In this file, we compute the graphs with minimum occupancy among all triangle – free 4 *– regular graphs of order bounded* **by** 19, **for** *every value of* lambda
Next, we check that there are graphs with higher order,
for *which the occupancy fraction is smaller on certain open intervals.*
For this, we insert all occupancy fractions **for** *the critical graphs of every order up* **to** 19
for *which a triangle – free* 4 *– regular graph exists.*
By Mantel's theorem, this **implies** *that n has* **to** *be at least* 8. *As the graph has* **to** *be regular,*
no such graph exists when $n = 9$.
After this, we conclude that every occupancy is at least $\min(p13, p19)$.
For some graphs with occupancy q, we just can conclude that $p13 < q$ **or** $p19 < q$ *on* R^+.
In other cases, we have that $p13 < q$ *on some interval I1* **and** $p19 < q$ *on I2 with I1 cup I2* $= R^+$.
Finally, we have examples on 20 **and** 22 *vertices* **for** *which* $q < \min(p13, p19)$.

```
> p8(x) := (x^3 + 3*x^2 + 3*x + 1)*x/(2*x^4 + 8*x^3 + 12*x^2 + 8*x + 1);
  p10(x) := (2*x^3 + 6*x^2 + 5*x + 1)*x/(5*x^4 + 20*x^3 + 25*x^2 + 10*x
      + 1);
  p11(x) := (4*x^3 + 9*x^2 + 6*x + 1)*x/(11*x^4 + 33*x^3 + 33*x^2 + 11*x
      + 1);
  p12(x) := (6*x^3 + 13*x^2 + 7*x + 1)*x/(18*x^4 + 52*x^3 + 42*x^2 + 12*x
      + 1);
  p13(x) := (12*x^3 + 18*x^2 + 8*x + 1)*x/(39*x^4 + 78*x^3 + 52*x^2 + 13*x
      + 1);
  p14(x) := 1/7*(30*x^4 + 150*x^3 + 168*x^2 + 63*x + 7)*x/(12*x^5 + 75*x^4
      + 112*x^3 + 63*x^2 + 14*x + 1);
  p15(x) := 1/3*(31*x^4 + 104*x^3 + 93*x^2 + 30*x + 3)*x/(31*x^5 + 130*x^4
      + 155*x^3 + 75*x^2 + 15*x + 1);
  p15b(x) := 1/15*(12*x^5 + 170*x^4 + 516*x^3 + 465*x^2 + 150*x + 15)*x/(2
      *x^6 + 34*x^5 + 129*x^4 + 155*x^3 + 75*x^2 + 15*x + 1);
```

$$p8 := x \mapsto \frac{(x^3 + 3\cdot x^2 + 3\cdot x + 1)\cdot x}{2\cdot x^4 + 8\cdot x^3 + 12\cdot x^2 + 8\cdot x + 1}$$

$$p10 := x \mapsto \frac{(2\cdot x^3 + 6\cdot x^2 + 5\cdot x + 1)\cdot x}{5\cdot x^4 + 20\cdot x^3 + 25\cdot x^2 + 10\cdot x + 1}$$

$$p11 := x \mapsto \frac{(4\cdot x^3 + 9\cdot x^2 + 6\cdot x + 1)\cdot x}{11\cdot x^4 + 33\cdot x^3 + 33\cdot x^2 + 11\cdot x + 1}$$

$$p12 := x \mapsto \frac{(6\cdot x^3 + 13\cdot x^2 + 7\cdot x + 1)\cdot x}{18\cdot x^4 + 52\cdot x^3 + 42\cdot x^2 + 12\cdot x + 1}$$

$$p13 := x \mapsto \frac{(12\cdot x^3 + 18\cdot x^2 + 8\cdot x + 1)\cdot x}{39\cdot x^4 + 78\cdot x^3 + 52\cdot x^2 + 13\cdot x + 1}$$

$$p14 := x \mapsto \frac{\left(\frac{30}{7}\cdot x^4 + \frac{150}{7}\cdot x^3 + 24\cdot x^2 + 9\cdot x + 1\right)\cdot x}{12\cdot x^5 + 75\cdot x^4 + 112\cdot x^3 + 63\cdot x^2 + 14\cdot x + 1}$$

$$p15 := x \mapsto \frac{\left(\frac{31}{3}\cdot x^4 + \frac{104}{3}\cdot x^3 + 31\cdot x^2 + 10\cdot x + 1\right)\cdot x}{31\cdot x^5 + 130\cdot x^4 + 155\cdot x^3 + 75\cdot x^2 + 15\cdot x + 1}$$

$$p15b := x \mapsto \frac{\left(\frac{4}{5}\cdot x^5 + \frac{34}{3}\cdot x^4 + \frac{172}{5}\cdot x^3 + 31\cdot x^2 + 10\cdot x + 1\right)\cdot x}{2\cdot x^6 + 34\cdot x^5 + 129\cdot x^4 + 155\cdot x^3 + 75\cdot x^2 + 15\cdot x + 1} \quad \textbf{(1)}$$

The other example for n=15,

```
> p16(x) := 1/4*(9*x^5 + 90*x^4 + 209*x^3 + 156*x^2 + 44*x + 4)*x/(6*x^6
      + 72*x^5 + 209*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p16b(x) := 1/4*(9*x^5 + 95*x^4 + 211*x^3 + 156*x^2 + 44*x + 4)*x/(6*x^6
      + 76*x^5 + 211*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p16c(x) := (25*x^4 + 54*x^3 + 39*x^2 + 11*x + 1)*x/(80*x^5 + 216*x^4 + 208
      *x^3 + 88*x^2 + 16*x + 1);
  p16d(x) := 1/8*(18*x^5 + 195*x^4 + 424*x^3 + 312*x^2 + 88*x + 8)*x/(6*x
      ^6 + 78*x^5 + 212*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p16e(x) := 1/4*(6*x^5 + 100*x^4 + 213*x^3 + 156*x^2 + 44*x + 4)*x/(4*x^6
      + 80*x^5 + 213*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p16f(x) := 1/2*(3*x^5 + 50*x^4 + 107*x^3 + 78*x^2 + 22*x + 2)*x/(4*x^6
      + 80*x^5 + 214*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p16g(x) := 1/4*(3*x^5 + 100*x^4 + 215*x^3 + 156*x^2 + 44*x + 4)*x/(2*x^6
      + 80*x^5 + 215*x^4 + 208*x^3 + 88*x^2 + 16*x + 1);
  p17(x) := 1/17*(96*x^5 + 780*x^4 + 1316*x^3 + 816*x^2 + 204*x + 17)*x/(16
      *x^6 + 156*x^5 + 329*x^4 + 272*x^3 + 102*x^2 + 17*x + 1);
  p18(x) := 1/9*(156*x^5 + 725*x^4 + 976*x^3 + 522*x^2 + 117*x + 9)*x/(52*x
      ^6 + 290*x^5 + 488*x^4 + 348*x^3 + 117*x^2 + 18*x + 1);
  p19(x) := 1/19*(28*x^6 + 822*x^5 + 2580*x^4 + 2812*x^3 + 1311*x^2 + 266*x
      + 19)*x/(4*x^7 + 137*x^6 + 516*x^5 + 703*x^4 + 437*x^3 + 133*x^2 + 19
      *x + 1);
  p19b(x) := 1/19*(28*x^6 + 894*x^5 + 2665*x^4 + 2828*x^3 + 1311*x^2 + 266
      *x + 19)*x/(4*x^7 + 149*x^6 + 533*x^5 + 707*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19c(x) := 1/19*(28*x^6 + 900*x^5 + 2675*x^4 + 2832*x^3 + 1311*x^2 + 266
      *x + 19)*x/(4*x^7 + 150*x^6 + 535*x^5 + 708*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19d(x) := 1/19*(28*x^6 + 924*x^5 + 2685*x^4 + 2832*x^3 + 1311*x^2 + 266
      *x + 19)*x/(4*x^7 + 154*x^6 + 537*x^5 + 708*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19e(x) := 1/19*(28*x^6 + 930*x^5 + 2700*x^4 + 2836*x^3 + 1311*x^2 + 266
      *x + 19)*x/(4*x^7 + 155*x^6 + 540*x^5 + 709*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19f(x) := 1/19*(21*x^6 + 924*x^5 + 2710*x^4 + 2840*x^3 + 1311*x^2 + 266
      *x + 19)*x/(3*x^7 + 154*x^6 + 542*x^5 + 710*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19g(x) := 1/19*(14*x^6 + 936*x^5 + 2730*x^4 + 2844*x^3 + 1311*x^2 + 266
      *x + 19)*x/(2*x^7 + 156*x^6 + 546*x^5 + 711*x^4 + 437*x^3 + 133*x^2
      + 19*x + 1);
  p19h(x) := 1/19*(966*x^5 + 2775*x^4 + 2856*x^3 + 1311*x^2 + 266*x + 19)*x
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/(161*x^6 + 555*x^5 + 714*x^4 + 437*x^3 + 133*x^2 + 19*x + 1);
p19i(x) := 1/19*(28*x^6 + 936*x^5 + 2705*x^4 + 2836*x^3 + 1311*x^2 + 266
*x + 19)*x/(4*x^7 + 156*x^6 + 541*x^5 + 709*x^4 + 437*x^3 + 133*x^2
+ 19*x + 1);
p19j(x) := 1/19*(28*x^6 + 954*x^5 + 2710*x^4 + 2836*x^3 + 1311*x^2 + 266*x
+ 19)*x/(4*x^7 + 159*x^6 + 542*x^5 + 709*x^4 + 437*x^3 + 133*x^2 + 19
*x + 1);
```

$$p16 := x \mapsto \frac{\left(\frac{9}{4}\cdot x^5 + \frac{45}{2}\cdot x^4 + \frac{209}{4}\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{6\cdot x^6 + 72\cdot x^5 + 209\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16b := x \mapsto \frac{\left(\frac{9}{4}\cdot x^5 + \frac{95}{4}\cdot x^4 + \frac{211}{4}\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{6\cdot x^6 + 76\cdot x^5 + 211\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16c := x \mapsto \frac{\left(25\cdot x^4 + 54\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{80\cdot x^5 + 216\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16d := x \mapsto \frac{\left(\frac{9}{4}\cdot x^5 + \frac{195}{8}\cdot x^4 + 53\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{6\cdot x^6 + 78\cdot x^5 + 212\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16e := x \mapsto \frac{\left(\frac{3}{2}\cdot x^5 + 25\cdot x^4 + \frac{213}{4}\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{4\cdot x^6 + 80\cdot x^5 + 213\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16f := x \mapsto \frac{\left(\frac{3}{2}\cdot x^5 + 25\cdot x^4 + \frac{107}{2}\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{4\cdot x^6 + 80\cdot x^5 + 214\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p16g := x \mapsto \frac{\left(\frac{3}{4}\cdot x^5 + 25\cdot x^4 + \frac{215}{4}\cdot x^3 + 39\cdot x^2 + 11\cdot x + 1\right)\cdot x}{2\cdot x^6 + 80\cdot x^5 + 215\cdot x^4 + 208\cdot x^3 + 88\cdot x^2 + 16\cdot x + 1}$$

$$p17 := x \mapsto \frac{\left(\frac{96}{17}\cdot x^5 + \frac{780}{17}\cdot x^4 + \frac{1316}{17}\cdot x^3 + 48\cdot x^2 + 12\cdot x + 1\right)\cdot x}{16\cdot x^6 + 156\cdot x^5 + 329\cdot x^4 + 272\cdot x^3 + 102\cdot x^2 + 17\cdot x + 1}$$

$$p18 := x \mapsto \frac{\left(\frac{52}{3}\cdot x^5 + \frac{725}{9}\cdot x^4 + \frac{976}{9}\cdot x^3 + 58\cdot x^2 + 13\cdot x + 1\right)\cdot x}{52\cdot x^6 + 290\cdot x^5 + 488\cdot x^4 + 348\cdot x^3 + 117\cdot x^2 + 18\cdot x + 1}$$

$$p19 := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{822}{19}\cdot x^5 + \frac{2580}{19}\cdot x^4 + 148\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 137\cdot x^6 + 516\cdot x^5 + 703\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19b := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{894}{19}\cdot x^5 + \frac{2665}{19}\cdot x^4 + \frac{2828}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 149\cdot x^6 + 533\cdot x^5 + 707\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19c := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{900}{19}\cdot x^5 + \frac{2675}{19}\cdot x^4 + \frac{2832}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 150\cdot x^6 + 535\cdot x^5 + 708\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19d := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{924}{19}\cdot x^5 + \frac{2685}{19}\cdot x^4 + \frac{2832}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 154\cdot x^6 + 537\cdot x^5 + 708\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19e := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{930}{19}\cdot x^5 + \frac{2700}{19}\cdot x^4 + \frac{2836}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 155\cdot x^6 + 540\cdot x^5 + 709\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19f := x \mapsto \frac{\left(\frac{21}{19}\cdot x^6 + \frac{924}{19}\cdot x^5 + \frac{2710}{19}\cdot x^4 + \frac{2840}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{3\cdot x^7 + 154\cdot x^6 + 542\cdot x^5 + 710\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19g := x \mapsto \frac{\left(\frac{14}{19}\cdot x^6 + \frac{936}{19}\cdot x^5 + \frac{2730}{19}\cdot x^4 + \frac{2844}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{2\cdot x^7 + 156\cdot x^6 + 546\cdot x^5 + 711\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19h := x \mapsto \frac{\left(\frac{966}{19}\cdot x^5 + \frac{2775}{19}\cdot x^4 + \frac{2856}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{161\cdot x^6 + 555\cdot x^5 + 714\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19i := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{936}{19}\cdot x^5 + \frac{2705}{19}\cdot x^4 + \frac{2836}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 156\cdot x^6 + 541\cdot x^5 + 709\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1}$$

$$p19j := x \mapsto \frac{\left(\frac{28}{19}\cdot x^6 + \frac{954}{19}\cdot x^5 + \frac{2710}{19}\cdot x^4 + \frac{2836}{19}\cdot x^3 + 69\cdot x^2 + 14\cdot x + 1\right)\cdot x}{4\cdot x^7 + 159\cdot x^6 + 542\cdot x^5 + 709\cdot x^4 + 437\cdot x^3 + 133\cdot x^2 + 19\cdot x + 1} \quad \textbf{(2)}$$

Remark that for p16b, one can also conclude from p16 and p16d.

```
> q(x) := p8(x);
  simplify(p13(x) - q(x));
  q(x) := p10(x);
  simplify(p13(x) - q(x));
  q(x) := p11(x);
  simplify(p13(x) - q(x));
  q(x) := p12(x);
  simplify(p13(x) - q(x));
```

$$q := x \mapsto p8(x)$$

$$-\frac{x^4\left(15\,x^4 + 63\,x^3 + 99\,x^2 + 64\,x + 14\right)}{\left(39\,x^4 + 78\,x^3 + 52\,x^2 + 13\,x + 1\right)\left(2\,x^4 + 8\,x^3 + 12\,x^2 + 8\,x + 1\right)}$$

$$q := x \mapsto p10(x)$$

$$-\frac{x^4\left(18x^4+60x^3+67x^2+32x+6\right)}{\left(39x^4+78x^3+52x^2+13x+1\right)\left(5x^4+20x^3+25x^2+10x+1\right)}$$

$$q := x \mapsto p11(x)$$

$$-\frac{x^4\left(24x^4+69x^3+66x^2+26x+4\right)}{\left(39x^4+78x^3+52x^2+13x+1\right)\left(11x^4+33x^3+33x^2+11x+1\right)}$$

$$q := x \mapsto p12(x)$$

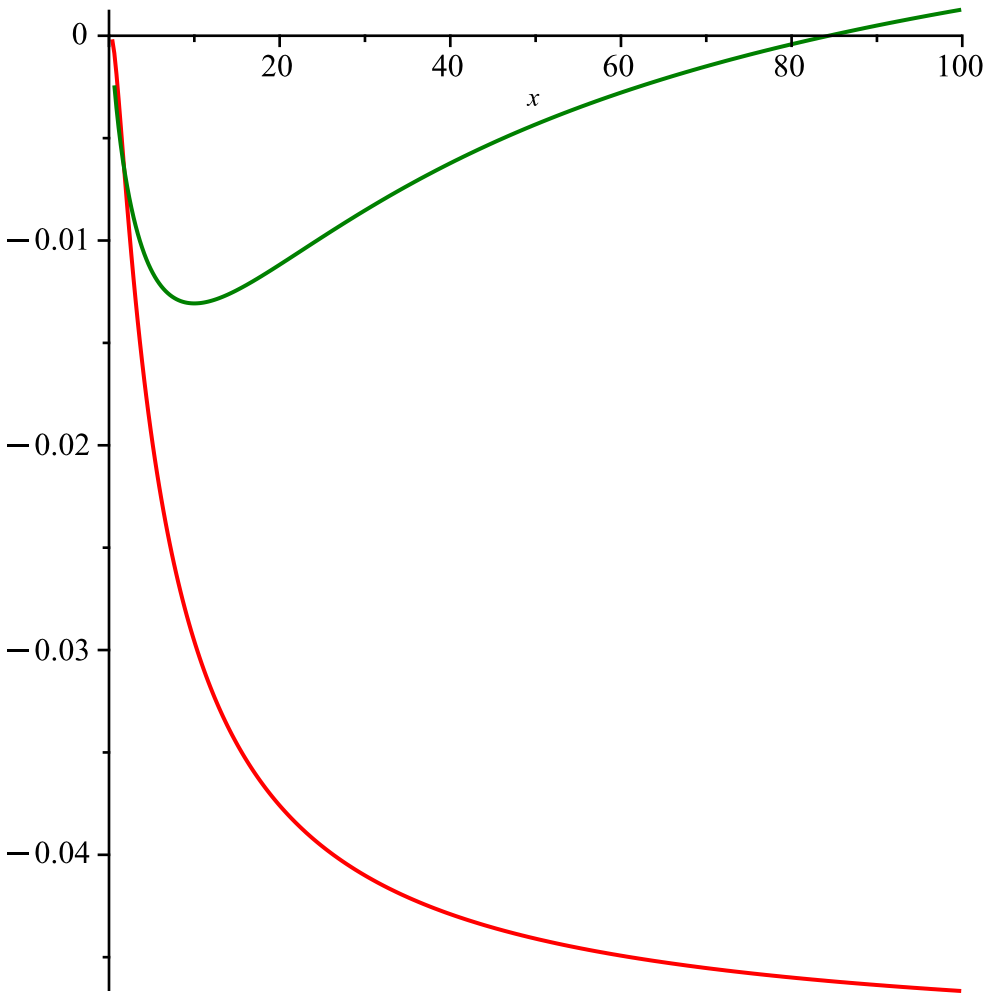
$$-\frac{x^4\left(18x^4+27x^3+15x^2+5x+1\right)}{\left(39x^4+78x^3+52x^2+13x+1\right)\left(18x^4+52x^3+42x^2+12x+1\right)} \quad \textbf{(3)}$$

```
> q(x) := p14(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..100, color = ["Red", "Green"]);
```

$$q := x \mapsto p14(x)$$

$$-\frac{x^4\left(162x^5+378x^4+282x^3+63x^2-8x-4\right)}{7\left(39x^4+78x^3+52x^2+13x+1\right)\left(12x^5+75x^4+112x^3+63x^2+14x+1\right)}$$

$$\left(x^4\left(72x^8-5742x^7-27116x^6-65898x^5-89843x^4-67350x^3-27147x^2-5538x-456\right)\right) \Big/ \left(133\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(12x^5+75x^4+112x^3+63x^2+14x+1\right)\right)$$

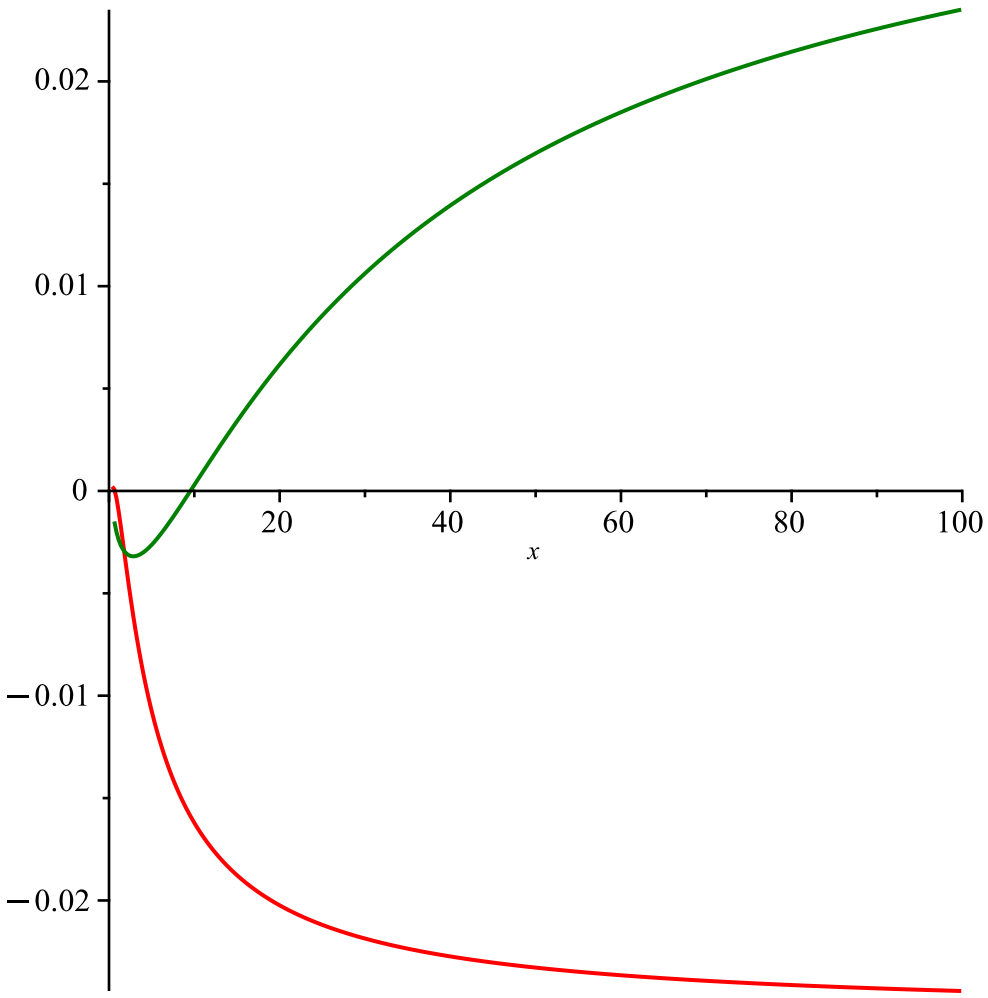



```
> q(x) := p15(x);
  simplify(p13(x) − q(x));
  simplify(p19(x) − q(x));
  plot([p13(x) − q(x), p19(x) − q(x)], x = 0..100, color = ["Red", "Green"]);
```

$$q := x \mapsto p15(x)$$

$$-\frac{x^4\left(93x^5+120x^4+7x^3-48x^2-24x-4\right)}{3\left(39x^4+78x^3+52x^2+13x+1\right)\left(31x^5+130x^4+155x^3+75x^2+15x+1\right)}$$

$$\left(x^4\left(248x^8-1231x^7-8164x^6-21796x^5-32098x^4-24797x^3-9912x^2-1950x-152\right)\right)\Big/$$
$$\left(57\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(31x^5+130x^4+155x^3\right.\right.$$
$$\left.\left.+75x^2+15x+1\right)\right)$$




```
>
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```
> simplify(p19(x) − p15b(x));
  q(x) := p16(x);
  simplify(p19(x) − q(x));
  q(x) := p16b(x);
  simplify(p19(x) − q(x));
  q(x) := p16d(x);
  simplify(p19(x) − q(x));
  q(x) := p16e(x);
  simplify(p19(x) − q(x));
  q(x) := p16f(x);
  simplify(p19(x) − q(x));
```

$$-\left(x^4\left(72x^9+5216x^8+48574x^7+149622x^6+242585x^5+237136x^4+140135x^3+48230x^2+8876x+684\right)\right)\Big/\left(285\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(2x^6+34x^5+129x^4+155x^3+75x^2+15x+1\right)\right)$$

$$q := x \mapsto p16(x)$$

$$-\left(x^4\left(12x^9+2475x^8+16326x^7+37440x^6+43345x^5+33406x^4+17962x^3+6164x^2+1175x+95\right)\right)\Big/\left(76\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(6x^6+72x^5+209x^4+208x^3+88x^2+16x+1\right)\right)$$

$$q := x \mapsto p16b(x)$$

$$-\left(x^4\left(12x^9+2407x^8+16117x^7+43810x^6+64106x^5+58163x^4+32459x^3+10591x^2+1840x+133\right)\right)\Big/\left(76\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(6x^6+76x^5+211x^4+208x^3+88x^2+16x+1\right)\right)$$

$$q := x \mapsto p16d(x)$$

$$-\left(x^4\left(24x^9+4746x^8+32025x^7+93990x^6+148973x^5+141083x^4+79415x^3+25609x^2+4345x+304\right)\right)\Big/\left(152\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(6x^6+78x^5+212x^4+208x^3+88x^2+16x+1\right)\right)$$

$$q := x \mapsto p16e(x)$$

$$-\left(x^4\left(8x^9+1106x^8+7136x^7+32605x^6+70446x^5+77467x^4+46025x^3+14961x^2+2505x+171\right)\right)\Big/\left(76\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^6+80x^5+213x^4+208x^3+88x^2+16x+1\right)\right)$$

$$q := x \mapsto p16f(x)$$

$$-\left(x^4\left(4x^9+553x^8+3550x^7+15960x^6+34965x^5+39788x^4+24542x^3+8212x^2+1395x+95\right)\right)\Big/\left(38\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^6+80x^5+214x^4+208x^3+88x^2+16x+1\right)\right) \quad \textbf{(4)}$$

> $q(x) := p16g(x);$
> $simplify(p19(x) - q(x));$
> $simplify(p13(x) - q(x));$

$$q := x \mapsto p16g(x)$$

$$-\left(x^4\left(4x^9-127x^8-1708x^7+13660x^6+54993x^5+76232x^4+51212x^3+17830x^2+3075x+209\right)\right)\Big/\left(76\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(2x^6+80x^5+215x^4+208x^3+88x^2+16x+1\right)\right)$$

$$-\frac{x^4\left(21x^6+150x^5+197x^4+61x^3-33x^2-25x-5\right)}{4\left(39x^4+78x^3+52x^2+13x+1\right)\left(2x^6+80x^5+215x^4+208x^3+88x^2+16x+1\right)} \quad \textbf{(5)}$$

$p19(x) - q(x)$<0 if x \le 1
$p13(x) - q(x) < 0$ **if** $x > 1$

$$\frac{\left(\frac{28}{19}x^6+\frac{822}{19}x^5+\frac{2580}{19}x^4+148x^3+69x^2+14x+1\right)x}{4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1}$$
$$-\frac{\left(\frac{3}{4}x^5+25x^4+\frac{215}{4}x^3+39x^2+11x+1\right)x}{2x^6+80x^5+215x^4+208x^3+88x^2+16x+1}$$

Error, unable to parse

$$p13(x)-q(x)<0 \text{ if } x>1$$

>

> $q(x):=p16c(x);$
$simplify(p19(x)-q(x));$
$simplify(p13(x)-q(x));$

$$q:=x\mapsto p16c(x)$$

$$\left(x^4\left(340x^8+2629x^7+1150x^6-10014x^5-18222x^4-13335x^3-4809x^2-840x-57\right)\right)/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(80x^5+216x^4+208x^3+88x^2+16x+1\right)\right)$$

$$\frac{-15x^9-24x^8-9x^7+4x^6+4x^5+x^4}{3120x^9+14664x^8+29120x^7+31928x^6+21192x^5+8783x^4+2262x^3+348x^2+29x+1} \quad (6)$$

$p19(x)-q(x)$<0 if x \le 1
$p13(x)-q(x)<0 \text{ if } x>1$

$$\frac{\left(\frac{28}{19}x^6+\frac{822}{19}x^5+\frac{2580}{19}x^4+148x^3+69x^2+14x+1\right)x}{4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1}$$
$$-\frac{\left(25x^4+54x^3+39x^2+11x+1\right)x}{80x^5+216x^4+208x^3+88x^2+16x+1}$$

Error, unable to parse

$$p13(x)-q(x)<0 \text{ if } x>1$$

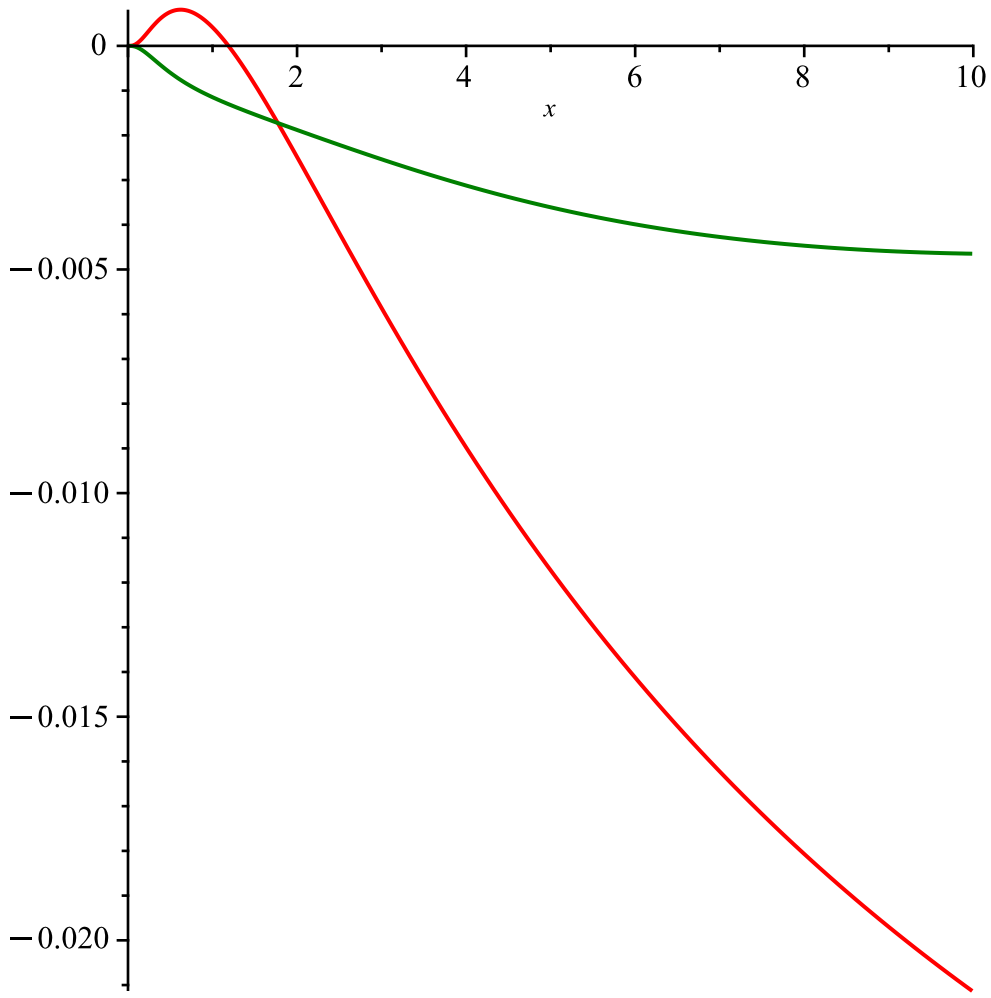
>

> $q(x):=p17(x);$
$simplify(p13(x)-q(x));$
$simplify(p19(x)-q(x));$
$plot([p13(x)-q(x),p19(x)-q(x)],x=0..10,color=["Red","Green"]);$

$$q:=x\mapsto p17(x)$$

$$-\frac{2x^4\left(240x^6+594x^5+64x^4-685x^3-582x^2-185x-22\right)}{17\left(39x^4+78x^3+52x^2+13x+1\right)\left(16x^6+156x^5+329x^4+272x^3+102x^2+17x+1\right)}$$

$$\left(2x^4\left(160x^9-5664x^8-16616x^7-41507x^6-81864x^5-94556x^4-59289x^3-19944x^2-3378x-228\right)\right)/\left(323\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(16x^6+156x^5+329x^4+272x^3+102x^2+17x+1\right)\right)$$



$p19(x) - q(x) < 0$ if x \le 10

$p13(x) - q(x) < 0$ **if** $x > 10$

$$\frac{\left(\frac{28}{19}x^6 + \frac{822}{19}x^5 + \frac{2580}{19}x^4 + 148x^3 + 69x^2 + 14x + 1\right)x}{4x^7 + 137x^6 + 516x^5 + 703x^4 + 437x^3 + 133x^2 + 19x + 1}$$

$$-\frac{\left(\frac{96}{17}x^5 + \frac{780}{17}x^4 + \frac{1316}{17}x^3 + 48x^2 + 12x + 1\right)x}{16x^6 + 156x^5 + 329x^4 + 272x^3 + 102x^2 + 17x + 1}$$

Error, unable to parse

$p13(x) - q(x) < 0$ **if** $x > 1$

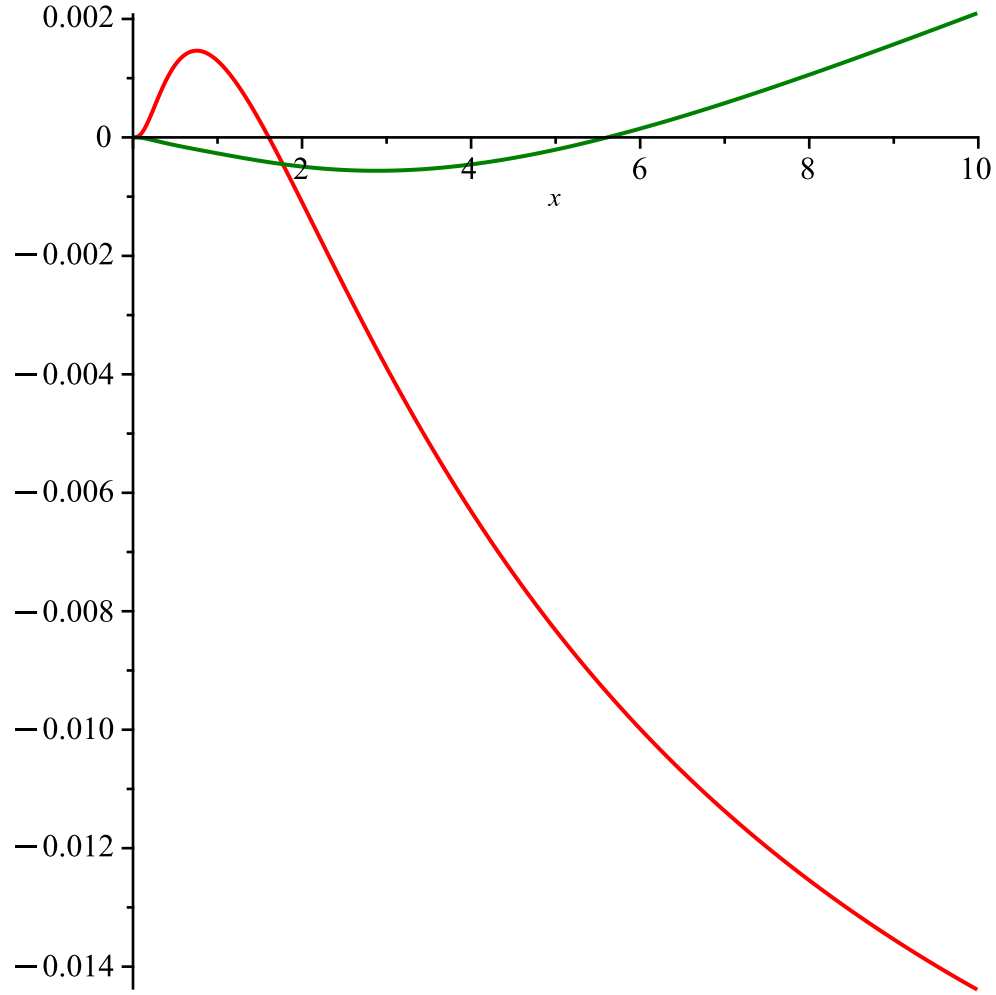
```
> q(x) := p18(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p18(x)$$

$$-\frac{x^4\left(468x^6 + 699x^5 - 702x^4 - 1774x^3 - 1146x^2 - 312x - 32\right)}{9\left(39x^4 + 78x^3 + 52x^2 + 13x + 1\right)\left(52x^6 + 290x^5 + 488x^4 + 348x^3 + 117x^2 + 18x + 1\right)}$$

$$\left(x^4\left(1248x^9 - 3392x^8 - 14939x^7 - 24056x^6 - 27239x^5 - 26078x^4 - 16564x^3 - 5904x^2\right.\right.$$

$$-1062\,x-76\big)\big)\Big/\big(171\,\big(4\,x^7+137\,x^6+516\,x^5+703\,x^4+437\,x^3+133\,x^2+19\,x+1\big)\,\big(52\,x^6+290\,x^5+488\,x^4+348\,x^3+117\,x^2+18\,x+1\big)\big)$$

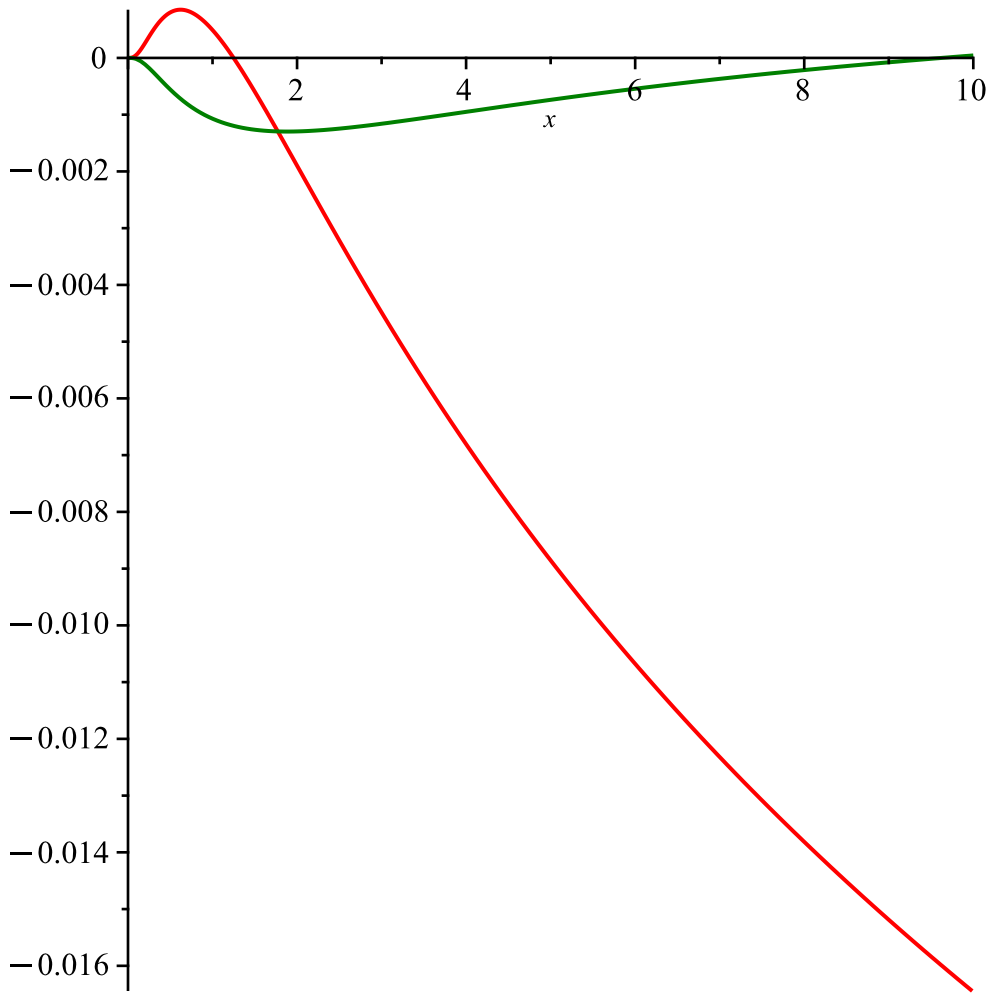



```
> q(x) := p19b(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19b(x)$$

$$-\big(x^4\,\big(180\,x^7+1710\,x^6+2033\,x^5-1192\,x^4-3334\,x^3-2142\,x^2-585\,x-60\big)\big)\Big/\big(19\,\big(39\,x^4+78\,x^3+52\,x^2+13\,x+1\big)\,\big(4\,x^7+149\,x^6+533\,x^5+707\,x^4+437\,x^3+133\,x^2+19\,x+1\big)\big)$$

$$\big(x^4\,\big(48\,x^9+136\,x^8-3815\,x^7-15776\,x^6-25619\,x^5-21242\,x^4-9671\,x^3-2428\,x^2-313\,x-16\big)\big)\Big/\big(19\,\big(4\,x^7+137\,x^6+516\,x^5+703\,x^4+437\,x^3+133\,x^2+19\,x+1\big)\,\big(4\,x^7+149\,x^6+533\,x^5+707\,x^4+437\,x^3+133\,x^2+19\,x+1\big)\big)$$

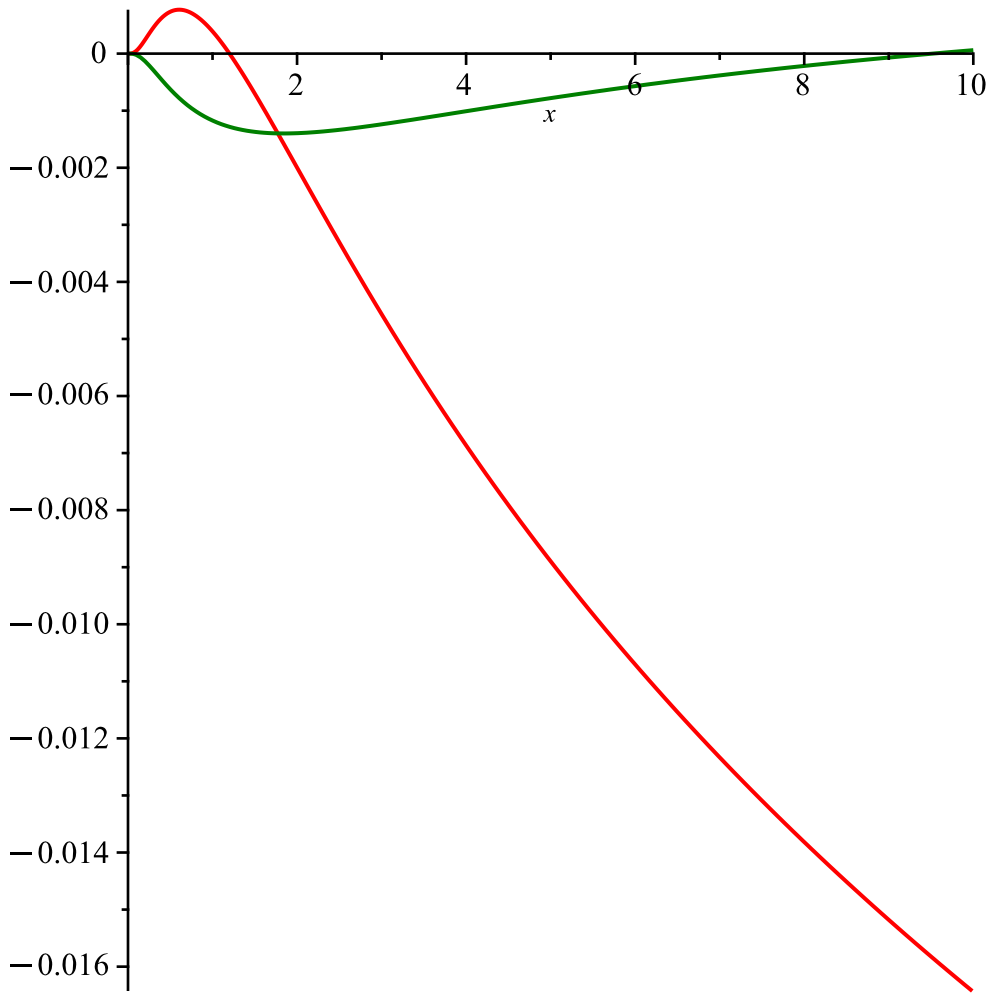



```
> q(x) := p19c(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19c(x)$$

$$-\left(x^4\left(180x^7+1716x^6+2093x^5-1008x^4-3089x^3-1988x^2-542x-56\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(4x^7+150x^6+535x^5+708x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(52x^9+152x^8-4045x^7-16908x^6-27820x^5-23522x^4-11001x^3-2852x^2-380x-20\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^7+150x^6+535x^5+708x^4+437x^3+133x^2+19x+1\right)\right)$$

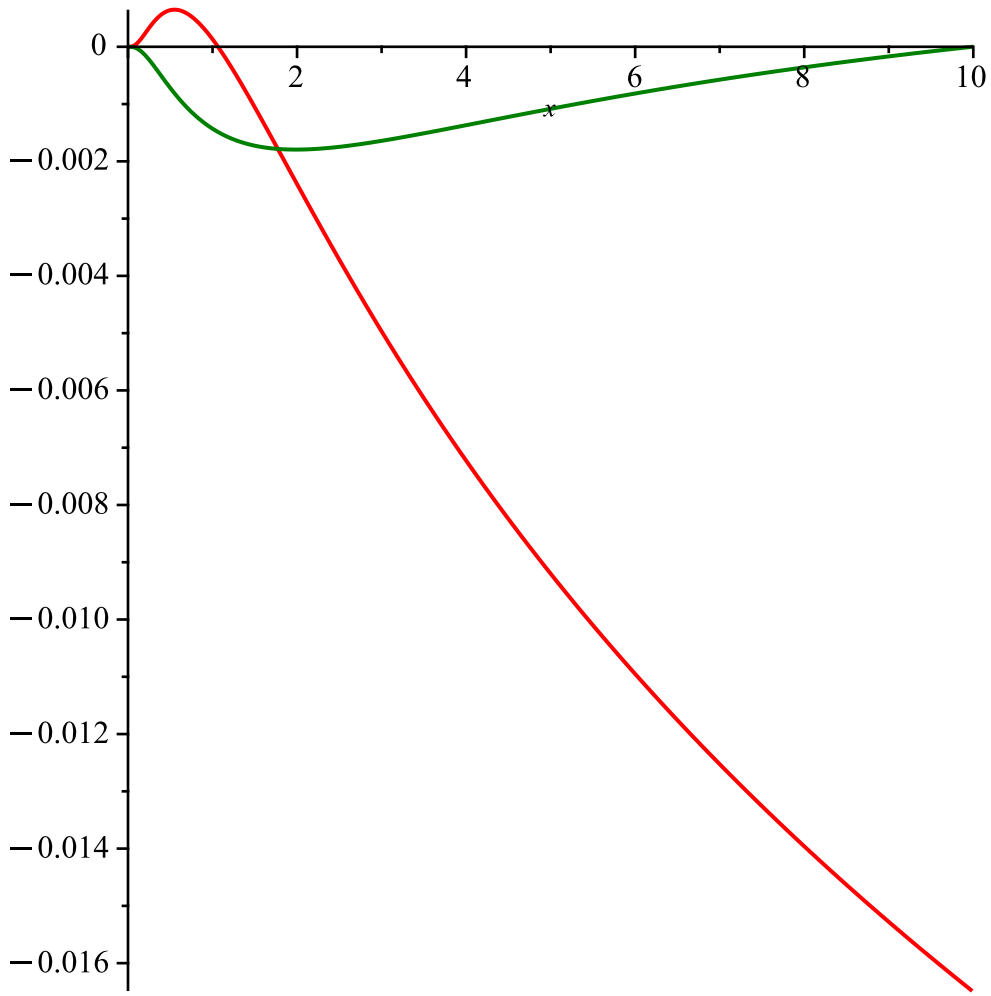



```
> q(x) := p19d(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19d(x)$$

$$-\left(x^4\left(180x^7+1740x^6+2531x^5-272x^4-2637x^3-1872x^2-532x-56\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(4x^7+154x^6+537x^5+708x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(68x^9+168x^8-5835x^7-22532x^6-34470x^5-27398x^4-12179x^3-3028x^2-390x-20\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^7+154x^6+537x^5+708x^4+437x^3+133x^2+19x+1\right)\right)$$

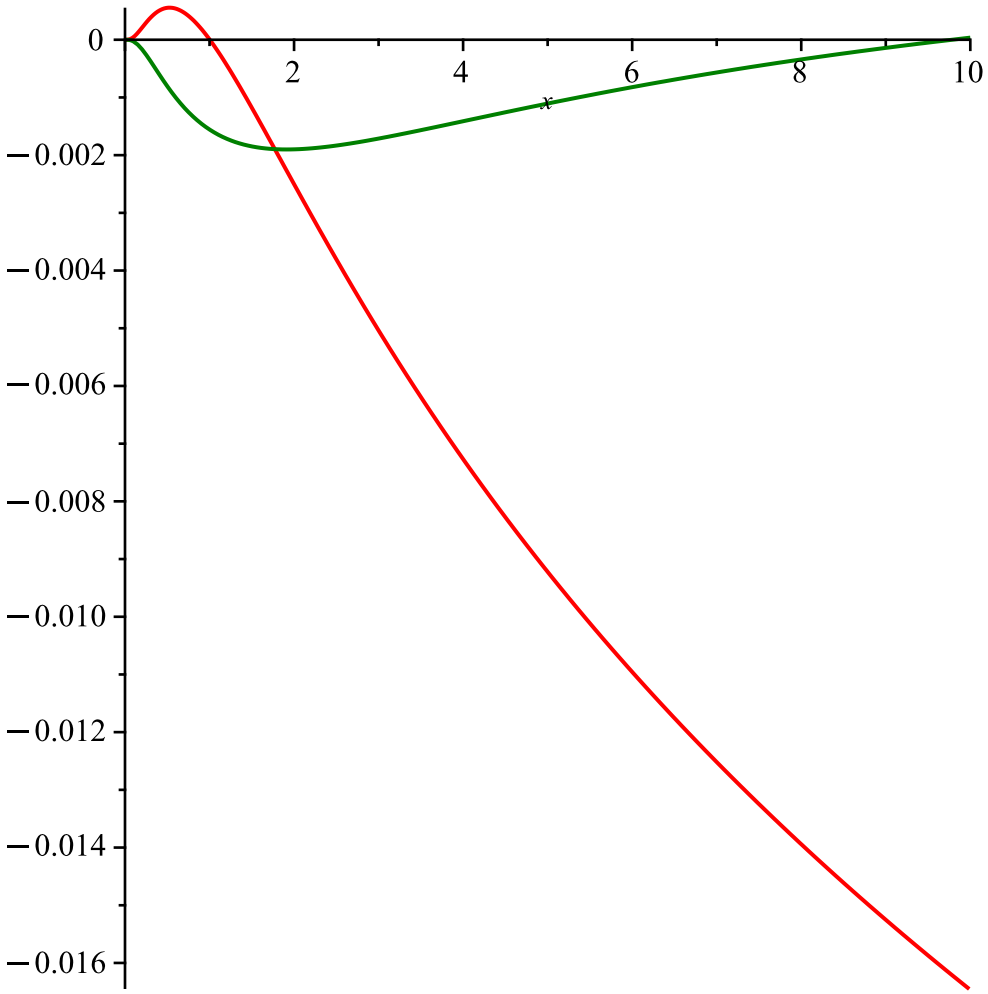



```
> q(x) := p19e(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19e(x)$$

$$-\left(2x^4\left(90x^7+873x^6+1279x^5-20x^4-1142x^3-836x^2-242x-26\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(4x^7+155x^6+540x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(6x^4\left(12x^9+32x^8-988x^7-3944x^6-6229x^5-5092x^4-2318x^3-588x^2-77x-4\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^7+155x^6+540x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$

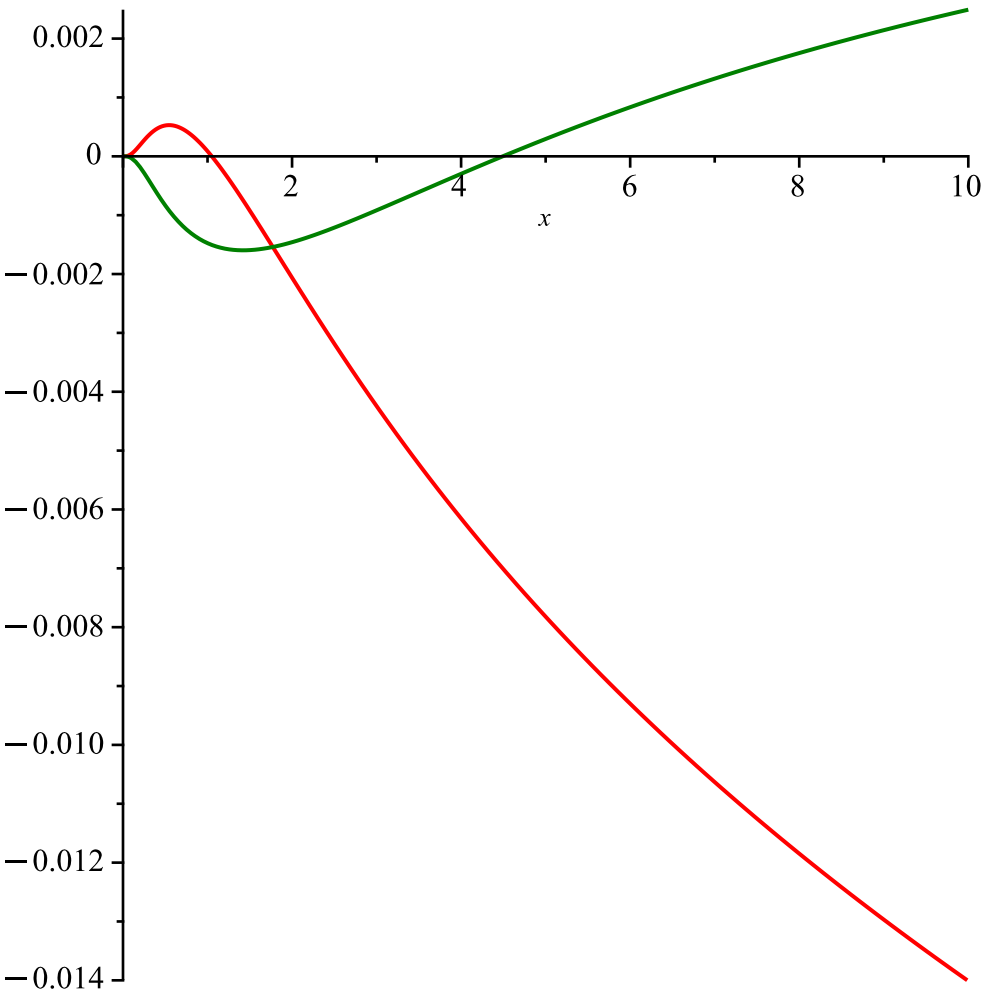



```
> q(x) := p19f(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19f(x)$$

$$-\left(x^4\left(135x^7+1536x^6+2154x^5-248x^4-2164x^3-1530x^2-441x-48\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(3x^7+154x^6+542x^5+710x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(205x^9+1240x^8-3017x^7-20236x^6-36288x^5-31654x^4-15041x^3-3940x^2-529x-28\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(3x^7+154x^6+542x^5+710x^4+437x^3+133x^2+19x+1\right)\right)$$

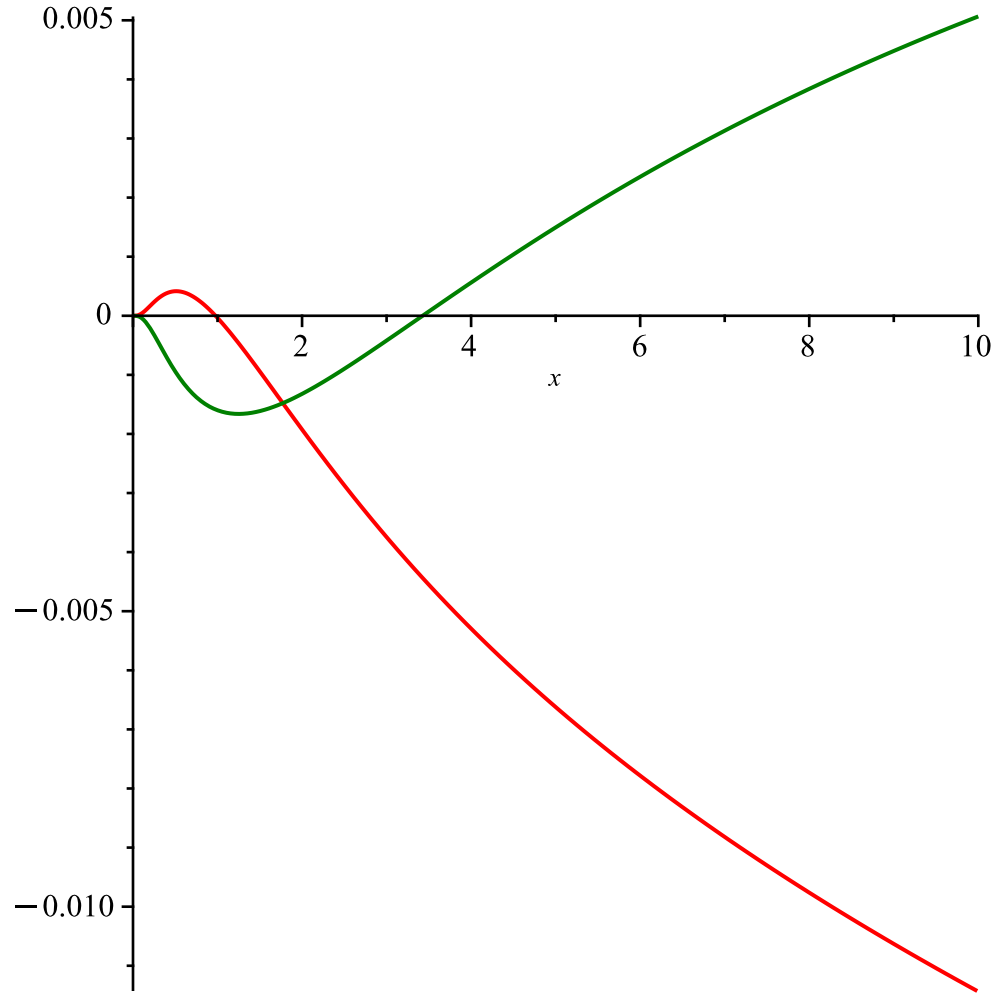



```
> q(x) := p19g(x);
  simplify(p13(x) − q(x));
  simplify(p19(x) − q(x));
  plot([p13(x) − q(x), p19(x) − q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19g(x)$$

$$-\left(x^4\left(90x^7+1344x^6+2062x^5+120x^4-1651x^3-1278x^2-388x-44\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(2x^7+156x^6+546x^5+711x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(350x^9+2304x^8-1380x^7-21026x^6-40541x^5-36100x^4-17257x^3-4522x^2-606x-32\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(2x^7+156x^6+546x^5+711x^4+437x^3+133x^2+19x+1\right)\right)$$

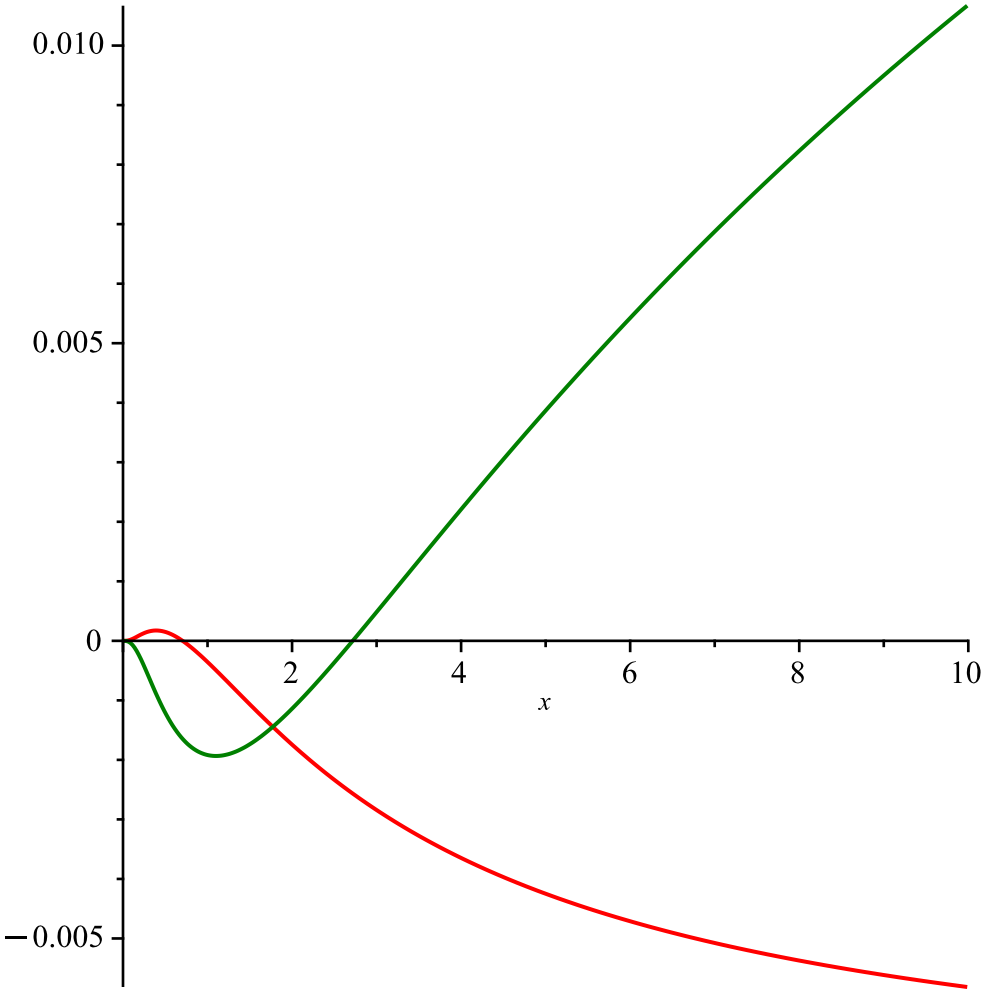



```
> q(x) := p19h(x);
  simplify(p13(x) − q(x));
  simplify(p19(x) − q(x));
  plot([p13(x) − q(x), p19(x) − q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19h(x)$$

$$-\left(x^4\left(966x^6+1971x^5+992x^4-488x^3-666x^2-244x-32\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(161x^6+555x^5+714x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(644x^9+4440x^8+1527x^7-23738x^6-50545x^5-46398x^4-22620x^3-6034x^2-822x-44\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(161x^6+555x^5+714x^4+437x^3+133x^2+19x+1\right)\right)$$

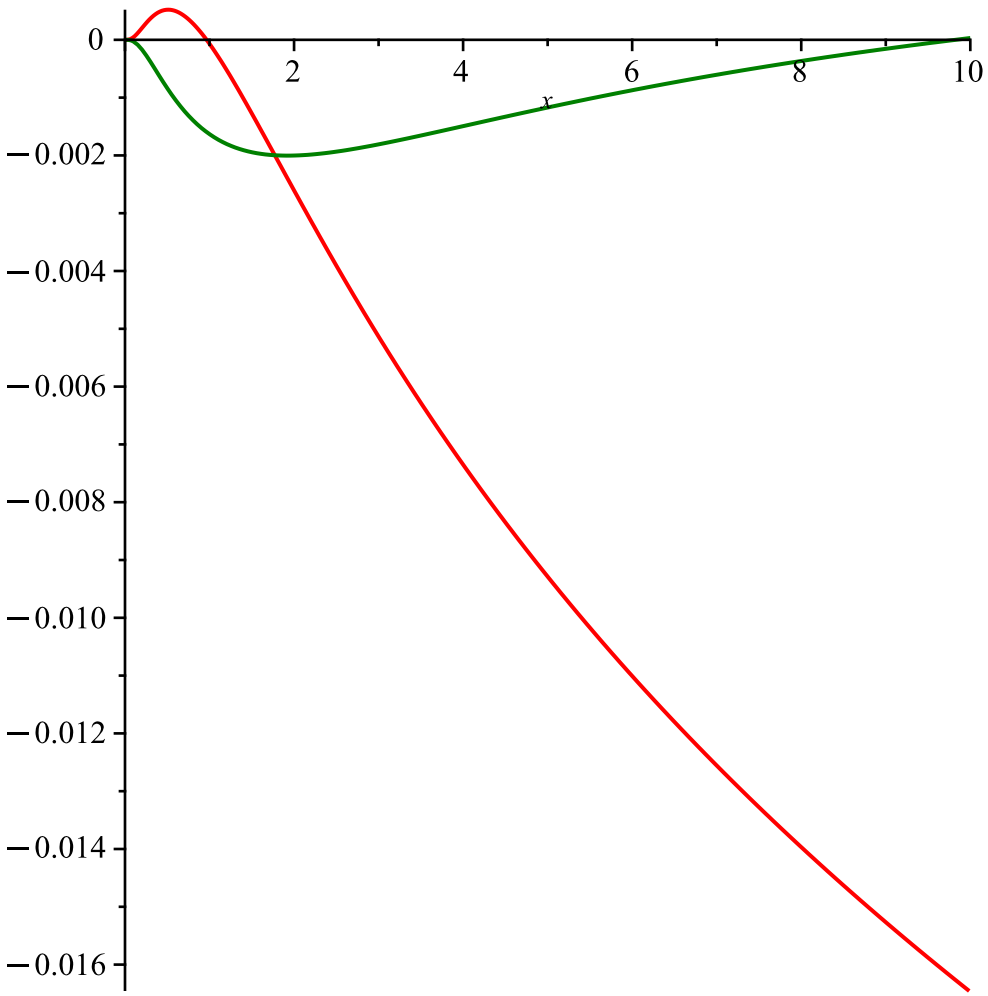



```
> q(x) := p19i(x);
  simplify(p13(x) − q(x));
  simplify(p19(x) − q(x));
  plot([p13(x) − q(x), p19(x) − q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19i(x)$$

$$-\left(x^4\left(180x^7+1752x^6+2651x^5+168x^4-2117x^3-1620x^2-479x-52\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(4x^7+156x^6+541x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(x^4\left(76x^9+200x^8-6307x^7-25070x^6-39388x^5-31958x^4-14402x^3-3610x^2-467x-24\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^7+156x^6+541x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$

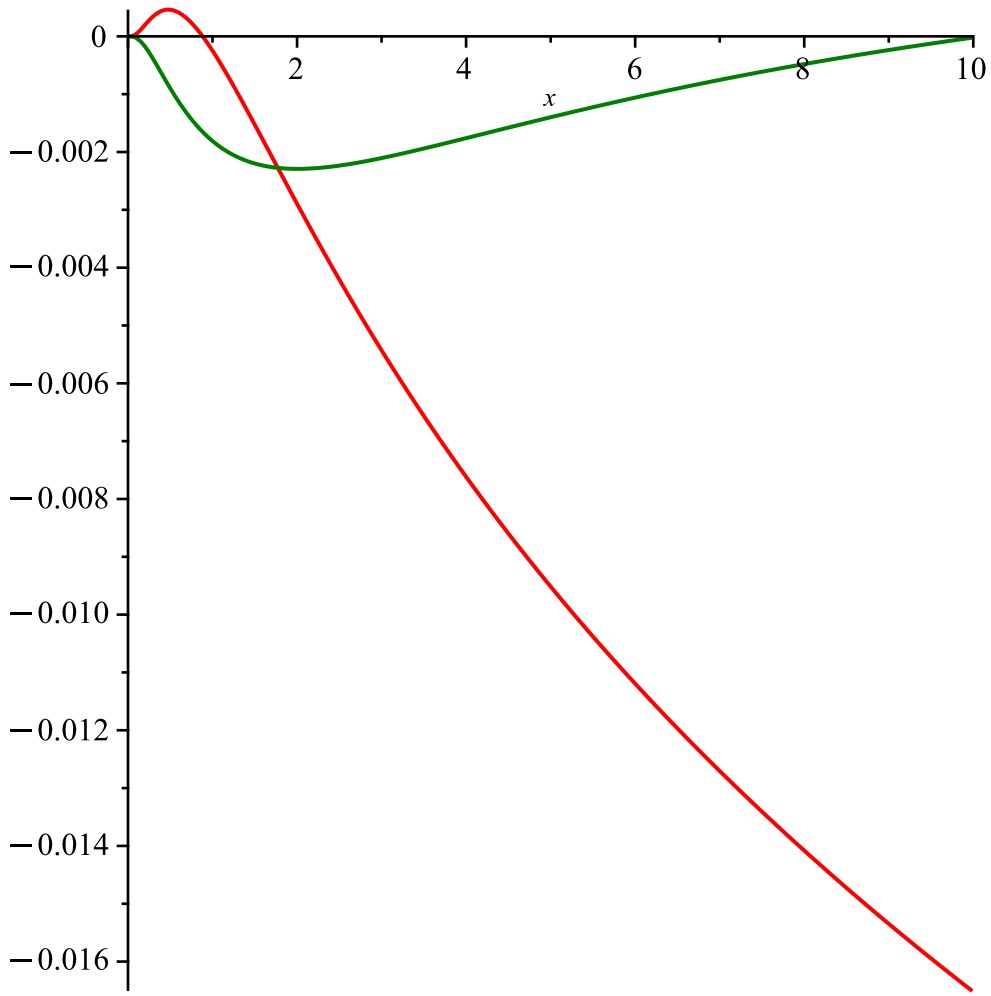



```
> q(x) := p19j(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p19j(x)$$

$$-\left(2x^4\left(90x^7+885x^6+1498x^5+348x^4-916x^3-778x^2-237x-26\right)\right)\Big/\left(19\left(39x^4+78x^3+52x^2+13x+1\right)\left(4x^7+159x^6+542x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$

$$\left(2x^4\left(44x^9+104x^8-3859x^7-14644x^6-22012x^5-17214x^4-7543x^3-1852x^2-236x-12\right)\right)\Big/\left(19\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(4x^7+159x^6+542x^5+709x^4+437x^3+133x^2+19x+1\right)\right)$$




```
> p20(x) := 1/5*(28*x^6 + 492*x^5 + 1110*x^4 + 988*x^3 + 405*x^2 + 75*x + 5)
  *x/(16*x^7 + 328*x^6 + 888*x^5 + 988*x^4 + 540*x^3 + 150*x^2 + 20*x
  + 1);
  p22(x) := 1/2*(8*x^7 + 168*x^6 + 672*x^5 + 975*x^4 + 652*x^3 + 216*x^2
  + 34*x + 2)*x/(11*x^8 + 264*x^7 + 1232*x^6 + 2145*x^5 + 1793*x^4 + 792
  *x^3 + 187*x^2 + 22*x + 1);
```

$$p20 := x \mapsto \frac{\left(\frac{28}{5}\cdot x^6 + \frac{492}{5}\cdot x^5 + 222\cdot x^4 + \frac{988}{5}\cdot x^3 + 81\cdot x^2 + 15\cdot x + 1\right)\cdot x}{16\cdot x^7 + 328\cdot x^6 + 888\cdot x^5 + 988\cdot x^4 + 540\cdot x^3 + 150\cdot x^2 + 20\cdot x + 1}$$

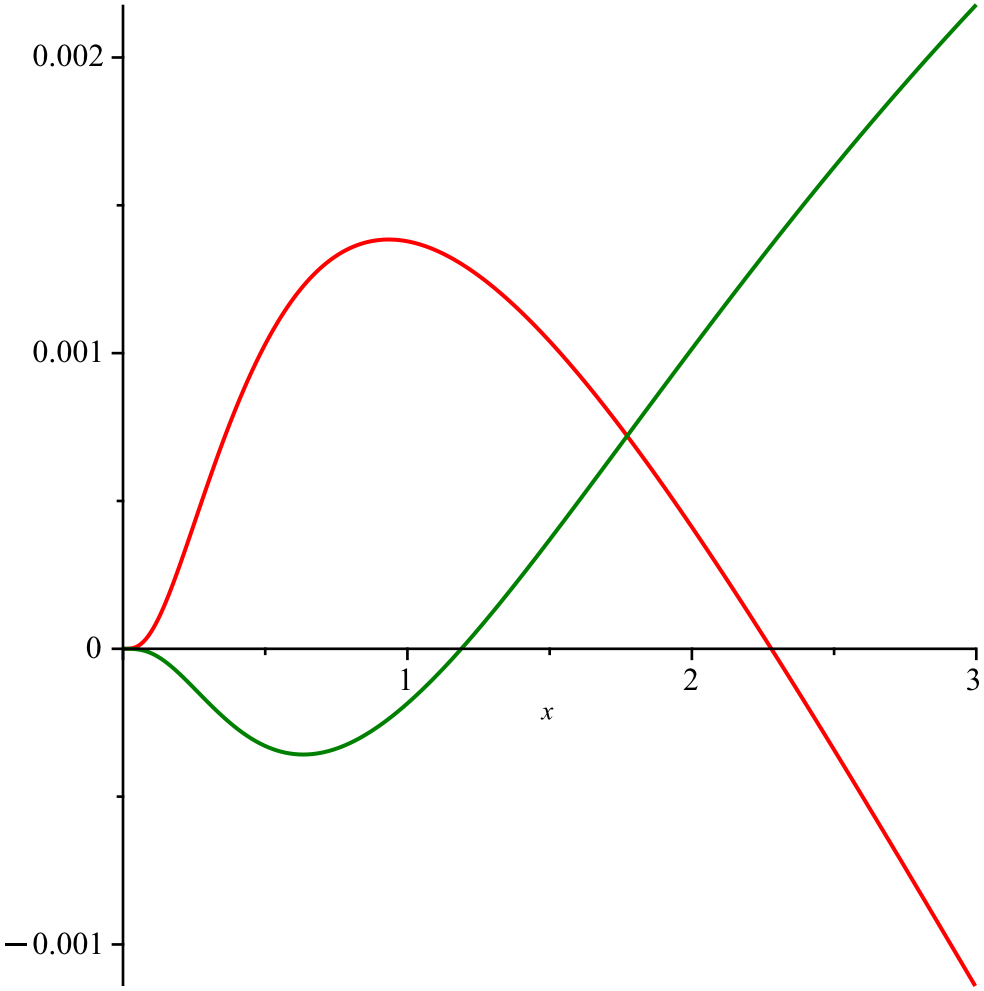
$$p22 := x \mapsto \frac{\left(4\cdot x^7 + 84\cdot x^6 + 336\cdot x^5 + \frac{975}{2}\cdot x^4 + 326\cdot x^3 + 108\cdot x^2 + 17\cdot x + 1\right)\cdot x}{11\cdot x^8 + 264\cdot x^7 + 1232\cdot x^6 + 2145\cdot x^5 + 1793\cdot x^4 + 792\cdot x^3 + 187\cdot x^2 + 22\cdot x + 1} \quad \textbf{(7)}$$

```
> q(x) := p20(x);
  simplify(p13(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..3, color = ["Red", "Green"]);
```

$$q := x \mapsto p20(x)$$

$$-\left(x^4\left(132\,x^7 + 252\,x^6 - 318\,x^5 - 1340\,x^4 - 1477\,x^3 - 747\,x^2 - 181\,x - 17\right)\right) \Big/ \left(5\left(39\,x^4 + 78\,x^3 + 52\,x^2 + 13\,x + 1\right)\left(16\,x^7 + 328\,x^6 + 888\,x^5 + 988\,x^4 + 540\,x^3 + 150\,x^2 + 20\,x + 1\right)\right)$$

$$\left(x^4\left(112x^{10}+1404x^9+39252x^8+82178x^7+18928x^6-86169x^5-97483x^4-46098x^3-11034x^2-1293x-57\right)\right)\Big/\left(95\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(16x^7+328x^6+888x^5+988x^4+540x^3+150x^2+20x+1\right)\right)$$



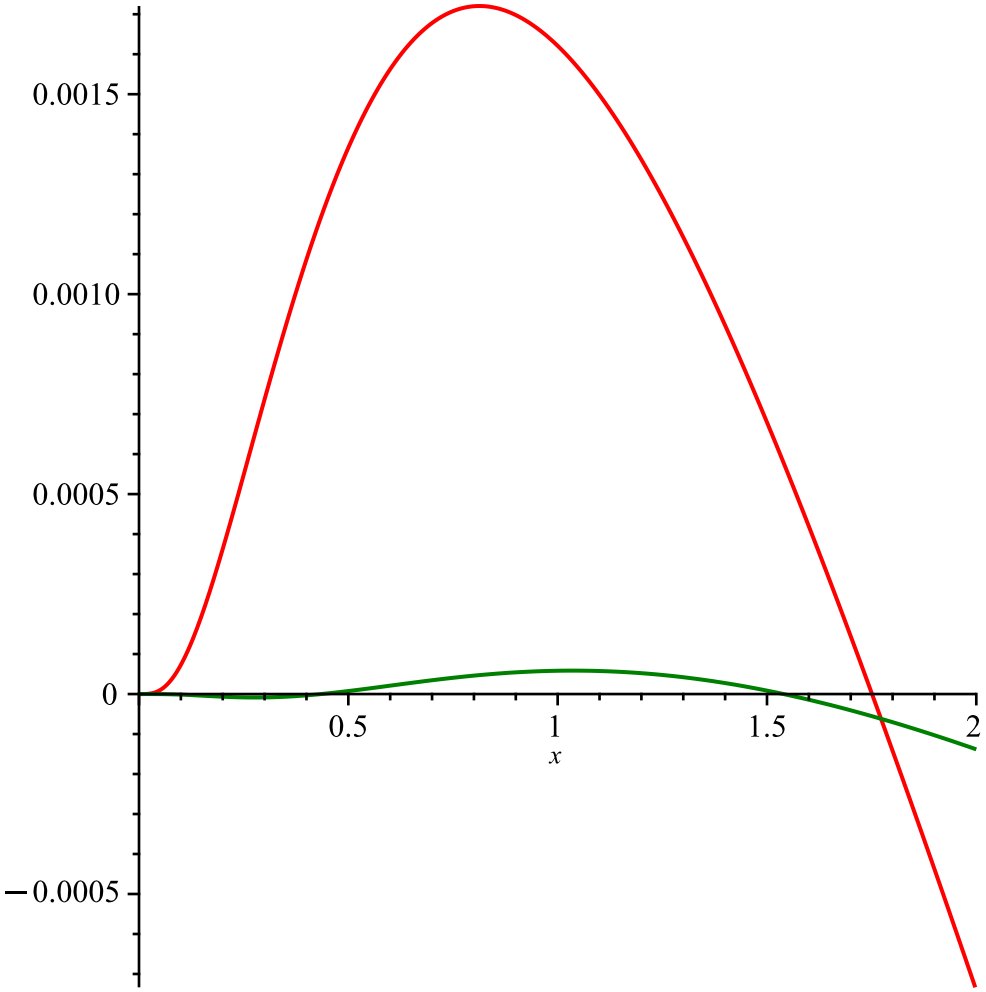

```
> q(x) := p22(x);
  simplify(p20(x) - q(x));
  simplify(p19(x) - q(x));
  plot([p13(x) - q(x), p19(x) - q(x)], x = 0..2, color = ["Red", "Green"]);
```

$$q := x \mapsto p22(x)$$

$$\left(-24x^{15}-952x^{14}-11612x^{13}-25296x^{12}-9658x^{11}+25646x^{10}+36870x^9+22158x^8+7266x^7+1356x^6+137x^5+6x^4\right)\Big/\left(1760\left(x+\frac{1}{2}\right)\left(x^8+24x^7+112x^6+195x^5+163x^4+72x^3+17x^2+2x+\frac{1}{11}\right)\left(x^6+20x^5+\frac{91}{2}x^4+39x^3+\frac{57}{4}x^2+\frac{9}{4}x+\frac{1}{8}\right)\right)$$

$$\left(x^5\left(8x^{10}-724x^9-7040x^8-7612x^7+11321x^6+23576x^5+11895x^4-593x^3-2265x^2-679x-65\right)\right)\Big/\left(38\left(4x^7+137x^6+516x^5+703x^4+437x^3+133x^2+19x+1\right)\left(11x^8\right.\right.$$

$$+ 264\,x^7 + 1232\,x^6 + 2145\,x^5 + 1793\,x^4 + 792\,x^3 + 187\,x^2 + 22\,x + 1)\,)$$



>

> $simplify(p13(x) - p19(x));$

$$-(2\,x^4\,(90\,x^7 + 819\,x^6 + 541\,x^5 - 1820\,x^4 - 2879\,x^3 - 1610\,x^2 - 401\,x - 38))\,/\,(19\,(39\,x^4 + 78\,x^3 + 52\,x^2 + 13\,x + 1)\,(4\,x^7 + 137\,x^6 + 516\,x^5 + 703\,x^4 + 437\,x^3 + 133\,x^2 + 19\,x + 1)\,) \quad \textbf{(8)}$$

> $simplify\left(\frac{1}{x^4}\,(-24\,x^{15} - 952\,x^{14} - 11612\,x^{13} - 25296\,x^{12} - 9658\,x^{11} + 25646\,x^{10} + 36870\,x^9 + 22158\,x^8 + 7266\,x^7 + 1356\,x^6 + 137\,x^5 + 6\,x^4)\right)$

$$-24\,x^{11} - 952\,x^{10} - 11612\,x^9 - 25296\,x^8 - 9658\,x^7 + 25646\,x^6 + 36870\,x^5 + 22158\,x^4 + 7266\,x^3 + 1356\,x^2 + 137\,x + 6 \quad \textbf{(9)}$$

>

>