



- 1 a) Since $a = \dot{v} = \ddot{s}$ we get the following:

$$\begin{aligned}\dot{x}_2 &= a = F \\ \dot{x}_1 &= v = x_2 \\ \Rightarrow \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\end{aligned}$$

- b) First calculate $e^{A_c\tau}$:

$$\begin{aligned}e^{A_c\tau} &= \sum_{k=0}^{\infty} \frac{1}{k!} (A_c\tau)^k \\ &= I + A_c\tau + \frac{A_c^2\tau^2}{2!} + \dots + \frac{A_c^n\tau^n}{n!} \\ A_c \cdot A_c &= \mathbf{0} \\ \Rightarrow e^{A_c\tau} &= I + A_c\tau = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}\end{aligned}$$

For A we then have:

$$e^{A_cT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

For b :

$$\begin{aligned}\int_0^T e^{A_c\tau} d\tau b_c &= \int_0^T e^{A_c\tau} b_c d\tau \\ &= \int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\ &= \int_0^T \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} \frac{\tau^2}{2} \\ \tau \end{bmatrix} \Big|_{\tau=0}^{\tau=T} \\ &= \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \\ &= \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}\end{aligned}$$

c) The Riccati-equation is given by:

$$P_t = Q + A^\top P_{t+1} (I + bR^{-1}b^\top P_t + 1)^{-1} A$$

$$P_N = Q$$

The solution of this (P_t) is used in the definition of the feedback gain matrix K_{t-1} :

$$K_{t-1} = R^{-1}b^\top P_t (I + bR^{-1}b^\top P_t)^{-1} A$$

The feedback gain matrix is then used in the controller u_t which minimizes $f(z)$:

$$u_t = -K_t x_t$$

The solution is illustrated in Figure 1.

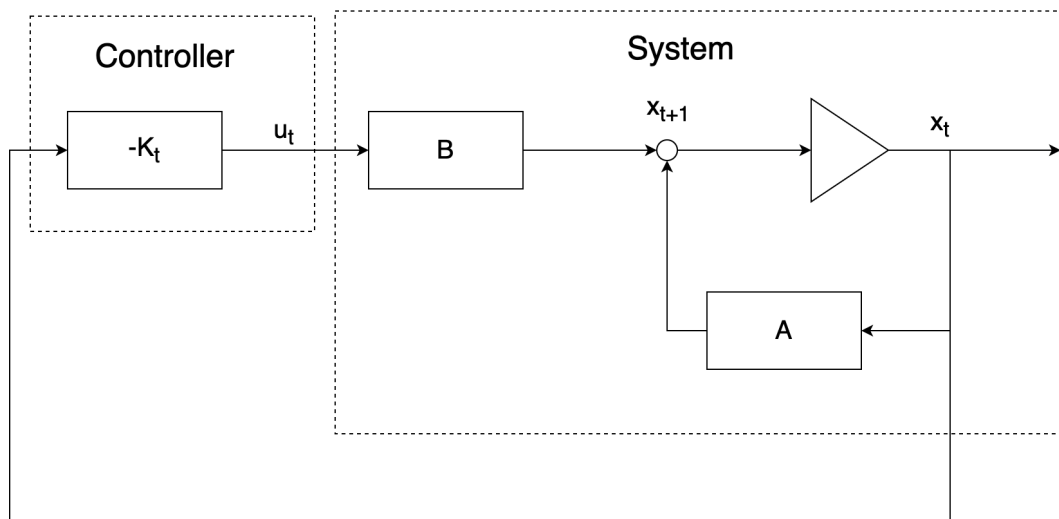


Figure 1: The system and controller.

d) To solve the stationary Riccati equation I used the following code:

```

1  Q = 1/2*[
2      2  0
3      0  2
4  ];
5  R = 2/2;
6  A = [
7      1  0.5
8      0  1
9  ];
10 B = [
11     0.125
12     0.5
13 ];
14 [K,S,e] = dlqr(A,B,Q,R,[]);

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15 disp(K) ;
16 disp(S) ;
17 disp(e) ;

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This gives that:

$$P = \begin{bmatrix} 4.0350 & 2.0616 \\ 2.0616 & 4.1438 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.6514 \\ 1.3142 \end{bmatrix}$$

$$\lambda = 0.6307 \pm 0.1628i$$

The resulting system is stable if $|\text{eig}(A - BK)| < 1$. Since

$$|\lambda_1| = |\lambda_2| = \sqrt{0.6307^2 + 0.1628^2} = 0.6514 < 1,$$

the closed-loop system is stable.

- e) The closed-loop system is stable if the system (A, B) is stabilizable and the system (A, D) is detectable, where D is defined by $Q = D^\top D$. The first part means that the system must be stabilizable, since we have a infinite horizon. The second part means that we must be able to influence all unstable modes of the system. In order for us to influence all unstable modes we must be able to detect them in our objective function (through D).

- 2 a) The stationary one-dimensional Riccati equation is given by:

$$p = q + ap(1 + \frac{b^2 p}{r})^{-1}a$$

Some rest

$$p = q + ap(1 + \frac{b^2 p}{r})^{-1}a$$

$$p = q + a^2 p \frac{1}{1 + \frac{b^2 p}{r}}$$

$$ps = q + \frac{a^2 pr}{r + b^2 p}$$

With $q = 2$, $a = 3$, $b = 2$, $r = 1$ we get:

$$p = 2 + \frac{3^2 p}{1 + 2^2 p}$$

$$p(1 + 4p) = 2(1 + 4p) + 9p$$

$$4p^2 - 16 - 2 = 0$$

$$\Rightarrow p = \frac{16 \pm \sqrt{16^2 - 4(-2)4}}{8} = 2 \pm \frac{3}{\sqrt{2}}$$

Since p must be above 0 (see equation 4.32d in "Merging Optimization and Control"), we choose $p = 2 + \frac{3}{\sqrt{2}}$

b) The feedback gain is defined by:

$$k = \frac{bp}{r} \left(1 + \frac{b^2 p}{r}\right)^{-1} a$$

Setting in all the values and $p = 2 + \frac{3}{\sqrt{2}}$ gives us:

$$k = \sqrt{2}$$

This gives us that the optimal control law is given by:

$$u_t = -\sqrt{2}x_t$$

c) Same answer as Task 1e.

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