4DM30 - Non-Linear Control

Assignment 1

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Contents

1 Backstepping

1.1 Exercise 1

The results is obtained by rewriting the given error variables and substituting these in the original system dynamics equations. The dynamics in x coordinates is then given by $\dot{x} = f(x) + g(x)u$ where:

$$f = \begin{bmatrix} -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \\ \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\, (x_2+1) + \frac{1}{2} \\ 0 \end{bmatrix} \quad and, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

1.2 Exercise 2

The stability of the x_1 subsystem (with $x_2 = 0$) is checked by means of a Lyapunov function. The proposed Lyapunov function and its time derivative is given by: 3

$$V_1 = \frac{1}{2}x_1^2 \tag{2}$$

$$\dot{V_1} = x_1\dot{x_1} = -\frac{1}{2}\sigma\, x_1\, \left((0+1)^2 + {x_1}^2 - 1\right) = -\frac{1}{2}\sigma x_1^4 \tag{3}$$

The function 3 is negative definite for all x_1 values and therefore the controller ($x_2 = 0$) stabilizes the subsystem x_1.

1.3 Exercise 3

Because in the previous exercise the Lyapunov function was stabilized for $x_2 = 0$, the resulting α is also zero. The z coordinates are introduced to perform the coordinate transformation.

$$z_1 = x_1 \tag{4}$$

$$z_2 = x_2 - \alpha = x_2 \tag{5}$$

$$\dot{z_1} = \dot{x_1} = -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \tag{6}$$

$$\dot{z_2} = \dot{x_2} = \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\, (x_2+1) + \frac{1}{2} \tag{7}$$

The virtual control input x_3^v can be solved using the Lyaponov function.

$$V_2 = V_1(z_1) + \frac{1}{2}z_2^2 \tag{8}$$

$$\dot{V}_2 = z_1\dot{z}_1 + z_2\dot{z}_2 \tag{9}$$

$$\dot{V}_2 = -c_1z_1^2 - c_2z_2^2 \tag{10}$$

$$\dot{V}_2 = z_2\left(x_3 + \frac{3\,z_2}{2} - \frac{(z_2+1)^3}{2} - {z_1}^2\,(3\,z_2+3) + \frac{1}{2}\right) - \frac{\sigma\,{z_1}^2\,\left((z_2+1)^2 + {z_1}^2 - 1\right)}{2} \tag{11}$$

In equation 9, x_3 will arise from the system dynamics via $\dot{z}_2$, see equation 11. The goal is to choose a x_3^v that replaces x_3 in $\dot{V}_2$ which ensures that the Lyapunov function becomes negative definite, this is done by demanding that $\dot{V}_2 = -c_1z_1^2 - c_2z_2^2$, where c_1, c_2 are positive and are used to place the poles in exercise 12. The resulting x_3^v controller is given by 12.

$$x_3^v = \frac{6\,{z_1}^2\,{z_2}^2 + \sigma\,{z_1}^4 + 6\,{z_1}^2\,z_2 - 2\,{z_1}^2 - 2\,{z_2}^2 + 3\,{z_2}^3 + {z_2}^4 + 2\,\sigma\,{z_1}^2\,z_2 + \sigma\,{z_1}^2\,{z_2}^2}{2\,z_2} \tag{12}$$

1.4 Exercise 4

Expanding the z system with:

$$z_3 = x_3 - x_3^v \tag{13}$$

$$\dot{z}_3 = \dot{x}_3 - \frac{\partial x_3^v}{\partial z}\dot{z} \tag{14}$$

To find a solution for the control input u we use the Lyapunov function:

$$V_3 = V_2 + \frac{1}{2}z_3^2 \tag{15}$$

$$\dot{V}_3 = z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 \tag{16}$$

The input can be solved using this Lyaponov function, this function must be semi-negative definite in z_3 domain.

$$\dot{V}_3 = z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 = -c_3z_3^2 \tag{17}$$

By solving equation 47 for u the controller is calculated. The solution is given in 46 in the appendix.

1.5 Exercise 5

The derivative of the Lyapunov function is given by:

$$\dot{V_2} = 2c_1 x_1 \dot{x_1} + \sigma x_2 \dot{x_2} \tag{18}$$

With the use of the proposed equation for x_3 the equation is rewritten to:

$$\dot{V_2}(x_1, x_2) = -\frac{1}{2}\sigma(6x_1^2 x_2^2 + 2c_1 x_2^2 + 2c_1 x_1^4 + x_2^2 + 4c_1 x_1^2 x_2 + 2c_1 x_1^2 x_2^2) \tag{19}$$

The (x_1, x_2)-subsystem is stable if $\dot{V_2}$ is negative definite for all x_1 and x_2. In the equation above all the terms without c_1 will not cause the equation to become positive. Therefore for the rest of this exercise these terms will be not be taken into account. The terms of $\dot{V_2}$ with only c_1 term is given by:

$$\dot{V}_{2(c_1)}(x_1, x_2) = -\frac{1}{2}\sigma(2c_1 x_2^2 + 2c_1 x_1^4 + 4c_1 x_1^2 x_2 + 2c_1 x_1^2 x_2^2) \tag{20}$$

which can be rewritten to:

$$\dot{V}_{2(c_1)}(x_1, x_2) = -\sigma c_1((x_1^2 + x_2)^2 + x_1^2 x_2^2) \tag{21}$$

Which is negative definite for all x, resulting in the proof that the proposed virtual input for x_3 and the given Lyapunov function will result in a stable (x_1, x_2)-subsystem.

1.6 Exercise 6

First, the coordinate transformation is performed with the same method as in Exercise 4, see equation 14. Then the new Lyapunov function is defined by:

$$V_3 = V_2 + \frac{1}{2}z_3^2 \tag{22}$$

The derivative is given by equation 23, the complete version can be found in the appendix 50. This function is forced to be negative definite to ensure stability. Here variable c_2 and c_3 are introduced which are used in exercise 12.

$$\dot{V_3} = \dot{V_2} + z_3 \dot{z_3} = -c_2 z_2^2 - c_3 z_3^2 \tag{23}$$

When this equation is solved for control input u, equation 49 is found.

2 Feedback Linearisation

2.1 Exercise 7

The conditions for feedback linearizability are given in the box below, where n denotes the state dimension. For the system that is considered, n = 3.

$$D_i = span(g_1(x), ..., g_m(x), ..., ad_f^i g_1(x), ..., ad_f^i g_m(x)), \text{ for i} = 0, 1, 2, 3 \quad (24)$$

1. Where each distribution D_i has constant dimension
2. D_{n-1} has dimension 3
3. Each distrubution D_i is involutive for $i = 0, 1, 2, 3$

The conditions 1,2 and 3 from 24 are checked for D_0:

$$D_0 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \tag{25}$$

Equation 25 shows that the distribution has a constant dimension 1. Furthermore this vector is involutive because it is constant and has dimension 1. Since the rank of D_0 does not equal 3, the next distribution (D_1) is checked:

$$D_1 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}\right\} \tag{26}$$

The distribution D_1 has a constant dimension. The involutiveness is checked by checking if the Lie brackets of D_1 are spanned by D_1:

$$\begin{aligned}[g_1, g_2] &= \frac{\partial g_2}{\partial x}g_1 - \frac{\partial g_1}{\partial x}g_2 = [0\ 0\ 0\ 0]^T \in D_0 \\ [g_2, g_1] &= \frac{\partial g_1}{\partial x}g_2 - \frac{\partial g_2}{\partial x}g_1 = [0\ 0\ 0\ 0]^T \in D_0\end{aligned} \tag{27}$$

Since the rank of D_1 does not equal 3 the next distribution (D_2) is checked on the 3 conditions (24).

$$D_2 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}-\frac{\sigma\, x_1\,(2\,x_2+2)}{2}\\ \frac{3}{2} - 3\,{x_1}^2 - \frac{3\,(x_2+1)^2}{2}\\ 0\end{pmatrix}\right\} \tag{28}$$

This distribution has a constant dimension if: $(x_1 \neq 0 \wedge x_2 \neq -1)$. Furthermore D_2 is involutive since the rank of the distribution equals the dimension. Thus every Lie bracket must be spanned D_2. Concluding that the system is feedback linearizable if: $(x_1 \neq 0 \wedge x_2 \neq -1)$.

2.2 Exercise 8

To determine the relative degree we use the definition. If in 29 for all x the relative degree is k.

$$L_g L_f^{k-1} h(x) \neq 0 \tag{29}$$

Now we compute the Lie derivatives:

$$L_g h(x) = 0 \tag{30}$$

$$L_g L_f h(x) = 1 \tag{31}$$

Therefore the relative degree is 2. Because the value of 31 is not dependent on any value of $\mathbf{x}$, the relative degree is globally defined.

2.3 Exercise 9

We apply the regular state feedback:

$$u = \left(L_g L_f h(x)\right)^{-1} \left(-L_f^2 h(x) + v\right) \tag{32}$$

$$u = v - L_f^2 h(x) \tag{33}$$

Equation 32 is rewritten to:

$$u = \alpha(x) + \beta(x) v \tag{34}$$

Therefore:

$$\begin{aligned}\alpha(x) = &\left(\frac{3\,(x_2+1)^2}{2} + 3\,{x_1}^2 - \frac{3}{2}\right)\left(\frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - {x_1}^2\,(3\,x_2+3)\right.\\ &\left. + \frac{1}{2}\right) - \sigma\,{x_1}^2\,(3\,x_2+3)\,\left((x_2+1)^2 + {x_1}^2 - 1\right)\end{aligned} \tag{35}$$

$$\beta(x) = 1 \tag{36}$$

Using this control feedback equation yields: $\ddot{x}_2 = v$. Therefore the system is feedback linearised.

2.4 Exercise 10

Using the coordinate transformation definition:

$$\mathbf{z} = \Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \eta \end{pmatrix} \tag{37}$$

Using the results from the previous exercise:

$$\Phi(x) = \begin{pmatrix} x_2 \\ \frac{3x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\ (x_2+1) + \frac{1}{2} \\ \eta \end{pmatrix} \tag{38}$$

$$\Phi^{-1}(z) = \begin{pmatrix} z_3 \\ z_1 \\ \frac{1}{2}z_1^3 + \frac{3}{2}z_1^2 + 3z_1z_3^2 + 3z_3^2 + z_2 \end{pmatrix} \tag{39}$$

$\eta = x_1$ is chosen because no algebraic constraints are needed for this choice of $\Phi(x)$. To verify if Φ and Φ^{-1} are globally defined the determinant of these matrices is calculated and checked if these are non-zero. This is the case and therefore these state transformations are globally defined. Now the dynamics in z coordinates becomes:

$$\dot{\mathbf{z}} = \begin{pmatrix} z_2 \\ v \\ -\frac{1}{2}\sigma z_3((z_1+1)^2 + z_3^2 - 1) \end{pmatrix} \tag{40}$$

2.5 Exercise 11

The definition of the zero dynamics ($y_d = 0$) for a relative degree of 2 is given by:

$$\ddot{e} + a_1\dot{e} + a_0 e = 0 \tag{41}$$

with: $e = y$

When this is substituted by the equations found in the previous exercise the following equations arise.

$$e = y = x_2 = z_1 \tag{42}$$

$$\dot{e} = \dot{y} = \dot{z_1} \tag{43}$$

$$\ddot{e} = \dot{z_2} = v \tag{44}$$

Then the zero-dynamics can be written as the following matrix.

$$A = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} \tag{45}$$

The system dynamics will be stable if the eigenvalues of this matrix are in the left-hand plane. This is the case if both a_0 and a_1 have a negative value.

3 Simulation based controller comparison

3.1 Exercise 12

3.2 Exercise 13

3.3 Exercise 14

4 Equation appendix

Exercise 4

$$u(z) = \frac{c_1\, {z_1}^2 + c_2\, {z_2}^2 - z_2\, z_3 - c_3\, {z_3}^2 + f_1\, z_3}{z_3} \tag{46}$$

$$\dot{V}_3 = -c_1\, {z_1}^2 - c_2\, {z_2}^2 + z_3\, z_2 - f_1\, z_3 + u\, z_3 \tag{47}$$

with:

$$\begin{aligned} f_1 &= \frac{\partial x_3^v}{\partial z} \\ &= \frac{\left(\sigma\, {z_1}^4 + \sigma\, {z_1}^2\, {z_2}^2 + 2\,\sigma\, {z_1}^2\, z_2 - 2\, c_1\, {z_1}^2 - 2\, c_2\, {z_2}^2 + 2\, z_3\, z_2\right)\, \left(6\, {z_1}^2\, {z_2}^2 + 2\, c_1\, {z_1}^2 - 2\, c_2\, {z_2}^2 - \sigma\, {z_1}^4 + 6\, {z_2}^3 + 3\, {z_2}^4 + \sigma\, {z_1}^2\, {z_2}^2\right)}{4\, {z_2}^3} \\ &\quad - \frac{\sigma\, {z_1}^2\, \left({z_1}^2 + {z_2}^2 + 2\, z_2\right)\, \left(6\, z_2 - 2\, c_1 + 2\,\sigma\, z_2 + 2\,\sigma\, {z_1}^2 + \sigma\, {z_2}^2 + 6\, {z_2}^2\right)}{2\, z_2} \end{aligned} \tag{48}$$

Exercise 6

$$u(z) = \frac{f_2\, z_3 - c_3\, {z_3}^2 - \sigma\, z_2\, \left(\frac{3\, z_2}{2} + z_3 - \frac{{(z_2+1)}^3}{2} - c_1\, z_2 - {z_1}^2\, \left(3\, z_2 + 3\right) + 3\, {z_1}^2 + \frac{3\, {z_2}^2}{2} + \frac{1}{2}\right) + c_1\, \sigma\, {z_1}^2\, \left({\left(z_2 + 1\right)}^2 + {z_1}^2 - 1\right)}{z_3} \tag{49}$$

$$\begin{aligned} \dot{V}_3 = \sigma\, z_2\, &\left(\frac{3\, z_2}{2} + z_3 - \frac{{\left(z_2 + 1\right)}^3}{2} - c_1\, z_2 - {z_1}^2\, \left(3\, z_2 + 3\right) + 3\, {z_1}^2 + \frac{3\, {z_2}^2}{2} + \frac{1}{2}\right) \\ &- z_3\, \left(f_2 - u\right) - c_1\, \sigma\, {z_1}^2\, \left({\left(z_2 + 1\right)}^2 + {z_1}^2 - 1\right) \end{aligned} \tag{50}$$

with:

$$f_2 = \frac{\partial x_3^v}{\partial z} = \frac{(c_1 - 3\,z_2)\,\left(6\,{z_1}^2\,z_2 + {z_2}^3 + 2\,c_1\,z_2 - 2\,z_3\right)}{2} - 3\,\sigma\,{z_1}^2\,\left({z_1}^2 + {z_2}^2 + 2\,z_2\right) \tag{51}$$