4DM30 - Non-Linear Control

Assignment 1

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1 Backstepping

1.1 Exercise 1

The dynamics in x variables is obtained by rewriting the given error variables and substituting these in the original system dynamics equations. The resulting dynamics in x coordinates is then given by $\dot{x} = f(x) + g(x)u$ where:

$$f = \begin{bmatrix} -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \\ \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\,(\,x_2+1) + \frac{1}{2} \\ 0 \end{bmatrix} \quad and, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

1.2 Exercise 2

The stability of the x_1 subsystem (with $x_2 = 0$) is checked by means of a Lyapunov function. The proposed Lyapunov function and its time derivative is given by: 2 and 3

$$V_1 = \frac{1}{2}x_1^2 \tag{2}$$

$$\dot{V_1} = x_1\dot{x_1} = -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \tag{3}$$

$$= -\frac{1}{2}\sigma\, x_1\, \left(1 + {x_1}^2 - 1\right) = -\frac{1}{2}\sigma x_1^4$$

The function 3 is negative definite for all x_1 values and therefore the controller ($x_2 = 0$) stabilizes the subsystem x_1.

1.3 Exercise 3

First we introduce a coordinate transformation by using the virtual control stated in question 2 ($x_2^v = \alpha(x_1) = 0$):

$$z_1 = x_1 \tag{4}$$

$$z_2 = x_2 - \alpha(x_1) = x_2 \tag{5}$$

Therefore the reverse transformation is given by:

$$x_1 = z_1 \tag{6}$$

$$x_2 = z_2 \tag{7}$$

The time derivative are calculated for this coordinate transformation:

$$\dot{z_1} = \dot{x_1} = -\frac{1}{2}\sigma\, x_1\, \left((x_2+1)^2 + {x_1}^2 - 1\right) \tag{8}$$

$$\dot{z_2} = \dot{x_2} = \frac{3\, x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\, (\, x_2 + 1) + \frac{1}{2} \tag{9}$$

By using the backward coordinate transformation (6 and 7) the z derivatives in z variables is given by:

$$\dot{z_1} = \dot{z_1} = -\frac{1}{2}\sigma\, z_1\, \left((z_2+1)^2 + {z_1}^2 - 1\right) \tag{10}$$

$$\dot{z_2} = \dot{z_2} = \frac{3\, z_2}{2} + x_3 - \frac{(z_2+1)^3}{2} - 3{z_1}^2\, (\, z_2 + 1) + \frac{1}{2} \tag{11}$$

The variable x_3 appears in the equations, which is the virtual control input which is needed. To guarantee stability in the z domain the Lyapunov function V_2 is introduced:

$$V_2 = V_1(z_1) + \frac{1}{2}z_2^2 \tag{12}$$

$$\dot{V_2} = z_1\dot{z_1} + z_2\dot{z_2} \tag{13}$$

The variables $\dot{z_1}$ and $\dot{z_2}$ from equation 10 and 11 are substituted in equation 13:

$$\dot{V_2} = z_2\, \left(x_3 + \frac{3\, z_2}{2} - \frac{(z_2+1)^3}{2} - {z_1}^2\, (3\, z_2 + 3) + \frac{1}{2}\right) - \frac{\sigma\, {z_1}^2\, \left((z_2+1)^2 + {z_1}^2 - 1\right)}{2} \tag{14}$$

The goal is to choose a x_3^v that replaces x_3 in $\dot{V_2}$ which ensures that the Lyapunov function becomes negative definite, which can be achieved by stating that:

$$\dot{V_2} = -c_1 z_1^2 - c_2 z_2^2 \tag{15}$$

$$\tag{16}$$

Here: c_1, c_2 are positive and are used to place the poles in exercise 12. The resulting x_3^v controller is given by 17.

$$x_3^v = \frac{6\, {z_1}^2\, {z_2}^2 - 2\, c_1\, {z_1}^2 - 2\, c_2\, {z_2}^2 + \sigma\, {z_1}^4 + 6\, {z_1}^2\, z_2 + 3\, {z_2}^3 + {z_2}^4 + 2\, \sigma\, {z_1}^2\, z_2 + \sigma\, {z_1}^2\, {z_2}^2}{2\, z_2} \tag{17}$$

We choose to cancel all terms despite some are already negative definite because this simplifies the next Lyapunov function (V_3).

1.4 Exercise 4

In order to obtain the final controller u the coordinate transformation is expanded with:

$$z_3 = x_3 - x_3^v(x) \tag{18}$$

And the reverse coordinate transformation:

$$x_3 = z_3 + x_3^v(z) \tag{19}$$

the time derivative of z_3 is given by:

$$\dot{z}_3 = \dot{x}_3 - \frac{\partial x_3^v}{\partial z}\dot{z} \tag{20}$$

To find a solution for the control input u we use the Lyapunov function:

$$V_3 = V_2 + \frac{1}{2}z_3^2 \tag{21}$$

The time derivative of V_3 is given by:

$$\dot{V}_3 = z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 \tag{22}$$

The input can be solved using this Lyaponov function, this function must be negative definite in z_3 domain. This is achieved by solving:

$$\dot{V}_3 = z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 = -c_3 z_3^2 \tag{23}$$

for the control input u. Variable x_3 is substituted by using the reverse coordinate transformation form z to x. The solution is given in 54 in the appendix.

1.5 Exercise 5

The derivative of the Lyapunov function is given by:

$$\dot{V}_2 = 2c_1 x_1 \dot{x}_1 + \sigma x_2 \dot{x}_2 \tag{24}$$

With the use of the proposed equation for x_3 the equation is rewritten to:

$$\dot{V}_2(x_1, x_2) = -\frac{1}{2}\sigma(6x_1^2x_2^2 + 2c_1x_2^2 + 2c_1x_1^4 + x_2^2 + 4c_1x_1^2x_2 + 2c_1x_1^2x_2^2) \tag{25}$$

The (x_1, x_2)-subsystem is stable if $\dot{V}_2$ is negative definite for all x_1 and x_2. In the equation above all the terms without c_1 will not cause the equation to become positive. Therefore for the rest of this exercise these terms will be not be taken into account. The terms of $\dot{V}_2$ with only c_1 term is given by:

$$\dot{V}_2(x_1, x_2) = -\frac{1}{2}\sigma(2c_1x_2^2 + 2c_1x_1^4 + 4c_1x_1^2x_2 + 2c_1x_1^2x_2^2) \tag{26}$$

which can be rewritten to:

$$\dot{V}_2(x_1, x_2) = -\sigma c_1((x_1^2 + x_2)^2 + x_1^2x_2^2) \tag{27}$$

Which is negative definite for all x_1 and x_2 due to the squared term, resulting in the proof that the proposed virtual input for x_3 and the given Lyapunov function will result in a stable (x_1, x_2)-subsystem.

1.6 Exercise 6

First, the coordinate transformation is performed as shown in Exercise 4, equations 18 and 20. The time derivative of the proposed virtual input is given by 58 which can be found in the appendix. The same Lyapunov function is used as in question 4 (equation 21) to guarantee stability. The derivative is given by equation 28, the complete version is equation 58 which can be found in the appendix. This function is forced to be negative definite to ensure stability. Here variable c_2 and c_3 are introduced which are used in exercise 12.

$$\dot{V}_3 = \dot{V}_2 + z_3\dot{z}_3 = -c_2z_2^2 - c_3z_3^2 \tag{28}$$

When this equation is solved for control input u, equation 57 is found.

2 Feedback Linearisation

2.1 Exercise 7

The conditions for feedback linearizability are given in the box below, where n denotes the state dimension. For the system that is considered, n = 3.

$$D_i = span(g_1(x), ..., g_m(x), ..., ad_f^i g_1(x), ..., ad_f^i g_m(x)), \text{ for i} = 0, 1, 2, 3 \quad (29)$$

1. Where each distribution D_i has constant dimension
2. D_{n-1} has dimension 3
3. Each distrubution D_i is involutive for $i = 0, 1, 2, 3$

The conditions 1,2 and 3 from 29 are checked for D_0:

$$D_0 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \tag{30}$$

Equation 30 shows that the distribution has a constant dimension 1. Furthermore this vector is involutive because it is constant and has dimension 1. Since the rank of D_0 does not equal 3, the next distribution (D_1) is checked:

$$D_1 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}\right\} \tag{31}$$

The distribution D_1 has a constant dimension. The involutiveness is checked by checking if the Lie brackets of D_1 are spanned by D_1:

$$\begin{aligned}[g_1, g_2] &= \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 = [0\ 0\ 0\ 0]^T \in D_0 \\ [g_2, g_1] &= \frac{\partial g_1}{\partial x} g_2 - \frac{\partial g_2}{\partial x} g_1 = [0\ 0\ 0\ 0]^T \in D_0\end{aligned} \tag{32}$$

Since the rank of D_1 does not equal 3 the next distribution (D_2) is checked on the 3 conditions (29).

$$D_2 = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}-\frac{\sigma\, x_1\,(2\,x_2+2)}{2}\\ \frac{3}{2} - 3\,{x_1}^2 - \frac{3\,(x_2+1)^2}{2}\\ 0\end{pmatrix}\right\} \tag{33}$$

This distribution has a constant dimension if: $(x_1 \neq 0 \wedge x_2 \neq -1)$. Furthermore D_2 is involutive since the rank of the distribution equals the dimension. Thus every Lie bracket must be spanned D_2. Concluding that the system is feedback linearizable if: $(x_1 \neq 0 \wedge x_2 \neq -1)$.

2.2 Exercise 8

To determine the relative degree we use the definition. If in 34 for all x the relative degree is k.

$$L_g L_f^{k-1} h(x) \neq 0 \tag{34}$$

Now we compute the Lie derivatives:

$$L_g h(x) = 0 \tag{35}$$
$$L_g L_f h(x) = 1 \tag{36}$$

Therefore the relative degree is 2. Because the value of equation 36 does not dependent on $\mathbf{x}$, the relative degree is globally defined.

2.3 Exercise 9

We apply the regular state feedback:

$$u = (L_g L_f h(x))^{-1} \left(-L_f^2 h(x) + v\right) \tag{37}$$

$$u = v - L_f^2 h(x) \tag{38}$$

Equation 37 is rewritten to:

$$u = \alpha(x) + \beta(x) v \tag{39}$$

Therefore:

$$\alpha(x) = \left(\frac{3\,(x_2+1)^2}{2} + 3\,{x_1}^2 - \frac{3}{2}\right)\left(\frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - {x_1}^2\,(3\,x_2+3) + \frac{1}{2}\right) - \sigma\,{x_1}^2\,(3\,x_2+3)\,\left((x_2+1)^2 + {x_1}^2 - 1\right) \tag{40}$$

$$\beta(x) = 1 \tag{41}$$

Using this control feedback equation yields: $\ddot{x}_2 = v$. Therefore the system is feedback linearised.

2.4 Exercise 10

Using the coordinate transformation definition:

$$\mathbf{z} = \Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \eta \end{bmatrix} \tag{42}$$

Using the results from the previous exercise:

$$\Phi(x) = \begin{bmatrix} x_2 \\ \frac{3\,x_2}{2} + x_3 - \frac{(x_2+1)^3}{2} - 3{x_1}^2\ (\,x_2 + 1) + \frac{1}{2} \\ \eta \end{bmatrix} \tag{43}$$

$$\Phi^{-1}(z) = \begin{bmatrix} z_3 \\ z_1 \\ \frac{1}{2}z_1^3 + \frac{3}{2}z_1^2 + 3z_1z_3^2 + 3z_3^2 + z_2 \end{bmatrix} \tag{44}$$

$\eta = x_1$ is chosen because no algebraic constraints are needed for this choice of $\Phi(x)$. To verify if Φ and Φ^{-1} are globally defined the determinant of these matrices is calculated and checked if these are non-zero. This is the case and therefore these state transformations are globally defined. Now the dynamics in z coordinates becomes:

$$\dot{\mathbf{z}} = \begin{bmatrix} z_2 \\ v \\ -\frac{1}{2}\sigma z_3((z_1+1)^2 + z_3^2 - 1) \end{bmatrix} \tag{45}$$

2.5 Exercise 11

The definition of the zero dynamics is that the reference output is zero ($y_d = x_2 = z_1 = 0$). This also implies that $\dot{y}_d = z_2 = 0$. This definition is substituted in 45. Giving the following system:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 \\ v \\ -\frac{1}{2}\sigma z_3^3 \end{bmatrix} \tag{46}$$

In equation 46 the controller v can be chosen as a PD-controller to stabilizes the system, using the equation:

$$\ddot{e} + a_1\dot{e} + a_0 e = 0 \tag{47}$$

With:

$$e = y = x_2 = z_1 \tag{48}$$
$$\dot{e} = \dot{y} = \dot{z_1} = z_2 \tag{49}$$
$$\ddot{e} = \dot{z_2} = v \tag{50}$$

Therefore:

$$v = -a_1 z_2 - a_0 z_1 \tag{51}$$

Substituting 51 in equation 46 gives the system:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\sigma z_3^3 \end{bmatrix} \tag{52}$$

Now the Lyapunov function $V_{3_2} = \frac{1}{2}z_3^2$ is used to show stability for the z system. This Lyapunov function is only a function of z_3 because $\dot{z_1} = 0$ and $\dot{z_2} = 0$.

$$\dot{V}_{3_2} = \dot{z_3} z_3 = -\frac{1}{2}\sigma z_3^4 \tag{53}$$

Since equation 53 is negative definite, this system is globally stable.

3 Simulation based controller comparison

3.1 Exercise 12

The systems are linearized around the equilibrium point ($x_1 = 0, x_2 = 0, x_3 = 0$). Unfortunately were unable to place the poles of the controlled closed loop systems, because the linearized system matrix A had a rank of 2 instead of 3 for both the controllers that are designed in exercise 4 and 6.This causes that it is hard to compare the controllers with each other. However to make a comparison all the gains $(c1, c2, c3)$ and (a_0, a_1) are set to 1. The resulting response of the system is given in Figure 1. The initial condition are set to $x_0 = [0.5, 0.5, 10]$ because the performance became clearly visible for these initial conditions.

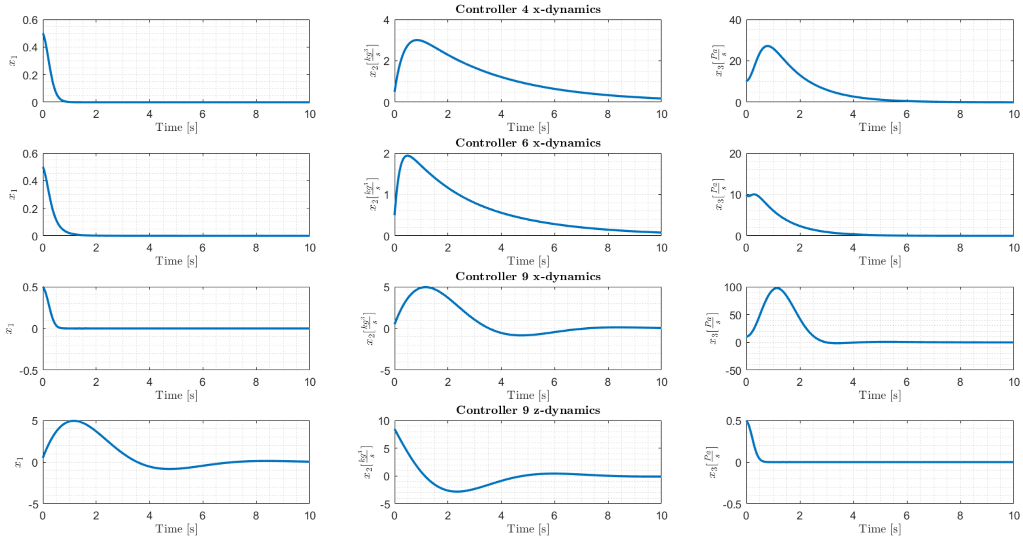

Figure 1: *Response of the system for the three different controllers with initial conditions* $x_0 = [0.5, 0.5, 10]$ *and* $(c1, c2, c3)$ *and* (a_0, a_1) *are set to* 1

This figure shows that controller 6 has the best performance in terms of overshoot. The settling time however is the lowest for controller 9, especially for the x_2 variable. We discovered that by tuning the parameters a_0 and a_1 the overshoot and settling time can drastically be reduced. Changing the parameters c_1, c_2, c_3 did not show an improvement of the same scale.

3.2 Exercise 13

Since in the system dynamics σ is only present in $\dot{x}_1$ the response for x_1 is investigated. When σ is increased to 2 the settling time clearly decreases for controllers 4 and 6 as can be seen in Figures 2 and 3. However this decrease in settling time does not hold for controller 9.

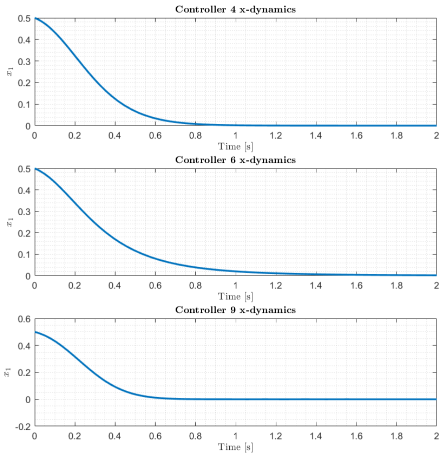

Figure 2: *Response for* $\sigma = 1$

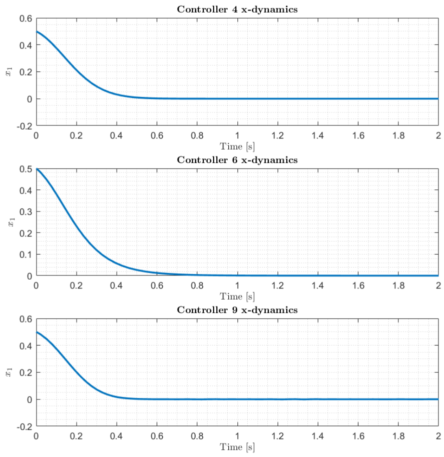

Figure 3: *Response for* $\sigma = 2$

Because the performance of controller 9 is less dependent on the system parameters this controller is preferred, since this controller is more robust. Furthermore the tuning for this controller turned out to be more easy.

3.3 Exercise 14

As can been seen in the subsystem x_1 if the value of x_1 approaches zero, the time derivative of x_1 approaches zero (all terms do depend on x_1). Therefore all controllers guarantee that if $x_1(0) > 0$ than $x_1(t) \geq 0$. The same holds for if $x_1(0) < 0$ is negative (The pressure rise is still positive) than $x_1(t) \leq 0$. Thus the controllers can still be used.

4 Equation appendix

Exercise 4

$$u(z) = \frac{c_1\, {z_1}^2 + c_2\, {z_2}^2 - z_2\, z_3 - c_3\, {z_3}^2 + f_1\, z_3}{z_3} \tag{54}$$

$$\dot{V}_3 = -c_1\, {z_1}^2 - c_2\, {z_2}^2 + z_3\, z_2 - f_1\, z_3 + u\, z_3 \tag{55}$$

with:

$$\begin{aligned} f_1 &= \frac{\partial x_3^v}{\partial z} \\ &= \frac{\left(\sigma\, {z_1}^4 + \sigma\, {z_1}^2\, {z_2}^2 + 2\,\sigma\, {z_1}^2\, z_2 - 2\, c_1\, {z_1}^2 - 2\, c_2\, {z_2}^2 + 2\, z_3\, z_2\right)\, \left(6\, {z_1}^2\, {z_2}^2 + 2\, c_1\, {z_1}^2 - 2\, c_2\, {z_2}^2 - \sigma\, {z_1}^4 + 6\, {z_2}^3 + 3\, {z_2}^4 + \sigma\, {z_1}^2\, {z_2}^2\right)}{4\, {z_2}^3} \\ &\quad - \frac{\sigma\, {z_1}^2\, \left({z_1}^2 + {z_2}^2 + 2\, z_2\right)\, \left(6\, z_2 - 2\, c_1 + 2\,\sigma\, z_2 + 2\,\sigma\, {z_1}^2 + \sigma\, {z_2}^2 + 6\, {z_2}^2\right)}{2\, z_2} \end{aligned} \tag{56}$$

Exercise 6

$$u(z) = \frac{f_2\, z_3 - c_3\, {z_3}^2 - \sigma\, z_2\, \left(\frac{3\, z_2}{2} + z_3 - \frac{{(z_2+1)}^3}{2} - c_1\, z_2 - {z_1}^2\, \left(3\, z_2 + 3\right) + 3\, {z_1}^2 + \frac{3\, {z_2}^2}{2} + \frac{1}{2}\right) + c_1\, \sigma\, {z_1}^2\, \left({\left(z_2 + 1\right)}^2 + {z_1}^2 - 1\right)}{z_3} \tag{57}$$

$$\begin{aligned} \dot{V}_3 = \sigma\, z_2\, &\left(\frac{3\, z_2}{2} + z_3 - \frac{{\left(z_2 + 1\right)}^3}{2} - c_1\, z_2 - {z_1}^2\, \left(3\, z_2 + 3\right) + 3\, {z_1}^2 + \frac{3\, {z_2}^2}{2} + \frac{1}{2}\right) \\ &- z_3\, \left(f_2 - u\right) - c_1\, \sigma\, {z_1}^2\, \left({\left(z_2 + 1\right)}^2 + {z_1}^2 - 1\right) \end{aligned} \tag{58}$$

with:

$$f_2 = \frac{\partial x_3^v}{\partial z} = \frac{(c_1 - 3\,z_2)\,\left(6\,{z_1}^2\,z_2 + {z_2}^3 + 2\,c_1\,z_2 - 2\,z_3\right)}{2} - 3\,\sigma\,{z_1}^2\,\left({z_1}^2 + {z_2}^2 + 2\,z_2\right) \tag{59}$$