# Formulation

We aim to find a solution to the electromagnetic wave equation given by:

The formulation based on [1] involves finding the function which minimizes the functional given by:

Where the source is a wave incident on with the form:

Where is the direction of propagation of the wave, and is the transverse mode of interest (in parametric coordinates). For this excitation, and are:

# Simplex Coordinates

The formulas for basis functions are expressed in simplex coordinates. We require a method of converting global coordinates to simplex coordinates to perform calculations.

## 2D

In 2d there are 3 simplex coordinates to define a point in a triangle: for . Cartesian coordinates can be converted to simplex coordinates using the equation:

Where coefficients are calculated by inverting a matrix of the tetrahedron coordinates:

The gradient of a simplex coordinate is given by:

To ensure consistency, the following numbering scheme is used.

|  |  |  |
| --- | --- | --- |
| Edge | Local Nodes | |
| 1 | 1 | 2 |
| 2 | 1 | 3 |
| 3 | 2 | 3 |

## 3D

In 3d there are 4 simplex coordinates to define a point in a tetrahedron: for . Cartesian coordinates can be converted to simplex coordinates using the equation:

Where coefficients are calculated by inverting a matrix of the tetrahedron coordinates:

The gradient of a simplex coordinate is given by:

To ensure consistency, the following numbering scheme is used.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Edge | Local Nodes | | Face | Local Nodes | | |
| 1 | 1 | 2 | 1 | 1 | 2 | 3 |
| 2 | 1 | 3 | 2 | 1 | 2 | 4 |
| 3 | 1 | 4 | 3 | 1 | 3 | 4 |
| 4 | 2 | 3 | 4 | 2 | 3 | 4 |
| 5 | 2 | 4 |  |  |  |  |
| 6 | 3 | 4 |  |  |  |  |

# Basis Functions

The problem will be discretized as tetrahedrons. The basis functions are the first 20 of Webb’s hierarchal functions. These formulas apply for both 3D and 2D basis functions.

|  |  |  |
| --- | --- | --- |
| Type | Number | Equation |
| Edge, | 1 per Edge |  |
| Edge, | 1 per Edge |  |
| Face, | 2 per Face | for {i; j; k}={1; 2; 3} and {2; 3; 1} |

The field in each element is then expressed as the sum of each of these 20 degrees of freedom:

Where contains , and and are the weighting coefficients.

# FEM Implementation

The variational form can be written in matrix form:

The matrices are defined as:

Where are the 2d basis functions across the surface (equivalent to .

The solution is the function which minimizes . This corresponds to the stationary point of the derivative with respect to the weighting coefficients:

This can be reduced to the standard linear form where and solved using standard techniques.

# Evaluation of Matrices

The matrices can be evaluated using an appropriate quadrature rule. For volume integrals an 11-point rule with degree of precision 4 is used [2]:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| 0.078933333333333 | 0.250000000000000 | 0.250000000000000 | 0.250000000000000 | 0.250000000000000 |
| 0.045733333333333 | 0.071428571428571 | 0.071428571428571 | 0.071428571428571 | 0.785714285714286 |
| 0.045733333333333 | 0.071428571428571 | 0.071428571428571 | 0.785714285714286 | 0.071428571428571 |
| 0.045733333333333 | 0.071428571428571 | 0.785714285714286 | 0.071428571428571 | 0.071428571428571 |
| 0.045733333333333 | 0.785714285714286 | 0.071428571428571 | 0.071428571428571 | 0.071428571428571 |
| 0.149333333333333 | 0.399403576166799 | 0.399403576166799 | 0.100596423833201 | 0.100596423833201 |
| 0.149333333333333 | 0.399403576166799 | 0.100596423833201 | 0.399403576166799 | 0.100596423833201 |
| 0.149333333333333 | 0.399403576166799 | 0.100596423833201 | 0.100596423833201 | 0.399403576166799 |
| 0.149333333333333 | 0.100596423833201 | 0.100596423833201 | 0.399403576166799 | 0.399403576166799 |
| 0.149333333333333 | 0.100596423833201 | 0.399403576166799 | 0.100596423833201 | 0.399403576166799 |
| 0.149333333333333 | 0.100596423833201 | 0.399403576166799 | 0.399403576166799 | 0.100596423833201 |

For surface integrals a 6-point rule with degree of precision 4 is used [3]:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| 0.223381589678011 | 0.108103018168070 | 0.445948490915965 | 0.445948490915965 |
| 0.223381589678011 | 0.445948490915965 | 0.108103018168070 | 0.445948490915965 |
| 0.223381589678011 | 0.445948490915965 | 0.445948490915965 | 0.108103018168070 |
| 0.109951743655322 | 0.816847572980459 | 0.091576213509771 | 0.091576213509771 |
| 0.109951743655322 | 0.091576213509771 | 0.816847572980459 | 0.091576213509771 |
| 0.109951743655322 | 0.091576213509771 | 0.091576213509771 | 0.816847572980459 |

Both these rules evaluate the integrals exactly and thus do not contribute any error into the solution (except floating point precision error). The rules are applied for surface integrals as:

And for volume integrals as:

To evaluate the integrals involving the terms, the curl can be precomputed using the formulas:

Where:

# Radiation Boundary Implementation

The radiation boundary is implemented using a PML. To modify a material to be a PML, the magnetic permeability and electric permittivity are multiplied by the following tensor [4]:

Where the must be placed on the dimension normal to the surface. This can be included simply in the above formulation by simply multiplying one of the vectors inside each of the nested dot products by this tensor [5].

# S-Parameter Evaluation

Using the above formulation, the S-parameters can be solved by taking the inner product of the computed field to the mode of interest:

Where is the resultant field when port is excited and is the mode of interest on port .

# Wave Port Implementation

## TE Modes

For TE modes, the port excitation is computed by solving this equation across the port surface.

This has multiple solutions depending on the value of . In variational form this becomes:

Which can be formulated into a generalized eigenproblem:

## TEM Modes

The solution to the wave equation for TEM modes is just the 2d fem solution across the face where .

# References

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| --- | --- |
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