

Lectures on Massive Stars *Series 2*
Stellar Winds in Massive Stars

Fundamentals of Radiative Transfer in Expanding Media

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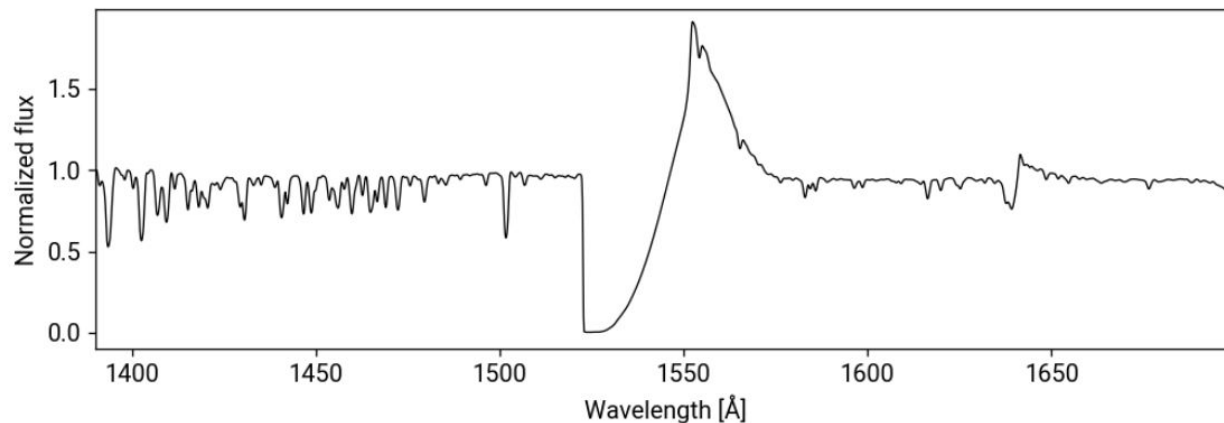
Radiation in hot stars interacts with matter in complex ways.

Need to understand radiative transfer in order to study:

1 Emergent spectra

2 Wind driving

→ Analysis of stellar atmospheres



Synthetic spectrum of a
WNh star computed with
PoWR

Radiative transfer – the basics

Specific intensity

$$dE = I_\nu(\mathbf{r}, \mathbf{n}, t) dA d\Omega dt d\nu$$

- Energy transported by radiation per (projected) **area**, per **solid angle**, per **time**, per **frequency**.
- Can generally depend on position, direction, time and frequency.

Unlike flux, **specific intensity does not dilute over distance**.

→ Only affected by interaction with matter → Radiative Transfer Equation

Radiative Transfer Equation (RTE)

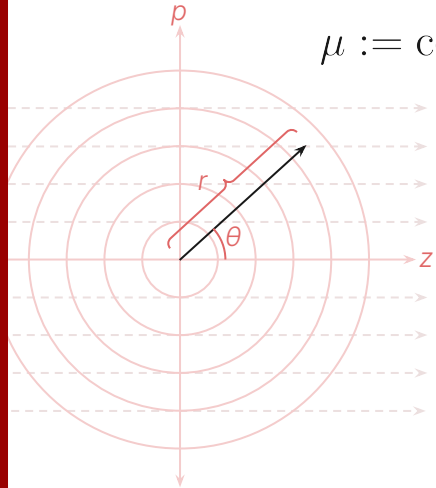
$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \mathbf{n} \cdot \nabla \right) I_\nu(\mathbf{r}, \mathbf{n}, t) = \eta_\nu(\mathbf{r}, \mathbf{n}, t) - \chi_\nu(\mathbf{r}, \mathbf{n}, t) I_\nu(\mathbf{r}, \mathbf{n}, t)$$

change in intensity = emission — absorption

Analytical solutions only for very special cases

Integrating full RTE is numerically expensive; accuracy depends on method

Spherical symmetry



$$\mu := \cos \theta$$

$$\mathbf{r} = (p, z) = (r \sin \theta, r \cos \theta) = (r \sqrt{1 - \mu^2}, r \mu)$$

Propagation of light: $\mathbf{n} := \hat{\mathbf{z}}$

$$\left(\mu \frac{\partial}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} \right) I_\nu(r, \mu) = \eta_\nu(r, \mu) - \chi_\nu(r, \mu) I_\nu(r, \mu)$$

Stationary RTE in spherical geometry

$$\frac{dI_\nu(p, z)}{dz} = \eta_\nu(p, z) - \chi_\nu(p, z) I_\nu(p, z)$$

Stationary RTE in p-z geometry

Common definitions

Optical depth

$$\tau_\nu(p, z) = \int_z^\infty \chi_\nu(p, z') \, dz'$$

Source function

$$S_\nu := \frac{\eta_\nu}{\chi_\nu}$$

Profile function

Opacity for one spectral line

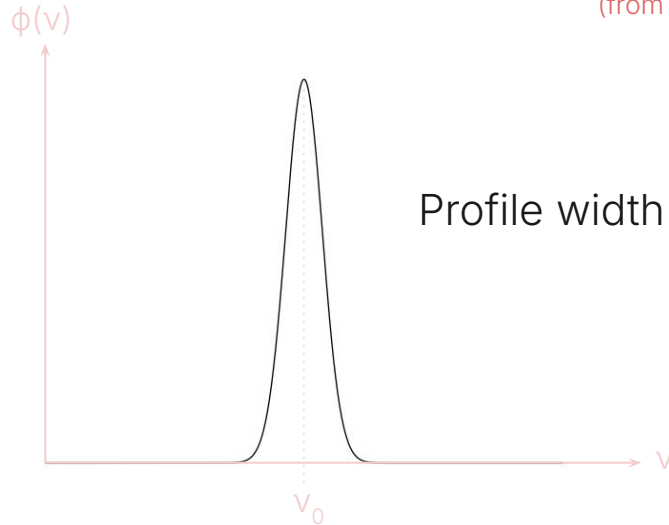
$$\chi_\nu(\mathbf{r}) = \bar{\chi}(\mathbf{r})\phi(\nu, \mathbf{r})$$

opacity

total line opacity
(from atomic data)

profile
function

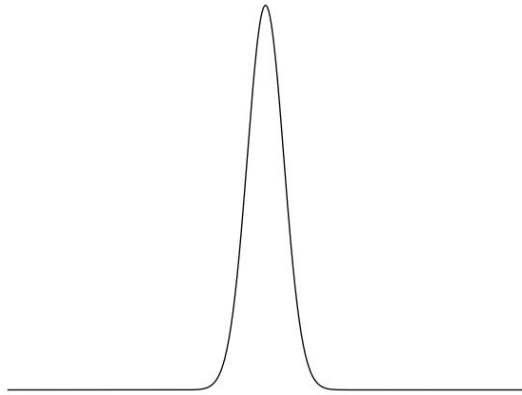
$$\bar{\chi} = \int \chi_\nu d\nu \quad \int \phi(\nu) d\nu = 1$$



Profile width $\Delta\nu$ due to thermal motion (and random motion)

$$\Delta\nu = \nu_0 \frac{v_{\text{th}}}{c}$$

Intrinsic line profile

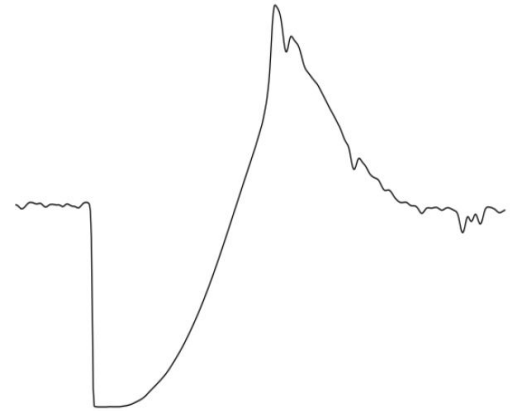


$$\Delta v \sim V_{\text{th}} \sim 10 \text{ km/s}$$

?



Emergent line profile



$$\Delta v \sim V_{\text{wind}} \sim 1000 \text{ km/s}$$

Formal integral

Integrating the RTE yields the “formal solution” for $I_\nu(\mathbf{r})$

$$I_\nu(\tau_\nu) = I_\nu(\tau_0)e^{-(\tau_0-\tau_\nu)} + \int_{\tau_\nu}^{\tau_0} S(\tau'_\nu) e^{-(\tau'_\nu-\tau_\nu)} d\tau'_\nu$$

Problems with numerical integration:

- Unknown source function?
- Highly resolved grids in space and frequency required (narrow lines vs. fast winds)
- Opacities and emissivities are non isotropic (due to Doppler-shift), and depend on r and μ .

→ Large errors when resolution not sufficient

Sobolev Theory

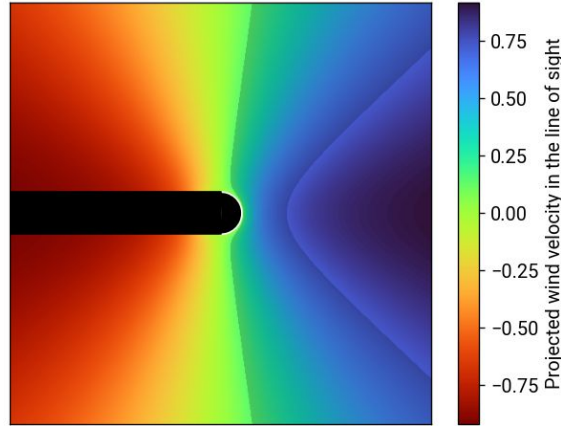
Sobolev (1960)

Rybicki & Hummer (1978)

Puls, Canary Winter-School (2017)

The Resonance Zone

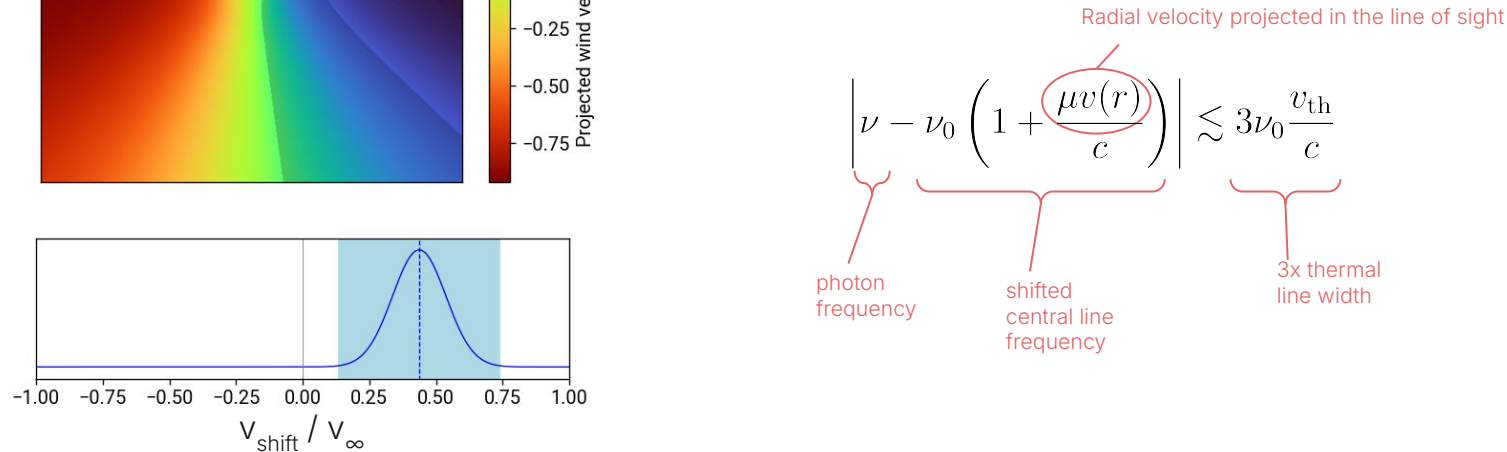
* This figure was an animated gif in the original presentation.



Consider a spectral line with restframe wavelength ν_0

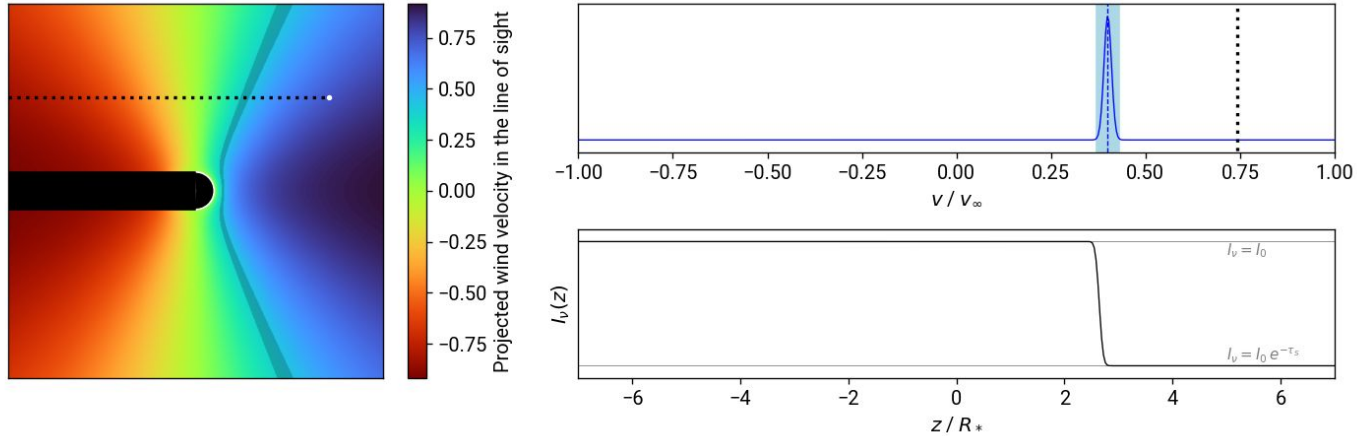
A photon with restframe wavelength ν can only be absorbed in the region where the line center is Doppler-shifted to within a few thermal widths of the photon.

→ **Resonance Zone**



The Resonance Zone

* This figure was an animated gif in the original presentation.



$$v_{\text{th}} \ll v_\infty$$

Assume the resonance zone is "narrow"

- macrovariables (opacity, source function, v gradient) are (almost) constant
- there is no interaction in most of the medium (if continuum is weak)

The Sobolev optical depth

$$\tau_S(\nu, p) = \int \bar{\chi}(p, z) \phi(\nu_{\text{CMF}}(\nu, p, z)) dz$$

Observed photon frequency

Impact parameter

Line opacity

Profile function

Photon frequency in the comoving frame

Integrate over photon path

$$\nu_{\text{CMF}}(\nu, r, \mu) = \nu \left(1 - \frac{\mu v(r)}{c} \right)$$

Position in polar (r, μ) coordinates

Local Doppler-shift

Transform integration variable from dz to $d\nu_{\text{CMF}}$:

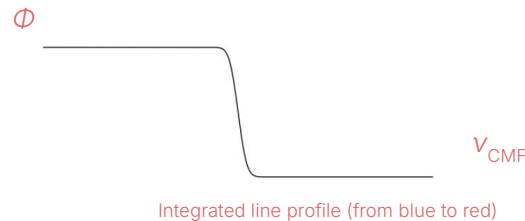
$$\frac{d\nu_{\text{CMF}}}{dz} = -\frac{\nu}{c} \left(\mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r} \right)$$

Geometry of coordinate systems

$$r(p, z) = \sqrt{p^2 + z^2}$$

$$\mu(p, z) = \frac{z}{\sqrt{p^2 + z^2}}$$

The Sobolev optical depth



$$\tau_S(\nu, p) = \int \underbrace{\frac{\bar{\chi}(p, z)}{\frac{\nu}{c} \left| \mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r} \right|}}_{\text{Sobolev approximation}} \phi(\nu_{\text{CMF}}) d\nu_{\text{CMF}} = \left. \frac{\bar{\chi}(p, z)}{\frac{\nu}{c} \left| \mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r} \right|} \right|_{\text{RZ}(\nu, p)} \int \phi(\nu_{\text{CMF}}) d\nu_{\text{CMF}}$$

Sobolev approximation: **macrovariables** are constant in the resonance zone!
(including dv/dr)

$$\tau_S(\nu, p) = \left. \frac{\bar{\chi}}{\frac{\nu}{c} \left| \mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r} \right|} \right|_{\text{RZ}(\nu, p)}$$

Evaluate in the resonance zone
(for each ν and p)

Formal solution with Sobolev approximation

- For each ν and p :
- ① Find the location of the RZ,
 - ② Compute the Sobolev optical depth.

Then:

Before passing the RZ:

$$I_\nu(p, z) = I_\nu(p, z_{\text{back}})$$

After passing the RZ:

$$I_\nu(p, z) = I_\nu(p, z_{\text{back}}) e^{-\tau_S(\text{RZ})} + S_{\text{RZ}}(1 - e^{-\tau_S(\text{RZ})})$$

General:

$$I_\nu(p, z) = I_\nu(p, z_{\text{back}}) e^{-\tau_S(\text{RZ})\underbrace{\Phi(\nu_{\text{CMF}})}_{\text{Integrated line profile}}} + S_{\text{RZ}}(1 - e^{-\tau_S(\text{RZ})\underbrace{\Phi(\nu_{\text{CMF}})}_{\text{Integrated line profile}}})$$

The specific intensity is calculated locally!



Profile-weighted mean intensity

From previous slide: $I_\nu(p, z) = I_\nu(p, z_{\text{back}}) e^{-\tau_S(RZ)\Phi(\nu_{\text{CMF}})} + S_{\text{RZ}}(1 - e^{-\tau_S(RZ)\Phi(\nu_{\text{CMF}})})$

$$\bar{I}(p, z) = \int I_\nu(p, z) \phi(\nu_{\text{CMF}}) d\nu_{\text{CMF}}$$

Needed for **rate equations** and computing **radiative acceleration**.

Only photons with a comoving frequency close to the line center contribute!

Change coordinate system ($p, z \rightarrow r, \mu$) and integrate over $d\Phi = -\phi(\nu_{\text{CMF}}) d\nu_{\text{CMF}}$

Profile-weighted mean intensity

$$\bar{I}(r, \mu) = I_{\text{back}}(p) \frac{1 - e^{-\tau_S(r, \mu)}}{\tau_S(r, \mu)} + S(r) \left(1 - \frac{1 - e^{-\tau_S(r, \mu)}}{\tau_S(r, \mu)} \right)$$

This is completely **local**

(i.e. no information is needed from any other part of the star)

Optically thick lines: $\bar{I}(\mathbf{r}) = S(\mathbf{r})$

from 1st-order expansion of τ_S

Optically thin lines: $\bar{I}(\mathbf{r}) = I_{\text{back}}(p)$

Mean intensity and Eddington flux

$$\bar{I}(r, \mu) = I_{\text{back}}(p) \frac{1 - e^{-\tau_S(r, \mu)}}{\tau_S(r, \mu)} + S(r) \left(1 - \frac{1 - e^{-\tau_S(r, \mu)}}{\tau_S(r, \mu)} \right)$$

*** S is mostly independent of μ since the angle-dependence is in the profile function which is usually the same for emissivity and opacity and therefore cancels out when considering $S = \eta/\chi$.

$$\bar{J}(r) = \frac{1}{2} \int \bar{I}(r, \mu) d\mu \quad \rightarrow \text{rate equations}$$

$$\bar{H}(r) = \frac{1}{2} \int \bar{I}(r, \mu) \mu d\mu \quad \rightarrow \text{radiative acceleration}$$

*Independent of source function in the Sobolev approximation
(since only μ^2 plays a role and thus the second term in \bar{I} is
fore-aft symmetrical)*

When is the Sobolev approximation valid?

When macroscopic variables are constant in the RZ.

$$\text{Scale-height of variable } X \left\{ \frac{X}{dX/dr} \gg \frac{v_{\text{th}}}{dv/dr} \right\} \text{ Sobolev length} = \text{width of the RZ}$$

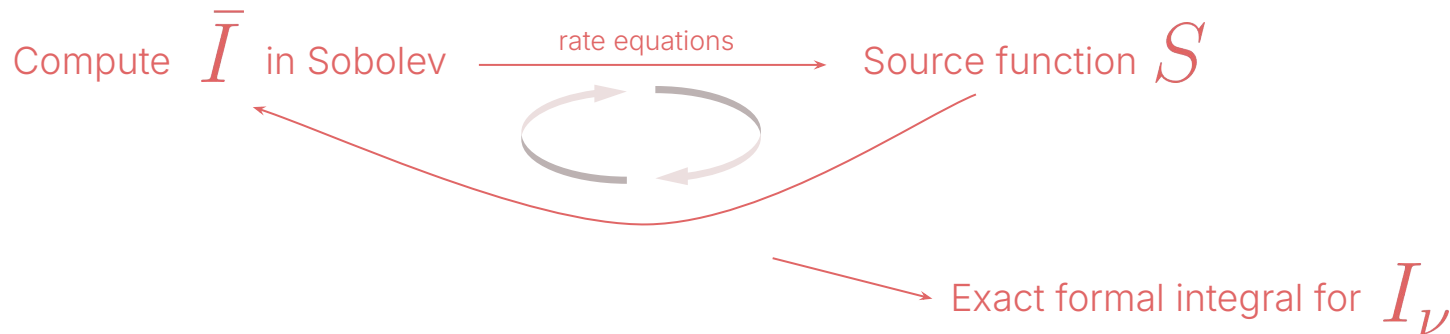
This is the case in fast-wind regimes: winds above the thermal point and SN remnants

It is not appropriate for lines formed below the sonic point, where $v \lesssim v_{\text{th}}$
(regions where $v=0$ would be inside the RZ, which then includes the entire stellar interior)

Also not appropriate for regions with high v curvature (where $dv/dr \neq \text{const.}$)
e.g. at the sonic point

Extensions of Sobolev theory

- Continuum Hummer & Rybicki (1985)
- Gradients of S Puls & Hummer (1988)
- Multiple lines Puls (1987)
- "Sobolev with Exact Integration" (SEI) Hamann (1981), Lamers et al. (1987)



The comoving frame (CMF) method

Lucy (1971),
Mihalas et al. (1975),
Hamann (1985)

Radiative transfer equation:

$$\pm \frac{dI^\pm}{dz} = \eta \left(r, \nu \left(1 - \frac{\mu v}{c} \right) \right) - \chi \left(r, \nu \left(1 - \frac{\mu v}{c} \right) \right) I^\pm$$

Simple transformation into CMF with $\nu_{\text{CMF}} = \nu \left(1 - \frac{\mu v}{c} \right)$ and $\frac{d}{dz} \Big|_\nu = \frac{\partial}{\partial z} \Big|_{\nu_{\text{CMF}}} + \frac{\partial \nu_{\text{CMF}}}{\partial z} \Big|_\nu \frac{\partial}{\partial \nu_{\text{CMF}}} \Big|_z$

geometrical factor $Q = \left| \mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r} \right|$

$$\pm \frac{\partial I^\pm}{\partial z} - \frac{\nu_{\text{CMF}} Q}{c} \frac{\partial I^\pm}{\partial \nu_{\text{CMF}}} = \eta(r, \nu_{\text{CMF}}) - \chi(r, \nu_{\text{CMF}}) I^\pm$$

($v \ll c$)

The comoving frame (CMF) method

$$\pm \frac{\partial I^{\pm}}{\partial z} - \frac{\nu_{\text{CMF}} Q}{c} \frac{\partial I^{\pm}}{\partial \nu_{\text{CMF}}} = \eta(r, \nu_{\text{CMF}}) - \chi(r, \nu_{\text{CMF}}) I^{\pm}$$

Numerical integration schemes: *implicit* (Mihalas+1975) or *semi-implicit* (Hamann 1981)

ADVANTAGES

- Only a small range of ν_{CMF} around the line center needs to be considered
- η and χ are **isotropic** in the CMF
- \mathcal{J} and \bar{H} don't need to be transformed into the observer's frame

POTENTIAL ISSUES

- Only covers the non-relativistic limit $v \ll c$
- Boundary conditions in space and initial conditions in frequency required.

From this equation, the Sobolev approximation can be exactly obtained by neglecting $\frac{\partial}{\partial z}$ -term.

Sobolev vs. CMF

Emergent line profile

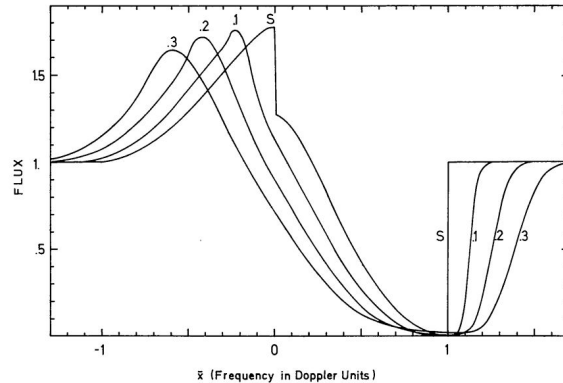
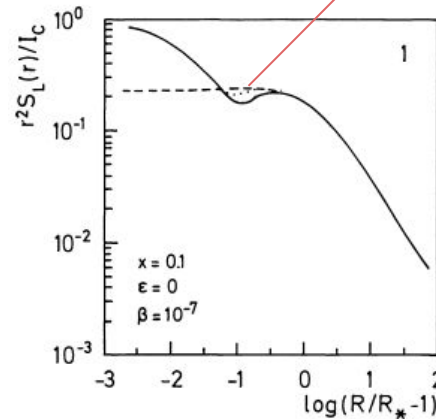


Fig. 2. Emergent flux profiles for the parameters $\kappa_0=10$, $\alpha=0$, $\beta=1/2$. The Sobolev result is labelled “S”, the comoving-frame results by their parameter $v_D/v_\infty=0.1, 0.2$ or 0.3 , respectively

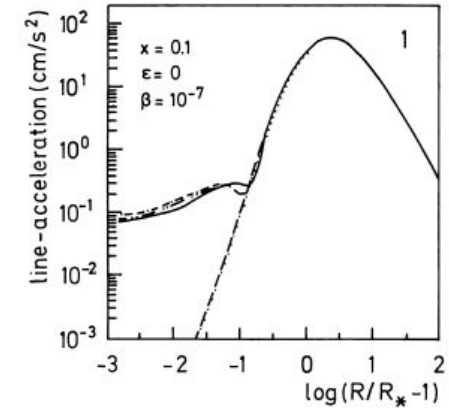
Hamann (1981)

Source function



Puls & Hummer (1988)

Radiative acceleration



— CMF
 ---- Sobolev
 Sobolev with Continuum

Conclusions

Sobolev theory allows for fast solution of radiative transfer for wind lines (above the thermal/sonic point or if $dv/dr = \text{const.}$)

Concept of the "Resonance Zone" reduces integrals to a local calculation.

→ Particularly powerful when including the extensions (continuum, multi-line, etc...)

For more general applications (e.g. quasi-photospheric lines), CMF is used.

Formal integral

Show formal integral solution

Problems:

- Source function 2D (r, μ) , anisotropic, inhomogeneous,
- mapping optical depth to position is non-precise
- large numerical errors from grids (irrelevant for J_{bar})

Everything is a resolution problem!

J_{bar} , I_{bar} calculated first \rightarrow source function, opacities \rightarrow formal solution

(because of precision)

Formal integral needed because of scale of v_{wind} vs. v_{therm}

Explain sobolev length

Goal: Calculate Intensity as a local quantity (without integrals!)

Show the main derivation steps to arrive at I and $\bar{I}(r, \mu)$ (maybe 3 slides?)

- Profile weighted mean intensity: Only consider Intensities at frequencies close to the local resonant frequency → Location of resonance zone becomes unimportant!!

Important: profile-weighted Flux is independent of S (for calculating g_{rad}) (if S fore-aft symmetrical)

Profile-weighted mean intensity represents the interaction of the radiation with the matter

→ needed for rate equations

Domains of applicability

good for large velocity gradients! e.g. SN remnant (constant $dv/dr \rightarrow$ no curvature effects), WR winds

bad for photospheric lines... below the sonic point where $v \sim v_{th}$, all of the interior is the resonance zone.

Connection point between quasi-hydrostatic & wind domain \rightarrow high v curvature

E.g. Formation of P-cygni lines (above sonic point)

Extensions of Sobolev:

- add continuum
- add S gradients
- multiline effects
- SEI (sobolev with exact integration) - Hamann 1981
 - Compute scattering integrals in Sobolev \rightarrow Source Function \rightarrow Exact formal solution.

CMF solution

CMF radiative transfer equation (with extra d/dx_0 term)

Which terms of (v/c) are necessary? (e.g. 1A SN: $v \sim 10\%c$)

Solved in the RZ \rightarrow Sobolev reproduced when d/dr neglected

Solution methods (implicit/semi-implicit \rightarrow numerically stable)

I_+ , I_- , u , v equations

Problem is outer boundary condition for optically thick conditions

Sobolev vs CMF

Exterior wind: Sobolev very good for J_{bar} (H_{bar} if extra gradient term), and continuum accounted for

Profiles: exact formal integral required

Puls & Hamann (1988), Fig.1 contains curves for CMF & Sobolev, Forces in Fig.3

Hamann (1981): CMF vs Sobolev