4DM30 - Non-Linear Control

Assignment 1

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Contents

1 Exercise 1

The results is obtained by rewriting the given error variables and substituting these in the original system dynamics equations. The dynamics in x coordinates is then given by $\dot{x} = f(x) + g(x)u$ where:

$$f = \begin{bmatrix} -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \\ \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\,(x_2+1) + \frac{1}{2} \\ 0 \end{bmatrix} \quad and, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

2 Exercise 2

The stability of the x_1 subsystem (with $x_2 = 0$) is checked by means of a Lyapunov function. The proposed Lyapunov function and its time derivative is given by: 3

$$V_1 = \frac{1}{2}x_1^2 \tag{2}$$

$$\dot{V_1} = x_1\dot{x_1} = -\frac{1}{2}\sigma x_1^4 \tag{3}$$

The function 3 is semi-negative definite for all x_1 values and therefore the controller ($x_2 = 0$) stabilizes the subsystem x_1.

3 Exercise 3

First, z coordinates are introduced:

$$z_1 = x_1 \tag{4}$$

$$z_2 = x_2 - x_1^2 \tag{5}$$

$$\dot{z_1} = \dot{x_1} = -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \tag{6}$$

$$\dot{z_2} = \dot{x_2} - 2x_1\dot{x_1} = \text{(see equation: (43))} \tag{7}$$

The virtual control input can be solved using the Lyaponov function:

$$V_2 = V_1(z_1) + \frac{1}{2}z_2^2 \tag{8}$$

$$\dot{V_2} = z_1\dot{z_1} + z_2\dot{z_2} = z_1 f_1(z_1, z_2) + z_2 \cdot (f_2(z_1, z_2) + \beta(z)) \tag{9}$$

The virtual control input as a funciton of variables z is given by equation 44 in appendix 12. In order to make the Lyaponuv function ($\dot{V}_2$) semi negative definite the entire equation is cancelled by $-f_2$ (this is the negative of equation 44). In order to stabilize the system extra terms are added:

$$\beta(z) = -f_2(z) - z_2 = (\text{see equation equation (45)}) = x_3^v \tag{10}$$

This controller in x coordinates is given by: 46 in appendix 12

4 Exercise 4

Expanding the z system with:

$$z_3 = x_3 - \beta(z)(\text{see equation 48}) \tag{11}$$

$$\dot{z}_3 = \dot{x}_3 - \frac{\partial\beta(z)}{\partial z}\dot{z} \tag{12}$$

To find a solution for the control input u we use the Lyapunov function:

$$V_3 = V_1 + V_2 + \frac{1}{2}z_3^2 \tag{13}$$

$$\dot{V}_3 = z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 \tag{14}$$

This function must be semi negative definite:

$$z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 < 0 \tag{15}$$

5 Exercise 5

The derivative of the lyapunov function is given by:

$$\dot{V}_2 = \dot{x}_1(2c_1x_1) + \dot{x}_2(\sigma x_2) \tag{16}$$

This equation can be rewritten to:

$$\dot{V}_2(x_1, x_2) = -\frac{1}{2}\sigma(6x_1^2x_2^2 + 2c_1x_2^2 + 2c_1x_1^4 + x_2^2 + 4c_1x_1^2x_2 + 2c_1x_1^2x_2^2) \tag{17}$$

All the terms without c_1 are stable and thus will not cause a problem. The terms of V_2 with only c_1 term are given by:

$$\dot{V}_{2(c_1)}(x_1, x_2) = -\frac{1}{2}\sigma(2c_1x_2^2 + 2c_1x_1^4 + 4c_1x_1^2x_2 + 2c_1x_1^2x_2^2) \tag{18}$$

which can be rewritten to:

$$\dot{V}_{2(c_1)}(x_1, x_2) = -\sigma c_1((x_1^2 + x_2)^2 + x_1^2 x_2^2) \tag{19}$$

Which is semi-negative definite for all x, resulting in the proof that the proposed virtual input for x_3 and the given Lyapunov function will result in a stable subsystem for (x_1, x_2).

6 Exercise 6

7 Exercise 7

Check the conditions for feedback linearizability of this system.
The conditions for feedback linearizability are given by 20, where n denotes the state dimension.

$$D_i = span(g_1(x), ..., g_m(x), ..., ad_f^i g_1(x), ..., ad_f^i g_m(x)), \text{ for i} = 0, 1, 2, 3 \tag{20}$$

1. Where each distribution D_i has constant dimension
2. D_{n-1} has dimension 3
3. Each distrubution D_i is involutive for $i = 0, 1, 2, 3$

The conditions 1,2 and 3 from 20 are checked for D_0:

$$D_0 = span\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \tag{21}$$

Equation 21 shows that the distribution has a constant dimension 1. Further more this vector is involutive because it is constant and has dimension 1. Since the rank of D_0 does not equal 3, the next distribution (D_1) is checked:

$$D_1 = span\left\{\begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\right\} \tag{22}$$

The distrubution D_1 has a constant dimension. The involutiveness is checked by checking if the Lie brackets of D_1 are spanned by D_1:

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 = [0\ 0\ 0\ 0]^T \in D_0 \tag{23}$$
$$[g_2, g_1] = \frac{\partial g_1}{\partial x} g_2 - \frac{\partial g_2}{\partial x} g_1 = [0\ 0\ 0\ 0]^T \in D_0$$

Since the rank of D_1 does not equal 3 the next distrubution (D_2) is checked on the 3 conditions (20).

$$D_2 = span \left\{ \begin{bmatrix} 0 & 0 & -\frac{\sigma\, x_1\, (2\, x_2+2)}{2} \\ 0 & -1 & \frac{3}{2} - 3\, {x_1}^2 - \frac{3\, (x_2+1)^2}{2} \\ 1 & 0 & 0 \end{bmatrix} \right\} \tag{24}$$

The distribution has not a constant dimension? For example is $x_1 = 0$ and $x_2 = \sqrt{15}/3 - 1$ the third column can be written as column 2.
Furthermore D_2 is involutive since the rank of the distribution equals the dimension. Thus every Lie bracket must be spanned D_2.
Concluding that the system is feedback linearizable.

8 Exercise 8

To determine the relative degree we the definition: if:

$$L_g L_f^{k-1} h(x) \neq 0 \tag{25}$$

for all x the relative degree is k. Now we compute the Lie derivatives:

$$L_g h(x) = 0 \tag{26}$$
$$L_g L_f h(x) = 1 \tag{27}$$

Therefore the relative degree is 2.

9 Exercise 9

We apply the regular state feedback:

$$u = (L_g \tag{28}$$

$$u = v + \left(\frac{3\, (x_2+1)^2}{2} + 3\, {x_1}^2 - \frac{3}{2} \right) \left(\frac{3\, x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - {x_1}^2\, (3\, x_2 + 3) + \frac{1}{2} \right) - \sigma\, {x_1}^2\, (3\, x_2 + \tag{29}$$

using this control feedback equation yields: $\ddot{x_2} = v$. Therefore the system is feedback linearised.
We rewrite equation 28 to:

$$u = \alpha(x) + \beta(x)v \tag{30}$$

Therefore:

$$\alpha(x) = \tag{31}$$

$$\beta(x) = 1 \tag{32}$$

10 Exercise 10

$$z = \phi(x) \tag{33}$$

$$\phi = \begin{bmatrix} h(x) \\ L_f h(x) \\ \eta \end{bmatrix} \tag{34}$$

$$\Phi = \begin{bmatrix} x_2 \\ \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\ (x_2 + 1) + \frac{1}{2} \\ \eta \end{bmatrix} \tag{35}$$

$$\Phi^{-1} = \begin{bmatrix} z_3 \\ z_1 \\ \frac{1}{2}z_1^3 + \frac{3}{2}z_1^2 + 3z_1z_3^2 + 3z_3^2 + z_2 \end{bmatrix} \tag{36}$$

We chose $\eta = x_1$ because no algebraic constraints are needed for this choice of $\phi(x)$. To verify if ϕ and ϕ^{-1} are globally defined the determinant of these matrices is calculated and checked if these are non zero. This is the case and therefore these state transformations are globally defined.
Now the dynamics in z coordinates becomes:

$$\dot{\mathbf{z}} = \begin{bmatrix} z_2 \\ v \\ -\frac{1}{2}\sigma z_3((z_1+1)^2 + z_3^2 - 1) \end{bmatrix} \tag{37}$$

11 Exercise 11

The definition of the zero dynamics ($y_d = 0$) for a relative degree of 2 is given by:

$$\ddot{e} + a_1\dot{e} + a_0 e = 0 \tag{38}$$

with: $e = y$
When this is substituded by the equations found in the previous exercise the following equations arise.

$$\begin{aligned} e = y = x_2 &= z_1 && (39) \\ \dot{e} = \dot{y} &= \dot{z}_1 && (40) \\ \ddot{e} = \dot{z}_2 &= v && (41) \end{aligned}$$

Then the zero-dynamics can be written as the following matrix.

$$A = \begin{bmatrix} a_0 & 0 \\ 0 & a_1 \end{bmatrix} \tag{42}$$

The system dynamics will be stable if the eigenvalues of this matrix are in the left-hand plane. This is the case if both a_0 and a_1 have a negative value.

12 Equation appendix

$$\dot{z_2} = x_3 + \frac{3\,z_2}{2} - \frac{\left({z_1}^2 + z_2 + 1\right)^3}{2} - {z_1}^2\,\left(3\,{z_1}^2 + 3\,z_2 + 3\right) + \frac{3\,{z_1}^2}{2} + \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2 + z_2 + 1\right)^2 + {z_1}^2 - 1\right)}{2} + \frac{1}{2} \tag{43}$$

$$x_3 = -\frac{z_2\,\left(\frac{3\,z_2}{2} - \frac{\left({z_1}^2+z_2+1\right)^3}{2} - {z_1}^2\,\left(3\,{z_1}^2 + 3\,z_2 + 3\right) + \frac{3\,{z_1}^2}{2} + \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2} + \frac{1}{2}\right) - \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2}}{z_2} \tag{44}$$

$$x_3^v(z) = -z_1 - z_2 - \frac{z_2\,\left(\frac{3\,z_2}{2} - \frac{\left({z_1}^2+z_2+1\right)^3}{2} - {z_1}^2\,\left(3\,{z_1}^2 + 3\,z_2 + 3\right) + \frac{3\,{z_1}^2}{2} + \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2} + \frac{1}{2}\right) - \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2}}{z_2} \tag{45}$$

$$x_3^v(x) = {x_1}^2 - x_2 - \frac{\left(x_2 - {x_1}^2\right)\,\left(\frac{3\,x_2}{2} - \frac{(x_2+1)^3}{2} - {x_1}^2\,\left(3\,x_2 + 3\right) + \frac{\sigma\,{x_1}^2\,\left((x_2+1)^2+{x_1}^2-1\right)}{2} + \frac{1}{2}\right) - \frac{\sigma\,{x_1}^2\,\left((x_2+1)^2+{x_1}^2-1\right)}{2}}{x_2 - {x_1}^2} - x_1 \tag{46}$$

$$\dot{z_3} = \dot{x_3} - \frac{\partial \beta(z)}{\partial z}\dot{\mathbf{z}} \tag{47}$$

$$\dot{z_3} = z_3 - z_2 - z_1 - \frac{z_2\,\left(\frac{3\,z_2}{2} - \frac{\left({z_1}^2+z_2+1\right)^3}{2} - {z_1}^2\,\left(3\,{z_1}^2 + 3\,z_2 + 3\right) + \frac{3\,{z_1}^2}{2} + \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2} + \frac{1}{2}\right) - \frac{\sigma\,{z_1}^2\,\left(\left({z_1}^2+z_2+1\right)^2+{z_1}^2-1\right)}{2}}{z_2} \tag{48}$$