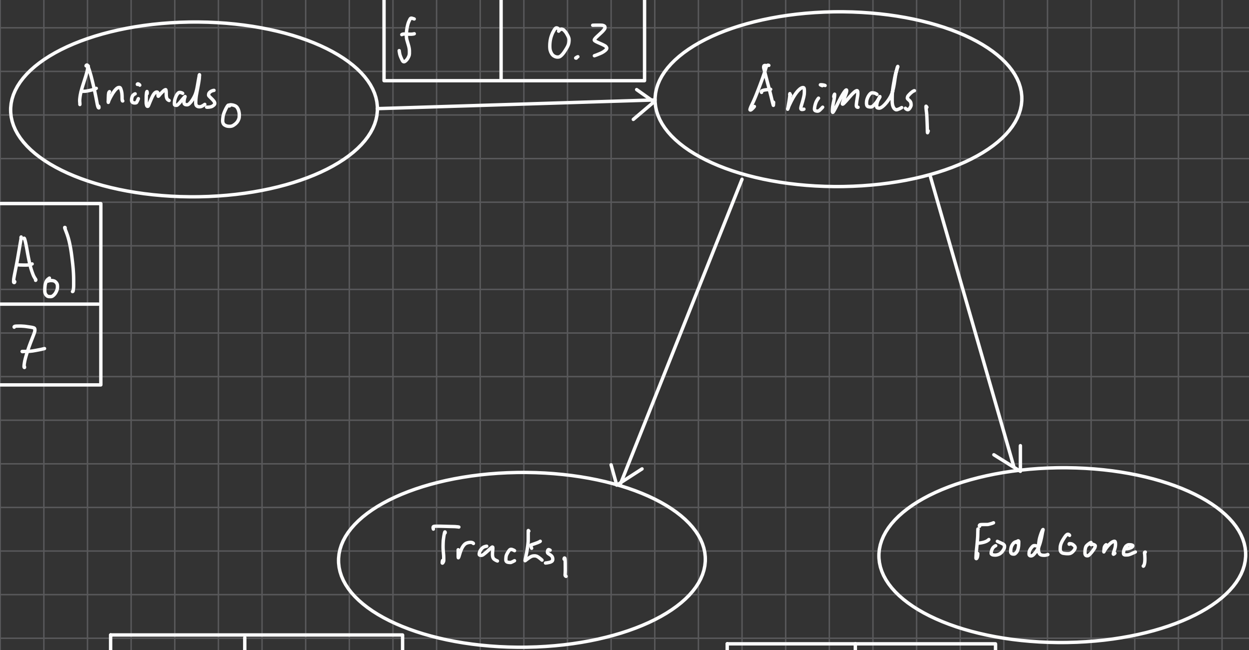
Task 2.

a)



A_0	$P(A_1)$
t	0.8
f	0.3

$P(A_0)$
0.7

A_1	$P(T_1)$
t	0.7
f	0.2

A_1	F_1
t	0.3
f	0.1

A = Animals, T = Tracks, F = Food Gone.

b) From equation 15.5 in the book we have that:

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})$$

In matrix-form we have:

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha O_{t+1} T^T P(X_t \mid e_{1:t})$$

Which is a recursive equation with the base case:

$$P(X_0 | e_{1:0}) = P(X_0) = P(A_0) = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

T is the transition model defined as:

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

O_t is the sensor-model.

$$O_t = \begin{bmatrix} P(t_t \cap f_t | a_t) & 0 \\ 0 & P(t_t \cap f_t | \neg a_t) \end{bmatrix}$$

Since T_t and F_t is conditionally independent given A_t we have:

$$O_t = \begin{bmatrix} P(t_t | a_t) P(f_t | a_t) & 0 \\ 0 & P(t_t | \neg a_t) P(f_t | \neg a_t) \end{bmatrix}$$

Now we can start calculating:

$t = 1$:

$$P(X_1 | e_{1:1}) = \alpha\, O_1 T^T P(X_0)$$

$$= \alpha \begin{bmatrix} 0.7 \cdot 0.3 & 0 \\ 0 & 0.2 \cdot 0.1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.21 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.1365 \\ 0.007 \end{bmatrix} = \begin{bmatrix} 39/41 \\ 2/41 \end{bmatrix} = f_{1:1}$$

$t=2$:

$$P(X_2 \mid e_{1:2}) = \alpha\, O_2\, T^T\, P(X_1 \mid e_{1:1})$$

$$= \alpha \begin{bmatrix} 0.3 \cdot 0.3 & 0 \\ 0 & 0.8 \cdot 0.1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 39/41 \\ 2/41 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.09 & 0 \\ 0 & 0.08 \end{bmatrix} \begin{bmatrix} 159/205 \\ 46/205 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1431/20500 \\ 92/5125 \end{bmatrix} \approx \begin{bmatrix} 0.79544 \\ 0.20456 \end{bmatrix} = f_{1:2}$$

$t=3$:

$$P(X_3 \mid e_{1:t}) = \alpha\, O_3\, T^T P(X_2 \mid e_{1:2})$$

$$= \alpha \begin{bmatrix} 0.3 \cdot 0.7 & 0 \\ 0 & 0.8 \cdot 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.79544 \\ 0.20456 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.21 & 0 \\ 0 & 0.72 \end{bmatrix} \begin{bmatrix} 0.69772 \\ 0.30228 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.14652 \\ 0.21764 \end{bmatrix} \approx \begin{bmatrix} 0.40235 \\ 0.59765 \end{bmatrix} = f_{1:3}$$

$t=4$:

$$P(X_4 | e_{1:4}) = \alpha\, O_4 T^T P(X_3 | e_{1:3})$$

$$= \alpha \begin{bmatrix} 0.7 \cdot 0.7 & 0 \\ 0 & 0.2 \cdot 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.40235 \\ 0.59765 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.49 & 0 \\ 0 & 0.18 \end{bmatrix} \begin{bmatrix} 0.501175 \\ 0.498825 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.2458575 \\ 0.0897885 \end{bmatrix} \approx \begin{bmatrix} 0.732 \\ 0.268 \end{bmatrix} = f_{1:4}$$

c) Prediction is the same as filtering without adding new evidence:

$$P(X_{t+k+1} | e_{1:t}) = \sum_{x_{t+k}} P(X_{t+k+1} | x_{t+k})\, P(x_{t+k} | e_{1:t})$$

We want to calculate this with $t=4$ and $k = 0, 1, 2, 3$.

We can rewrite this using matrices:

$$P(X_{t+k+1} \mid e_{1:t}) = T^T f_{1:t+k} = f_{1:t+k+1}$$

$k=0$:

gotten from previous task

$$P(X_5 \mid e_{1:4}) = T^T f_{1:4}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.732266 \\ 0.267733 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.666133 \\ 0.333867 \end{bmatrix} = f_{1:5}$$

$k=1$:

$$P(X_6 \mid e_{1:4}) = T^T f_{1:5}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.666133 \\ 0.333867 \end{bmatrix} \approx \begin{bmatrix} 0.63307 \\ 0.36693 \end{bmatrix} = f_{1:6}$$

$k=2$:

$$P(X_7 \mid e_{1:4}) = T^T f_{1:6}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.633066 \\ 0.366934 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.61653 \\ 0.38347 \end{bmatrix} = f_{1:7}$$

$k = 3$:

$$P(X_8 \mid e_{1:4}) = T^T f_{1:7}$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.616533 \\ 0.383467 \end{bmatrix} \approx \begin{bmatrix} 0.60827 \\ 0.39173 \end{bmatrix} = f_{1:8}$$

d) $$\lim_{t \to \infty} P(X_t \mid e_{1:4}) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$P(X_t \mid e_{1:4}) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\Rightarrow P(X_{t+1} \mid e_{1:4}) = T^T P(X_t \mid e_{1:4})$$

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Thus $P(X_t \mid e_{1:4})$ has converged to $[0.6 \;\; 0.4]^T$.

Next I will demonstrate why it converges to $[0.6 \;\; 0.4]^T$

We have that T^T is a regular stochastic matrix. Then T^T has a unique equilibrium vector q. For every vector x_0 where the Markov-process starts, the Markovchain $\{x_t\}$ will converge towards q when $t \to \infty$.

This q can be found by finding the eigenvector corresponding to the eigenvalue 1:

$$T^T - I_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} q = 0$$

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \sim \begin{bmatrix} -0.2 & 0.3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -0.2q_1 + 0.3q_2 = 0$$

$$q_1 = 1.5q_2$$

$$\Rightarrow \quad q = t \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

We want q to be a probability vector:

$$\Rightarrow \quad q = \frac{1}{1+1.5} \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

e) $$P(X_t | e_{1:4}) = \alpha P(X_t | e_{1:t}) P(e_{t+1:4} | X_t)$$

$$= \alpha f_{1:t} \times b_{t+1:4} \quad \text{(pointwise multiplication)}$$

$$b_{k+1:t} = P(e_{k+1:t} | X_k)$$

$$= \sum_{x_{k+1}} \underbrace{P(e_{k+1} | x_{k+1})}_{\text{sensor model}} \underbrace{P(e_{k+2:t} | x_{k+1})}_{b_{k+2:t}} \underbrace{P(x_{k+1} | X_k)}_{\text{transition model}}$$

$$= T O_{k+1} b_{k+2:t}$$

The base case for this recursive process is

$$b_{t+1:t} = P(e_{t+1:t} | X_t) = P(| X_t) 1 = [1 \quad 1]^T$$

$t = 3$:

have from previous task

$$P(X_3 | e_{1:4}) = \alpha f_{1:3} \times b_{4:4}$$

$$b_{4:4} = T O_4 b_{5:4}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.49 & 0 \\ 0 & 0.18 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 \\ 0.72 \end{bmatrix} = \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$\Rightarrow \quad P(X_3 \mid e_{1:4}) = \alpha \begin{bmatrix} 0.40235191 \\ 0.59764809 \end{bmatrix} \times \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.1722066 \\ 0.1631579 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.513491 \\ 0.486509 \end{bmatrix}$$

$t = 2:$

$$P(X_2 \mid e_{1:4}) = \alpha\, f_{1:2} \times b_{3:4}$$

$$b_{3:4} = T O_3\, b_{4:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 & 0 \\ 0 & 0.72 \end{bmatrix} \begin{bmatrix} 0.428 \\ 0.273 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.08988 \\ 0.19656 \end{bmatrix}$$

$$= \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$\Rightarrow \quad P(X_2 \mid e_{1:4}) = \alpha \begin{bmatrix} 0.7954419 \\ 0.2045581 \end{bmatrix} \times \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0884659 \\ 0.0336613 \end{bmatrix} \approx \begin{bmatrix} 0.724375 \\ 0.275625 \end{bmatrix}$$

$t=1:$

$$P(X_1 \mid e_{1:4}) = \alpha f_{1:1} \times b_{2:4}$$

$$b_{2:4} = TO_2\, b_{3:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.09 & 0 \\ 0 & 0.08 \end{bmatrix} \begin{bmatrix} 0.111216 \\ 0.164556 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.01000944 \\ 0.01316448 \end{bmatrix} = \begin{bmatrix} 0.01064045 \\ 0.01221797 \end{bmatrix}$$

$$\Rightarrow \quad P(X_1 \mid e_{1:4}) = \alpha \begin{bmatrix} 0.95121951 \\ 0.04878049 \end{bmatrix} \times \begin{bmatrix} 0.01064045 \\ 0.01221797 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0101214 \\ 0.0005960 \end{bmatrix} \approx \begin{bmatrix} 0.94439 \\ 0.05561 \end{bmatrix}$$

$t=0:$

$$P(X_0 \mid e_{1:4}) = \alpha f_{1:0} \times b_{1:4}$$

$$b_{1:4} = TO_1\, b_{2:4} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.21 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.0106405 \\ 0.0122180 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.0022345 \\ 0.00024436 \end{bmatrix} = \begin{bmatrix} 0.001836472 \\ 0.000841402 \end{bmatrix}$$

$$\Rightarrow \quad P(X_0 \mid e_{1:4}) = \alpha \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \times \begin{bmatrix} 0.001836472 \\ 0.000841402 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.00128553 \\ 0.00025242 \end{bmatrix} \approx \begin{bmatrix} 0.83587 \\ 0.16413 \end{bmatrix}$$