In this file, we compute the graphs with minimum occupancy among all triangle − free cubic graphs of order bounded **by** 24 **for** *every value of* lambda.
For this, we insert all occupancy fractions **for** *the critical graphs of every order up* **to** 24.
We prove that **for** *all these criticial graphs, with one exception, the occupancy q satisfies* $q < \min(p10, p14)$.
The verification of that can be done **by** *taking the difference* $p13 - q, p19 - q$ **and** *determine intervals where it is negative.*

```
>
> p6(x) := (x^2 + 2*x + 1)*x/(2*x^3 + 6*x^2 + 6*x + 1);
  p8(x) := (3*x^2 + 4*x + 1)*x/(8*x^3 + 16*x^2 + 8*x + 1);
  p10(x) := (2*x^3 + 9*x^2 + 6*x + 1)*x/(5*x^4 + 30*x^3 + 30*x^2 + 10*x
      + 1);
  p12(x) := 1/4*(5*x^4 + 56*x^3 + 76*x^2 + 32*x + 4)*x/(3*x^5 + 42*x^4 + 76
      *x^3 + 48*x^2 + 12*x + 1);
  p14(x) := 1/7*(120*x^4 + 294*x^3 + 231*x^2 + 70*x + 7)*x/(48*x^5 + 147*x
      ^4 + 154*x^3 + 70*x^2 + 14*x + 1);
  p16(x) := 1/8*(135*x^5 + 580*x^4 + 752*x^3 + 408*x^2 + 96*x + 8)*x/(45*x
      ^6 + 232*x^5 + 376*x^4 + 272*x^3 + 96*x^2 + 16*x + 1);
  p24b(x) := 1/6*(5*x^9 + 576*x^8 + 4462*x^7 + 11697*x^6 + 14832*x^5 + 10390
      *x^4 + 4212*x^3 + 978*x^2 + 120*x + 6)*x/(2*x^10 + 256*x^9 + 2231*x^8
      + 6684*x^7 + 9888*x^6 + 8312*x^5 + 4212*x^4 + 1304*x^3 + 240*x^2 + 24
      *x + 1);
  p24(x) := 1/2*(198*x^8 + 1528*x^7 + 3941*x^6 + 4959*x^5 + 3465*x^4 + 1404
      *x^3 + 326*x^2 + 40*x + 2)*x/(264*x^9 + 2292*x^8 + 6756*x^7 + 9918*x
      ^6 + 8316*x^5 + 4212*x^4 + 1304*x^3 + 240*x^2 + 24*x + 1);
  p24c(x) := 1/24*(10*x^9 + 2628*x^8 + 18600*x^7 + 47348*x^6 + 59490*x^5
      + 41575*x^4 + 16848*x^3 + 3912*x^2 + 480*x + 24)*x/(x^10 + 292*x^9
      + 2325*x^8 + 6764*x^7 + 9915*x^6 + 8315*x^5 + 4212*x^4 + 1304*x^3
      + 240*x^2 + 24*x + 1);
  p22d(x) := 1/11*(81*x^8 + 1460*x^7 + 6349*x^6 + 11538*x^5 + 10520*x^4
      + 5214*x^3 + 1419*x^2 + 198*x + 11)*x/(18*x^9 + 365*x^8 + 1814*x^7
      + 3846*x^6 + 4208*x^5 + 2607*x^4 + 946*x^3 + 198*x^2 + 22*x + 1);
  p22f(x) := 1/11*(18*x^8 + 1312*x^7 + 6335*x^6 + 11580*x^5 + 10530*x^4
      + 5214*x^3 + 1419*x^2 + 198*x + 11)*x/(4*x^9 + 328*x^8 + 1810*x^7
      + 3860*x^6 + 4212*x^5 + 2607*x^4 + 946*x^3 + 198*x^2 + 22*x + 1);
  p22c(x) := 1/2*(232*x^7 + 1148*x^6 + 2106*x^5 + 1915*x^4 + 948*x^3 + 258*x
      ^2 + 36*x + 2)*x/(319*x^8 + 1804*x^7 + 3861*x^6 + 4213*x^5 + 2607*x^4
      + 946*x^3 + 198*x^2 + 22*x + 1);
  p22b(x) := 1/22*(117*x^8 + 2648*x^7 + 12530*x^6 + 23064*x^5 + 21045*x^4
      + 10428*x^3 + 2838*x^2 + 396*x + 22)*x/(13*x^9 + 331*x^8 + 1790*x^7
      + 3844*x^6 + 4209*x^5 + 2607*x^4 + 946*x^3 + 198*x^2 + 22*x + 1);
  p22(x) := 1/11*(81*x^8 + 1332*x^7 + 6237*x^6 + 11514*x^5 + 10520*x^4 + 5214
      *x^3 + 1419*x^2 + 198*x + 11)*x/(18*x^9 + 333*x^8 + 1782*x^7 + 3838*x
      ^6 + 4208*x^5 + 2607*x^4 + 946*x^3 + 198*x^2 + 22*x + 1);
  p20(x) := (2*x^7 + 112*x^6 + 372*x^5 + 478*x^4 + 302*x^3 + 99*x^2 + 16*x
      + 1)*x/(5*x^8 + 320*x^7 + 1240*x^6 + 1912*x^5 + 1510*x^4 + 660*x^3
      + 160*x^2 + 20*x + 1);
```

```
p20b(x) := 1/5*(50*x^7 + 581*x^6 + 1848*x^5 + 2385*x^4 + 1510*x^3 + 495*x
    ^2 + 80*x + 5)*x/(25*x^8 + 332*x^7 + 1232*x^6 + 1908*x^5 + 1510*x^4
    + 660*x^3 + 160*x^2 + 20*x + 1);
p18(x) := 1/2*(21*x^6 + 202*x^5 + 415*x^4 + 356*x^3 + 146*x^2 + 28*x + 2)
    *x/(27*x^7 + 303*x^6 + 747*x^5 + 801*x^4 + 438*x^3 + 126*x^2 + 18*x
    + 1);
p18b(x) := 1/3*(42*x^6 + 302*x^5 + 620*x^4 + 534*x^3 + 219*x^2 + 42*x + 3)
    *x/(36*x^7 + 302*x^6 + 744*x^5 + 801*x^4 + 438*x^3 + 126*x^2 + 18*x
    + 1);
p18c(x) := 1/9*(126*x^6 + 921*x^5 + 1865*x^4 + 1602*x^3 + 657*x^2 + 126*x
    + 9)*x/(36*x^7 + 307*x^6 + 746*x^5 + 801*x^4 + 438*x^3 + 126*x^2 + 18
    *x + 1);
```

$$p6 := x \mapsto \frac{\left(x^2 + 2\cdot x + 1\right)\cdot x}{2\cdot x^3 + 6\cdot x^2 + 6\cdot x + 1}$$

$$p8 := x \mapsto \frac{\left(3\cdot x^2 + 4\cdot x + 1\right)\cdot x}{8\cdot x^3 + 16\cdot x^2 + 8\cdot x + 1}$$

$$p10 := x \mapsto \frac{\left(2\cdot x^3 + 9\cdot x^2 + 6\cdot x + 1\right)\cdot x}{5\cdot x^4 + 30\cdot x^3 + 30\cdot x^2 + 10\cdot x + 1}$$

$$p12 := x \mapsto \frac{\left(\frac{5}{4}\cdot x^4 + 14\cdot x^3 + 19\cdot x^2 + 8\cdot x + 1\right)\cdot x}{3\cdot x^5 + 42\cdot x^4 + 76\cdot x^3 + 48\cdot x^2 + 12\cdot x + 1}$$

$$p14 := x \mapsto \frac{\left(\frac{120}{7}\cdot x^4 + 42\cdot x^3 + 33\cdot x^2 + 10\cdot x + 1\right)\cdot x}{48\cdot x^5 + 147\cdot x^4 + 154\cdot x^3 + 70\cdot x^2 + 14\cdot x + 1}$$

$$p16 := x \mapsto \frac{\left(\frac{135}{8}\cdot x^5 + \frac{145}{2}\cdot x^4 + 94\cdot x^3 + 51\cdot x^2 + 12\cdot x + 1\right)\cdot x}{45\cdot x^6 + 232\cdot x^5 + 376\cdot x^4 + 272\cdot x^3 + 96\cdot x^2 + 16\cdot x + 1}$$

$$p24b := x \mapsto \left(\left(\frac{5}{6}\cdot x^9 + 96\cdot x^8 + \frac{2231}{3}\cdot x^7 + \frac{3899}{2}\cdot x^6 + 2472\cdot x^5 + \frac{5195}{3}\cdot x^4 + 702\cdot x^3 + 163\cdot x^2 + 20\cdot x + 1\right)\cdot x\right) \Big/ \left(2\cdot x^{10} + 256\cdot x^9 + 2231\cdot x^8 + 6684\cdot x^7 + 9888\cdot x^6 + 8312\cdot x^5 + 4212\cdot x^4 + 1304\cdot x^3 + 240\cdot x^2 + 24\cdot x + 1\right)$$

$$p24 := x \mapsto \left(\left(99\cdot x^8 + 764\cdot x^7 + \frac{3941}{2}\cdot x^6 + \frac{4959}{2}\cdot x^5 + \frac{3465}{2}\cdot x^4 + 702\cdot x^3 + 163\cdot x^2 + 20\cdot x + 1\right)\cdot x\right) \Big/ \left(264\cdot x^9 + 2292\cdot x^8 + 6756\cdot x^7 + 9918\cdot x^6 + 8316\cdot x^5 + 4212\cdot x^4 + 1304\cdot x^3 + 240\cdot x^2 + 24\cdot x + 1\right)$$

$$p24c := x \mapsto \left(\left(\frac{5}{12}\cdot x^9 + \frac{219}{2}\cdot x^8 + 775\cdot x^7 + \frac{11837}{6}\cdot x^6 + \frac{9915}{4}\cdot x^5 + \frac{41575}{24}\cdot x^4 + 702\cdot x^3\right.\right.$$

$$+ 163\cdot x^2 + 20\cdot x + 1\Big)\cdot x\Big) \Big/ \left(x^{10} + 292\cdot x^9 + 2325\cdot x^8 + 6764\cdot x^7 + 9915\cdot x^6 + 8315\cdot x^5 + 4212\cdot x^4 + 1304\cdot x^3 + 240\cdot x^2 + 24\cdot x + 1\right)$$

$$p22d := x \mapsto \left(\left(\frac{81}{11}\cdot x^8 + \frac{1460}{11}\cdot x^7 + \frac{6349}{11}\cdot x^6 + \frac{11538}{11}\cdot x^5 + \frac{10520}{11}\cdot x^4 + 474\cdot x^3 + 129\cdot x^2 + 18\cdot x + 1\right)\cdot x\right) \Big/ \left(18\cdot x^9 + 365\cdot x^8 + 1814\cdot x^7 + 3846\cdot x^6 + 4208\cdot x^5 + 2607\cdot x^4 + 946\cdot x^3 + 198\cdot x^2 + 22\cdot x + 1\right)$$

$$p22f := x \mapsto \left(\left(\frac{18}{11}\cdot x^8 + \frac{1312}{11}\cdot x^7 + \frac{6335}{11}\cdot x^6 + \frac{11580}{11}\cdot x^5 + \frac{10530}{11}\cdot x^4 + 474\cdot x^3 + 129\cdot x^2 + 18\cdot x + 1\right)\cdot x\right) \Big/ \left(4\cdot x^9 + 328\cdot x^8 + 1810\cdot x^7 + 3860\cdot x^6 + 4212\cdot x^5 + 2607\cdot x^4 + 946\cdot x^3 + 198\cdot x^2 + 22\cdot x + 1\right)$$

$$p22c := x \mapsto \frac{\left(116\cdot x^7 + 574\cdot x^6 + 1053\cdot x^5 + \frac{1915}{2}\cdot x^4 + 474\cdot x^3 + 129\cdot x^2 + 18\cdot x + 1\right)\cdot x}{319\cdot x^8 + 1804\cdot x^7 + 3861\cdot x^6 + 4213\cdot x^5 + 2607\cdot x^4 + 946\cdot x^3 + 198\cdot x^2 + 22\cdot x + 1}$$

$$p22b := x \mapsto \left(\left(\frac{117}{22}\cdot x^8 + \frac{1324}{11}\cdot x^7 + \frac{6265}{11}\cdot x^6 + \frac{11532}{11}\cdot x^5 + \frac{21045}{22}\cdot x^4 + 474\cdot x^3 + 129\cdot x^2 + 18\cdot x + 1\right)\cdot x\right) \Big/ \left(13\cdot x^9 + 331\cdot x^8 + 1790\cdot x^7 + 3844\cdot x^6 + 4209\cdot x^5 + 2607\cdot x^4 + 946\cdot x^3 + 198\cdot x^2 + 22\cdot x + 1\right)$$

$$p22 := x \mapsto \left(\left(\frac{81}{11}\cdot x^8 + \frac{1332}{11}\cdot x^7 + 567\cdot x^6 + \frac{11514}{11}\cdot x^5 + \frac{10520}{11}\cdot x^4 + 474\cdot x^3 + 129\cdot x^2 + 18\cdot x + 1\right)\cdot x\right) \Big/ \left(18\cdot x^9 + 333\cdot x^8 + 1782\cdot x^7 + 3838\cdot x^6 + 4208\cdot x^5 + 2607\cdot x^4 + 946\cdot x^3 + 198\cdot x^2 + 22\cdot x + 1\right)$$

$$p20 := x \mapsto \frac{\left(2\cdot x^7 + 112\cdot x^6 + 372\cdot x^5 + 478\cdot x^4 + 302\cdot x^3 + 99\cdot x^2 + 16\cdot x + 1\right)\cdot x}{5\cdot x^8 + 320\cdot x^7 + 1240\cdot x^6 + 1912\cdot x^5 + 1510\cdot x^4 + 660\cdot x^3 + 160\cdot x^2 + 20\cdot x + 1}$$

$$p20b := x \mapsto \frac{\left(10\cdot x^7 + \frac{581}{5}\cdot x^6 + \frac{1848}{5}\cdot x^5 + 477\cdot x^4 + 302\cdot x^3 + 99\cdot x^2 + 16\cdot x + 1\right)\cdot x}{25\cdot x^8 + 332\cdot x^7 + 1232\cdot x^6 + 1908\cdot x^5 + 1510\cdot x^4 + 660\cdot x^3 + 160\cdot x^2 + 20\cdot x + 1}$$

$$p18 := x \mapsto \frac{\left(\frac{21}{2}\cdot x^6 + 101\cdot x^5 + \frac{415}{2}\cdot x^4 + 178\cdot x^3 + 73\cdot x^2 + 14\cdot x + 1\right)\cdot x}{27\cdot x^7 + 303\cdot x^6 + 747\cdot x^5 + 801\cdot x^4 + 438\cdot x^3 + 126\cdot x^2 + 18\cdot x + 1}$$

$$p18b := x \mapsto \frac{\left(14\cdot x^6 + \frac{302}{3}\cdot x^5 + \frac{620}{3}\cdot x^4 + 178\cdot x^3 + 73\cdot x^2 + 14\cdot x + 1\right)\cdot x}{36\cdot x^7 + 302\cdot x^6 + 744\cdot x^5 + 801\cdot x^4 + 438\cdot x^3 + 126\cdot x^2 + 18\cdot x + 1}$$

$$p18c := x \mapsto \frac{\left(14\cdot x^6 + \frac{307}{3}\cdot x^5 + \frac{1865}{9}\cdot x^4 + 178\cdot x^3 + 73\cdot x^2 + 14\cdot x + 1\right)\cdot x}{36\cdot x^7 + 307\cdot x^6 + 746\cdot x^5 + 801\cdot x^4 + 438\cdot x^3 + 126\cdot x^2 + 18\cdot x + 1} \quad \textbf{(1)}$$

We next check except from p20, these functions are bounded by p10 and p14.
For some of these functions, it is sufficient to compare with one of them.

```
>
simplify(p10(x) - p6(x));
simplify(p14(x) - p8(x)) ;
q(x) := p16(x);
simplify(p14(x) - q(x));
q(x) := p24(x);
simplify(p14(x) - q(x));
q(x) := p24c(x);
simplify(p14(x) - q(x));
q(x) := p22c(x);
simplify(p14(x) - q(x));
q(x) := p22f(x);
simplify(p14(x) - q(x));
q(x) := p18c(x);
simplify(p14(x) - q(x));
```

$$-\frac{x^4\left(x^3 + 10\,x^2 + 17\,x + 6\right)}{10\,x^7 + 90\,x^6 + 270\,x^5 + 385\,x^4 + 272\,x^3 + 96\,x^2 + 16\,x + 1}$$

$$-\frac{x^4\left(48\,x^4 + 159\,x^3 + 174\,x^2 + 83\,x + 14\right)}{7\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1\right)\left(8\,x^3 + 16\,x^2 + 8\,x + 1\right)}$$

$$q := x \mapsto p16(x)$$

$$-\left(x^5\left(2160\,x^6 + 5235\,x^5 + 5238\,x^4 + 2878\,x^3 + 982\,x^2 + 201\,x + 20\right)\right) \Big/ \left(56\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1\right)\left(45\,x^6 + 232\,x^5 + 376\,x^4 + 272\,x^3 + 96\,x^2 + 16\,x + 1\right)\right)$$

$$q := x \mapsto p24(x)$$

$$-\left(x^5\left(3168\,x^9 + 11838\,x^8 + 18828\,x^7 + 17005\,x^6 + 10101\,x^5 + 4519\,x^4 + 1634\,x^3 + 437\,x^2 + 71\,x + 5\right)\right) \Big/ \left(14\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1\right)\left(264\,x^9 + 2292\,x^8 + 6756\,x^7 + 9918\,x^6 + 8316\,x^5 + 4212\,x^4 + 1304\,x^3 + 240\,x^2 + 24\,x + 1\right)\right)$$

$$q := x \mapsto p24c(x)$$

$$-\left(x^5\left(480\,x^{10} + 45282\,x^9 + 202696\,x^8 + 380164\,x^7 + 386720\,x^6 + 235056\,x^5 + 89459\,x^4 + 21974\,x^3 + 3606\,x^2 + 404\,x + 25\right)\right) \Big/ \left(168\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x\right.\right.$$

$$+1)\left(x^{10}+292x^9+2325x^8+6764x^7+9915x^6+8315x^5+4212x^4+1304x^3+240x^2+24x+1\right)\Big)$$

$$q := x \mapsto p22c(x)$$

$$-\left(x^5\left(1392x^8+3924x^7+4234x^6+2242x^5+855x^4+460x^3+220x^2+54x+5\right)\right)\Big/\left(14\left(48x^5+147x^4+154x^3+70x^2+14x+1\right)\left(319x^8+1804x^7+3861x^6+4213x^5+2607x^4+946x^3+198x^2+22x+1\right)\right)$$

$$q := x \mapsto p22f(x)$$

$$-\left(x^5\left(768x^9+13458x^8+37896x^7+47483x^6+32516x^5+13302x^4+3524x^3+685x^2+108x+10\right)\right)\Big/\left(77\left(48x^5+147x^4+154x^3+70x^2+14x+1\right)\left(4x^9+328x^8+1810x^7+3860x^6+4212x^5+2607x^4+946x^3+198x^2+22x+1\right)\right)$$

$$q := x \mapsto p18c(x)$$

$$-\left(x^5\left(3456x^7+12294x^6+17331x^5+12006x^4+4220x^3+707x^2+55x+5\right)\right)\Big/\left(63\left(48x^5+147x^4+154x^3+70x^2+14x+1\right)\left(36x^7+307x^6+746x^5+801x^4+438x^3+126x^2+18x+1\right)\right) \quad \textbf{(2)}$$

```
>
```

```
> q(x) := p22d(x);
  simplify(p10(x) - q(x));
```

$$q := x \mapsto p22d(x)$$

$$\left(-9x^{13}+82x^{12}-744x^{11}-4284x^{10}-6822x^9-5024x^8-1905x^7-358x^6-26x^5\right)\Big/\left(990\left(x^4+6x^3+6x^2+2x+\frac{1}{5}\right)\left(x^4+16x^3+27x^2+10x+1\right)\left(x+\frac{1}{2}\right)\left(x^2+3x+1\right)\left(x^2+\frac{7}{9}x+\frac{1}{9}\right)\right) \quad \textbf{(3)}$$

It can be checked that
$-9x^{13}+82x^{12}-744x^{11}-4284x^{10}-6822x^9-5024x^8-1905x^7-358x^6-26x^5$ *has no positive roots.*

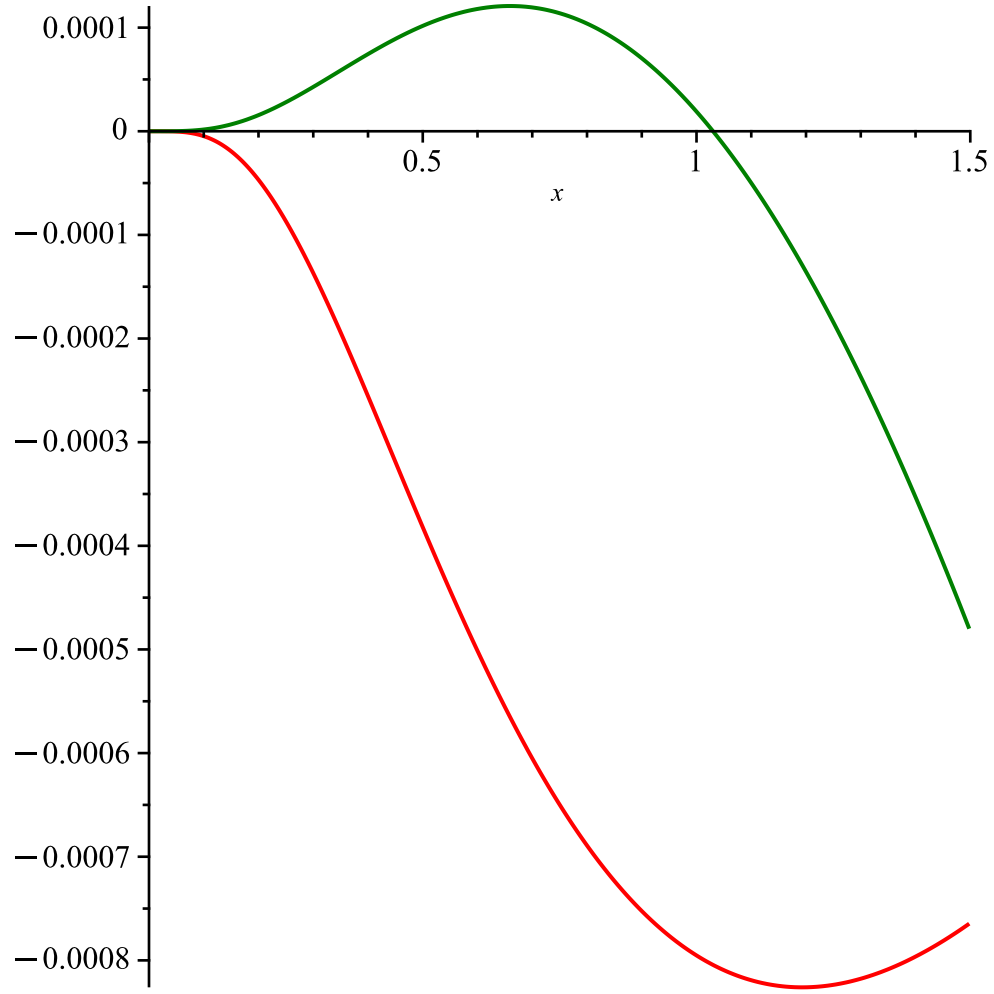
```
>
```

```
> q(x) := p12(x);
  simplify(p10(x) - q(x));
  simplify(p14(x) - q(x));
  plot([p10(x) - q(x), p14(x) - q(x)], x = 0..1.5, color = ["Red", "Green"]);
```

$$q := x \mapsto p12(x)$$

$$-\frac{x^5\left(x^4-14x^3+18x^2+30x+9\right)}{60x^9+1200x^8+6920x^7+15240x^6+16812x^5+10428x^4+3784x^3+792x^2+88x+4}$$

$$-\frac{x^5\left(240x^5+273x^4-94x^3-290x^2-146x-25\right)}{28\left(48x^5+147x^4+154x^3+70x^2+14x+1\right)\left(3x^5+42x^4+76x^3+48x^2+12x+1\right)}$$



The largest root of $\left(240x^5+273x^4-94x^3-290x^2-146x-25\right)$ *is roughly* 1.03

$x^4-14x^3+18x^2+30x+9>0$ *when* $x<\frac{18}{14}$

>

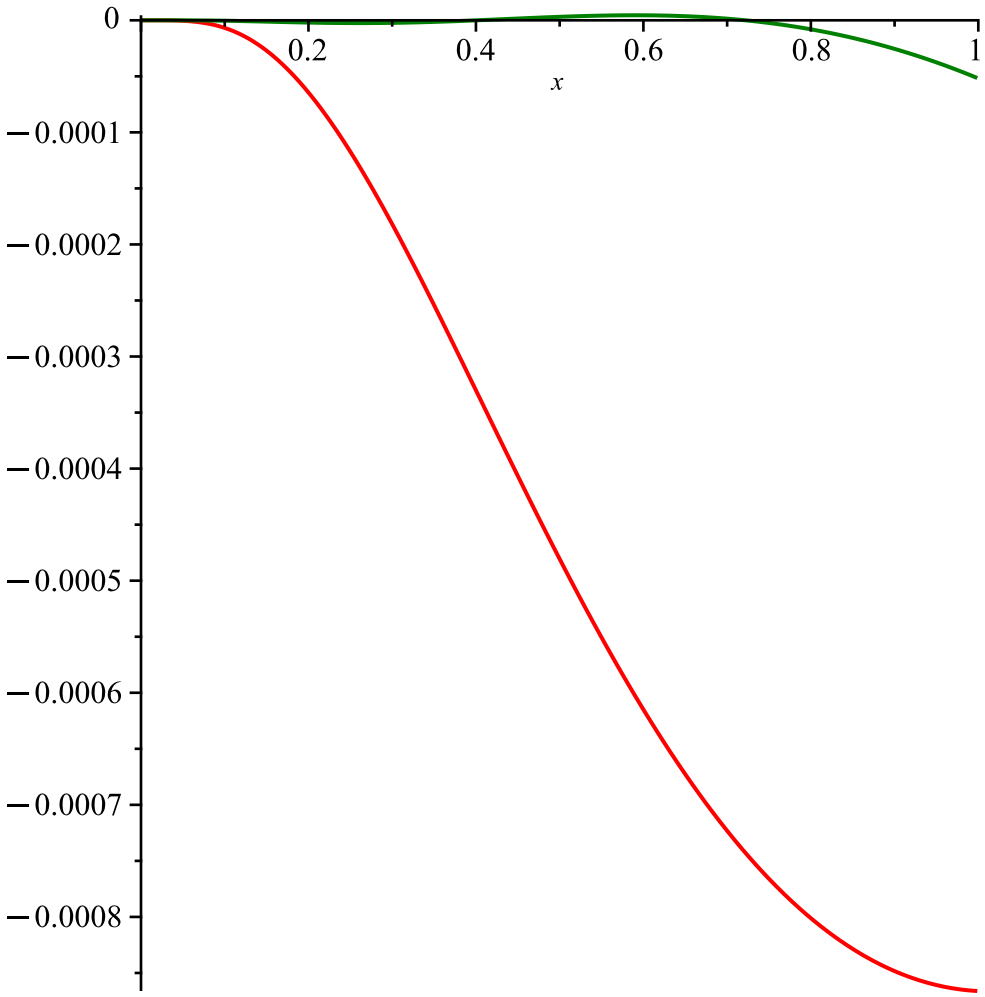
>

```
> q(x) := p18(x);
  simplify(p10(x) - q(x));
  simplify(p14(x) - q(x));
  plot([p10(x) - q(x), p14(x) - q(x)], x = 0..1, color = ["Red", "Green"]);
```

$$q := x \mapsto p18(x)$$

$$\left(x^5\left(3x^6+58x^5+x^4-160x^3-161x^2-58x-7\right)\right)/\left(2\left(5x^4+30x^3+30x^2+10x+1\right)\left(27x^7+303x^6+747x^5+801x^4+438x^3+126x^2+18x+1\right)\right)$$

$$-\left(x^5\left(576x^7+885x^6+18x^5-545x^4-232x^3+35x^2+34x+5\right)\right)/\left(14\left(48x^5+147x^4+154x^3+70x^2+14x+1\right)\left(27x^7+303x^6+747x^5+801x^4+438x^3+126x^2+18x+1\right)\right)$$



$576\,x^7 + 885\,x^6 + 18\,x^5 - 545\,x^4 - 232\,x^3 + 35\,x^2 + 34\,x + 5$>0 if x>1

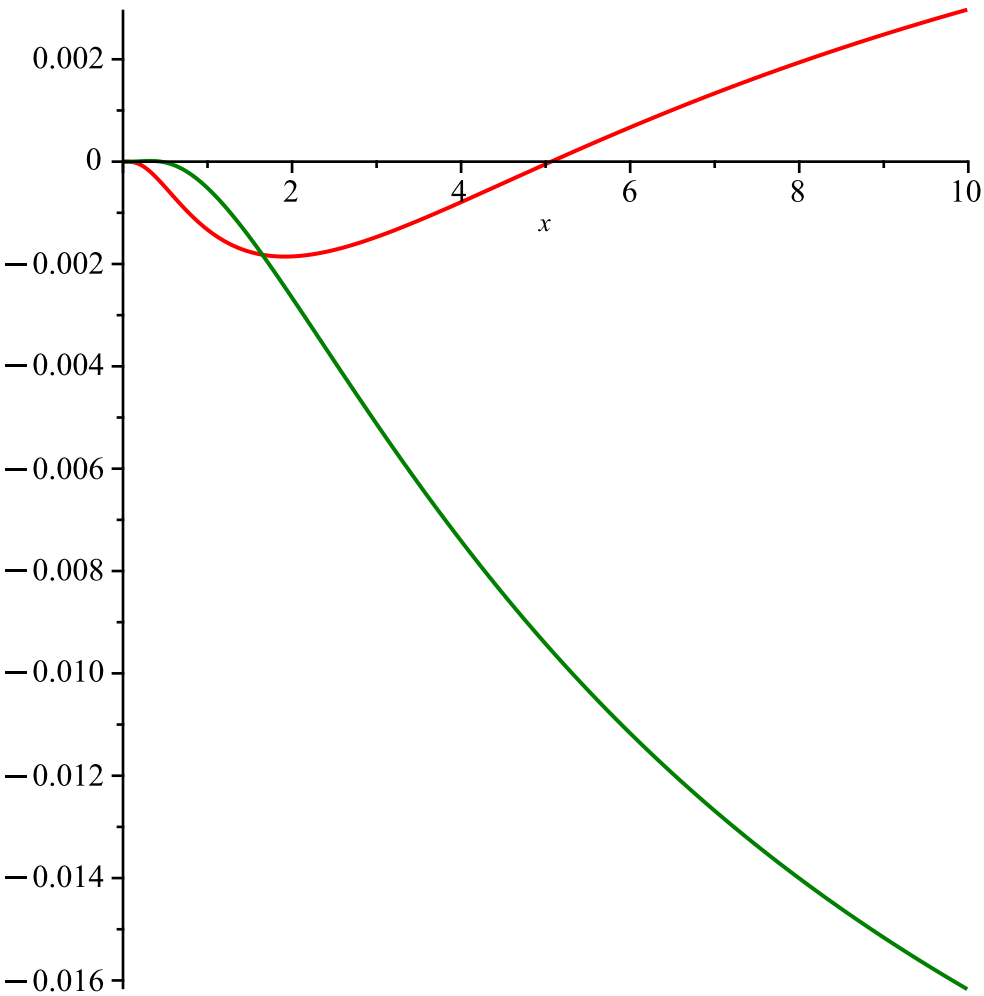
$(3\,x^6 + 58\,x^5 + x^4 - 160\,x^3 - 161\,x^2 - 58\,x - 7)$<0 if x \le 1

```
> q(x) := p18b(x);
  simplify(p10(x) - q(x));
  simplify(p14(x) - q(x));
  plot([p10(x) - q(x), p14(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p18b(x)$$

$$(2\,x^5\,(3\,x^6 + 7\,x^5 - 77\,x^4 - 156\,x^3 - 112\,x^2 - 35\,x - 4)) / (3\,(5\,x^4 + 30\,x^3 + 30\,x^2 + 10\,x + 1)\,(36\,x^7 + 302\,x^6 + 744\,x^5 + 801\,x^4 + 438\,x^3 + 126\,x^2 + 18\,x + 1))$$

$$-(2\,x^5\,(576\,x^7 + 2109\,x^6 + 2601\,x^5 + 1063\,x^4 - 202\,x^3 - 273\,x^2 - 69\,x - 5)) / (21\,(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1)\,(36\,x^7 + 302\,x^6 + 744\,x^5 + 801\,x^4 + 438\,x^3 + 126\,x^2 + 18\,x + 1))$$



$3x^6 + 7x^5 - 77x^4 - 156x^3 - 112x^2 - 35x - 4$<0 if x \le 1

$576x^7 + 2109x^6 + 2601x^5 + 1063x^4 - 202x^3 - 273x^2 - 69x - 5 > 0$ **if** $x > 1$

>

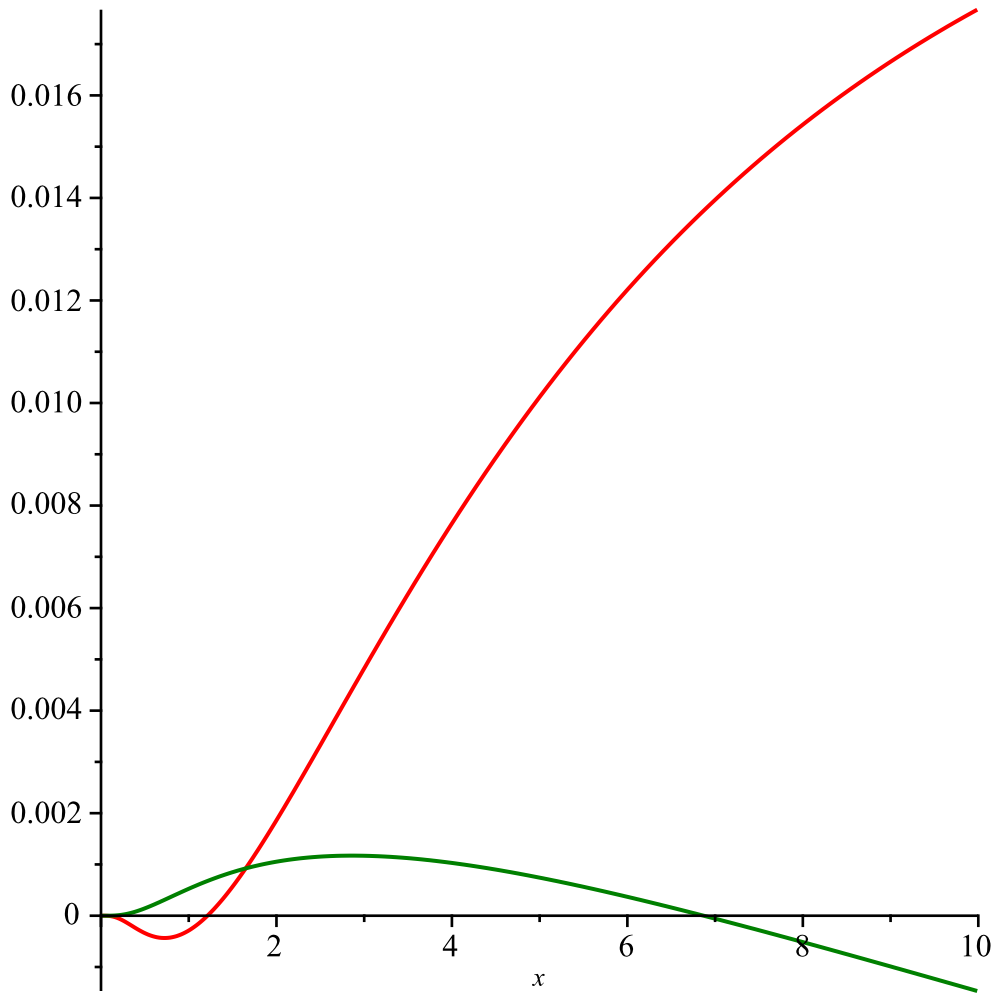
>

> $q(x) := p20(x);$
$simplify(p10(x) - q(x));$
$simplify(p14(x) - q(x));$
$plot([p10(x) - q(x), p14(x) - q(x)], x = 0..10, color = ["Red", "Green"]);$

$$q := x \mapsto p20(x)$$

$$(65x^{11} + 110x^{10} - 21x^9 - 144x^8 - 105x^7 - 30x^6 - 3x^5) \Big/ \left(25\left(x^8 + 64x^7 + 248x^6 + \frac{1912}{5}x^5 + 302x^4 + 132x^3 + 32x^2 + 4x + \frac{1}{5}\right)\left(x^4 + 6x^3 + 6x^2 + 2x + \frac{1}{5}\right)\right)$$

$$(-72x^{13} + 180x^{12} + 1639x^{11} + 3158x^{10} + 2777x^9 + 1276x^8 + 307x^7 + 34x^6 + x^5) \Big/ (7(48x^5 + 147x^4 + 154x^3 + 70x^2 + 14x + 1)(5x^8 + 320x^7 + 1240x^6 + 1912x^5 + 1510x^4 + 660x^3 + 160x^2 + 20x + 1))$$



The following simplifies the polynomials that are necessary to determine the roots (the values of lambda for which DOD beats the 2 other graphs

> $simplify\left(\frac{(65x^{11}+110x^{10}-21x^{9}-144x^{8}-105x^{7}-30x^{6}-3x^{5})}{x^{5}}\right)$

$$65x^{6}+110x^{5}-21x^{4}-144x^{3}-105x^{2}-30x-3 \quad \textbf{(4)}$$

> $simplify\left(\frac{1}{x^{5}}(-72x^{13}+180x^{12}+1639x^{11}+3158x^{10}+2777x^{9}+1276x^{8}+307x^{7}+34x^{6}+x^{5})\right)$

$$-72x^{8}+180x^{7}+1639x^{6}+3158x^{5}+2777x^{4}+1276x^{3}+307x^{2}+34x+1 \quad \textbf{(5)}$$

>

> $q(x) := p22(x);$
> $simplify(p10(x)-q(x));$

$$q := x \mapsto p22(x)$$

$$-\left(x^{5}\left(9x^{8}-18x^{7}+216x^{6}+2420x^{5}+4718x^{4}+3936x^{3}+1641x^{2}+334x+26\right)\right) / \left(11\left(5x^{4}+30x^{3}+30x^{2}+10x+1\right)\left(18x^{9}+333x^{8}+1782x^{7}+3838x^{6}+4208x^{5}\right.\right. \quad \textbf{(6)}$$

$$+ 2607\,x^4 + 946\,x^3 + 198\,x^2 + 22\,x + 1)\,)$$

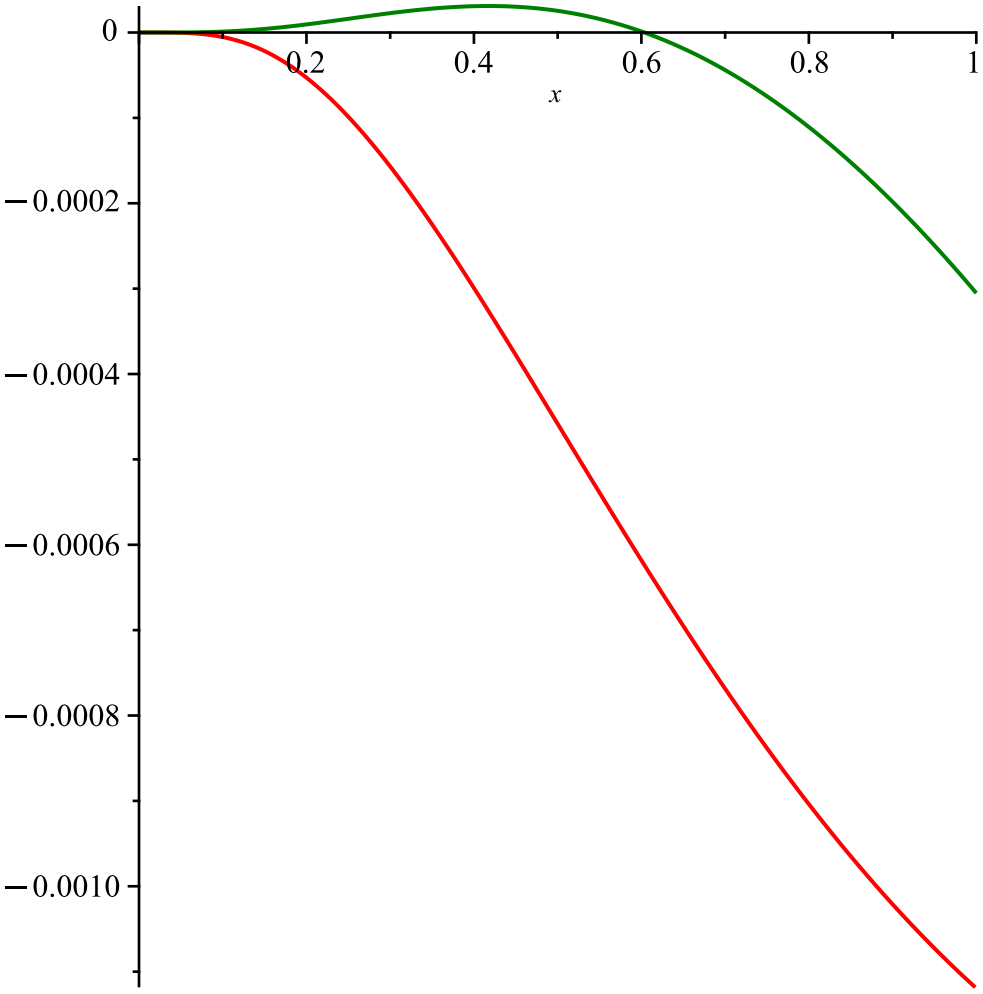
$9\,x^8 - 18\,x^7 + 9\,x^6$ \ge 0 by AM-GM.

```
> q(x) := p22b(x);
  simplify(p10(x) - q(x));
  simplify(p14(x) - q(x));
  plot([p10(x) - q(x), p14(x) - q(x)], x = 0..1, color = ["Red", "Green"]);
```

$$q := x \mapsto p22b(x)$$

$$-\left(x^5\left(13\,x^8 - 388\,x^7 - 414\,x^6 + 4296\,x^5 + 9772\,x^4 + 8480\,x^3 + 3584\,x^2 + 732\,x + 57\right)\right) / \left(22\left(5\,x^4 + 30\,x^3 + 30\,x^2 + 10\,x + 1\right)\left(13\,x^9 + 331\,x^8 + 1790\,x^7 + 3844\,x^6 + 4209\,x^5 + 2607\,x^4 + 946\,x^3 + 198\,x^2 + 22\,x + 1\right)\right)$$

$$-\left(x^5\left(4992\,x^9 + 52197\,x^8 + 128424\,x^7 + 126706\,x^6 + 37028\,x^5 - 26946\,x^4 - 26748\,x^3 - 9284\,x^2 - 1464\,x - 85\right)\right) / \left(154\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1\right)\left(13\,x^9 + 331\,x^8 + 1790\,x^7 + 3844\,x^6 + 4209\,x^5 + 2607\,x^4 + 946\,x^3 + 198\,x^2 + 22\,x + 1\right)\right)$$



The root around 0.6 is the only positive one of
$4992\,x^9 + 52197\,x^8 + 128424\,x^7 + 126706\,x^6 + 37028\,x^5 - 26946\,x^4 - 26748\,x^3 - 9284\,x^2 - 1464\,x - 85$.

For $x < 1$, $13\,x^8 - 388\,x^7 - 414\,x^6 + 4296\,x^5 + 9772\,x^4 + 8480\,x^3 + 3584\,x^2 + 732\,x$

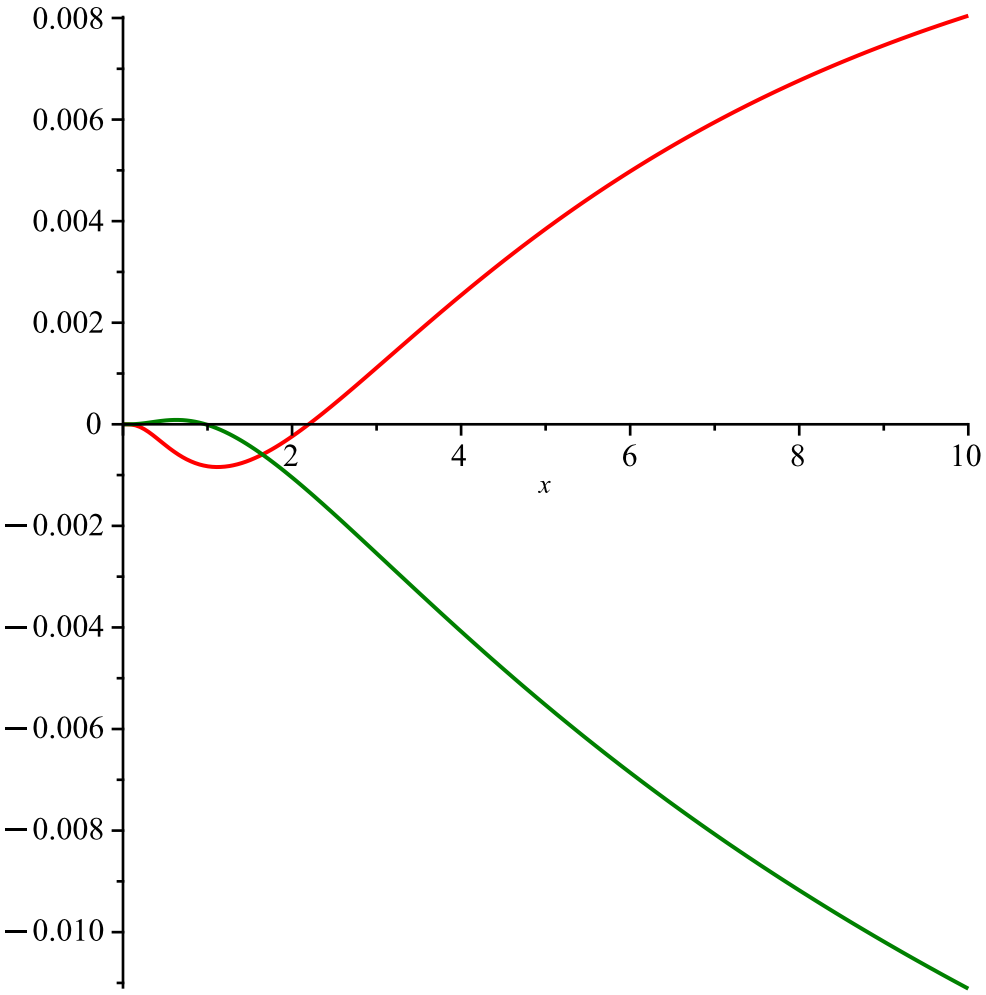
$+ 57$ *is always negative.*

```
>
>
> q(x) := p24b(x);
  simplify(p10(x) - q(x));
  simplify(p14(x) - q(x));
  plot([p10(x) - q(x), p14(x) - q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p24b(x)$$

$$-\left(x^5\left(x^9 - 150\,x^8 - 928\,x^7 - 235\,x^6 + 3251\,x^5 + 5310\,x^4 + 3688\,x^3 + 1311\,x^2 + 232\,x + 16\right)\right) \Big/ \left(60\left(x^{10} + 128\,x^9 + \frac{2231}{2}\,x^8 + 3342\,x^7 + 4944\,x^6 + 4156\,x^5 + 2106\,x^4 + 652\,x^3 + 120\,x^2 + 12\,x + \frac{1}{2}\right)\left(x^4 + 6\,x^3 + 6\,x^2 + 2\,x + \frac{1}{5}\right)\right)$$

$$-\left(x^5\left(240\,x^{10} + 10833\,x^9 + 36650\,x^8 + 41348\,x^7 + 2825\,x^6 - 33471\,x^5 - 32482\,x^4 - 14572\,x^3 - 3491\,x^2 - 424\,x - 20\right)\right) \Big/ \left(42\left(48\,x^5 + 147\,x^4 + 154\,x^3 + 70\,x^2 + 14\,x + 1\right)\left(2\,x^{10} + 256\,x^9 + 2231\,x^8 + 6684\,x^7 + 9888\,x^6 + 8312\,x^5 + 4212\,x^4 + 1304\,x^3 + 240\,x^2 + 24\,x + 1\right)\right)$$



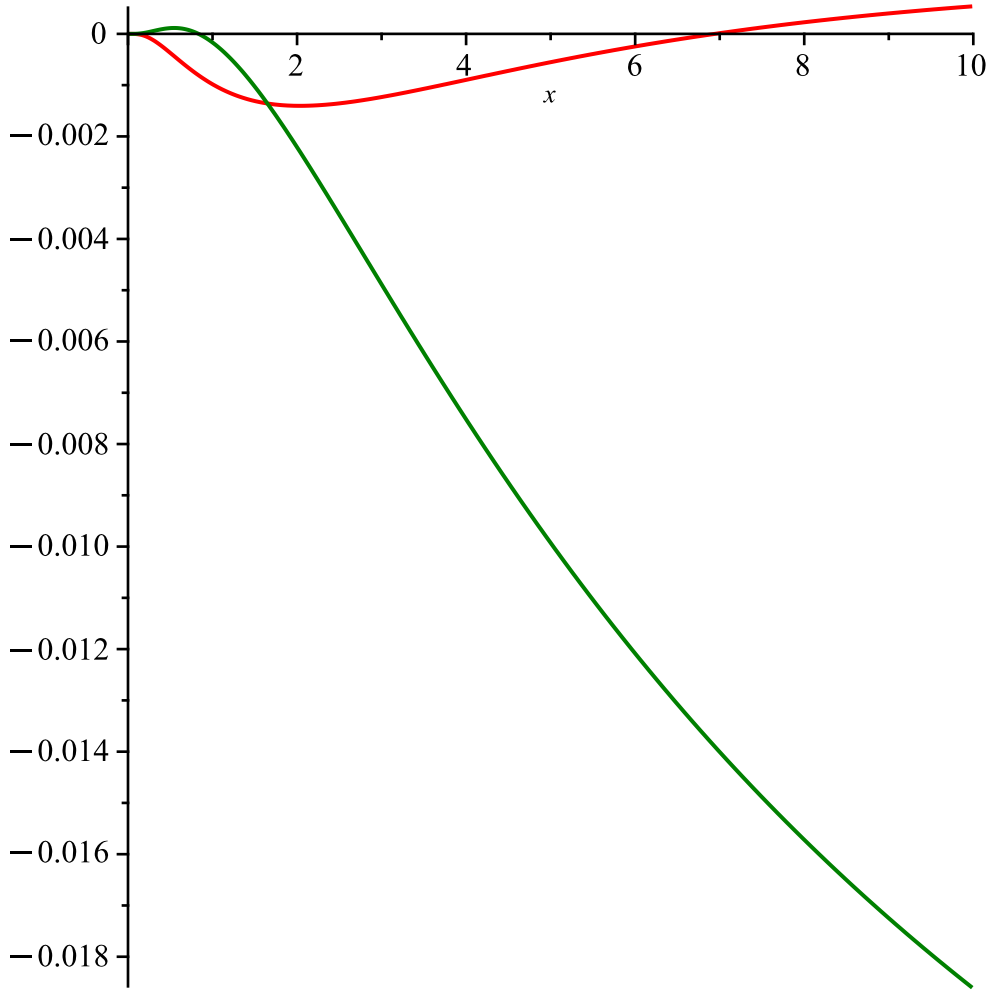

```
> q(x) := p20b(x);
```

```
simplify(p10(x) − q(x));
simplify(p14(x) − q(x));
plot([p10(x) − q(x), p14(x) − q(x)], x = 0..10, color = ["Red", "Green"]);
```

$$q := x \mapsto p20b(x)$$

$$\left(2x^5\left(20x^6 - 80x^5 - 345x^4 - 410x^3 - 218x^2 - 54x - 5\right)\right) / \left(5\left(5x^4 + 30x^3 + 30x^2 + 10x + 1\right)\left(25x^8 + 332x^7 + 1232x^6 + 1908x^5 + 1510x^4 + 660x^3 + 160x^2 + 20x + 1\right)\right)$$

$$- \left(2x^5\left(900x^8 + 5358x^7 + 8281x^6 + 2860x^5 - 3768x^4 - 4156x^3 - 1667x^2 - 302x - 20\right)\right) / \left(35\left(48x^5 + 147x^4 + 154x^3 + 70x^2 + 14x + 1\right)\left(25x^8 + 332x^7 + 1232x^6 + 1908x^5 + 1510x^4 + 660x^3 + 160x^2 + 20x + 1\right)\right)$$



$900x^8 + 5358x^7 + 8281x^6 + 2860x^5 - 3768x^4 - 4156x^3 - 1667x^2 - 302x - 20 > 0$ if x \ge 1

$20x^6 - 80x^5 - 345x^4 - 410x^3 - 218x^2 - 54x - 5$ **if** $x < 1$

>

>

>

>