Normal Mode Frequencies in a Cylindrical Cavity

(Version 1.0)

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Abstract

Separation of variables is used to solve the wave equation for the normal mode frequencies of sound waves in a cylindrical cavity. Normal modes arise because of the boundary conditions which demand that the gradient of the pressure normal to all interior surfaces vanishes.

Statement of the problem

Here we find the normal modes for sound waves in a closed cylindrical cavity of radius a and height b. The air pressure, p, in the cavity is a function of location and time. In Cartesian coordinates

$$p \to p(x, y, z, t).$$

Symmetry dictates we use cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \end{aligned}$$

in which case

$$p \to p(r, \theta, z, t).$$

We will choose a coordinate system in which the z axis coincides with the longitudinal axis of the cylinder, with $z = 0$ coinciding with the "floor" of the cylinder, putting the "ceiling" of the cylinder at $z = b$. The outward radial direction is the direction of $+r$, with points on the longitudinal axis at $r = 0$, and points on the inner curved surface of the cylinder at $r = a$. The polar angle θ varies from 0 to 2π.

The pressure, a scalar function of coordinates and time, satisfies the scalar wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{1}$$

in which c is the speed of sound. The pressure p is the *gauge* pressure, so it gives the value of the pressure *relative* to atmospheric pressure. Therefore, $p(r, \theta, z, t) = 0$ is a valid and physical solution, corresponding to the case in which the pressure is uniformly the same (atmospheric pressure) throughout the cavity. For us, this is an uninteresting solution because it is the one for which there are no sound waves in the cavity.

In a closed cylindrical cavity the pressure must also satisfy certain boundary conditions. At the rigid walls of the cavity the component of the air's velocity perpendicular to each surface must equal zero. Any component of the velocity is proportional to the gradient of the pressure in that direction, e.g.

$$v_s \propto \frac{\partial p}{\partial s}.$$

Thus, the boundary condition is that the gradient of the pressure at each surface, in the direction normal to each surface, must be zero. In cylindrical coordinates this condition can be stated mathematically with three equations, one each for the two flat circular surfaces, and one more for the curved "side" surface:

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = 0 \qquad \left.\frac{\partial p}{\partial z}\right|_{z=b} = 0 \qquad \left.\frac{\partial p}{\partial r}\right|_{r=a} = 0. \tag{2}$$

A fourth condition—a periodic boundary condition—will also be imposed. It is required that the pressure function be single-valued so that

$$p(\theta) = p(\theta + m2\pi) \qquad m = 0, \pm 1, \pm 2 \ldots.$$

As will be seen, only certain wave frequencies, the normal mode frequencies, are supported given the geometry and these boundary conditions. The goal of this analysis is to find the spectrum of normal mode frequencies.

Separation of Variables

This partial differential equation with boundary conditions is amenable to a solution by separation of variables. Because the pressure is a function of four variables three "separations" will be conducted, with a distinct separation constant introduced for each.

Separating the time variable

First, the spatial coordinates will be separated from the time coordinate. The functions $F \to F(r, \theta, z)$ and $T \to T(t)$ are introduced. It is supposed that there are some solutions to the wave equation for which the pressure can be written as a product of the two:

$$p(r, \theta, z, t) = F(r, \theta, z)T(t). \tag{3}$$

Substituting this into the wave equation gives

$$\nabla^2 \left[F(r, \theta, z)T(t)\right] = \frac{1}{c^2}\frac{\partial^2 \left[F(r, \theta, z)T(t)\right]}{\partial t^2}.$$

Since T is not a function of the spatial variables it is treated as a constant by the gradient operator, and since F is not a function of time, it is treated as a constant by the time derivative. This gives

$$T(t)\nabla^2 F(r,\theta,z) = \frac{F(r,\theta,z)}{c^2}\frac{d^2T(t)}{dt^2}.$$

Moving spatial variables to the *lhs* and time variables to the *rhs* gives

$$\frac{\nabla^2 F(r,\theta,z)}{F(r,\theta,z)} = \frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2}.$$

The equality must hold at given values of r, θ, and z for all values of t, and for a given value of t at all combinations of r, θ, and z. The only way this could possibly be the case is if both sides are equal to the same constant. Although the separation constant can be given any symbol, with experience and hindsight it has been found convenient to name it $-k^2$. So we have

$$\frac{\nabla^2 F(r,\theta,z)}{F(r,\theta,z)} = \frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2} = -k^2.$$

For the time coordinate this gives

$$\frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2} = -k^2.$$

With a slight rearrangement this is

$$\frac{d^2T(t)}{dt^2} + c^2k^2T(t) = 0.$$

It will be useful to define a new parameter,

$$\omega \equiv ck, \tag{4}$$

which, in this context, will be the angular frequency of the normal mode, in radians per unit time, with k being the wave number. This claim will be justified once we find a solution for the wave's time dependence. The normal mode frequency, f, in cycles per unit time, is related to the angular frequency in the usual way by

$$\omega = 2\pi f. \tag{5}$$

In terms of the speed of sound and the wave number, the normal mode frequency, is therefore,

$$f = \frac{ck}{2\pi}. \tag{6}$$

For the time being, this second order ordinary differential equation for T will be expressed in terms of the angular frequency; it is

$$\boxed{\frac{d^2T(t)}{dt^2} + \omega^2T(t) = 0.} \tag{7}$$

The Helmholtz equation

For the spatial coordinates the separation gives

$$\frac{\nabla^2 F(r,\theta,z)}{F(r,\theta,z)} = -k^2,$$

which, with a small rearrangement gives

$$\nabla^2 F(r,\theta,z) + k^2 F(r,\theta,z) = 0.$$

This is the well known partial differential equation called the Helmholtz equation.

Separating the angle variable

At this point, the time coordinate has been successfully separated from the spatial coordinates. The next task is to separate the three spatial coordinates. This can be accomplished by supposing there are some solutions to the wave equation for which F can be expressed as a product of three functions: $R \to R(r)$, $\Theta \to \Theta(\theta)$, and $\Psi \to \Psi(z)$, each a function of only one spatial variable:

$$F(r,\theta,z) = R(r)\Theta(\theta)\Psi(z). \tag{8}$$

Substituting in the Helmholtz equation gives

$$\nabla^2 \left[R(r)\Theta(\theta)\Psi(z)\right] + k^2 \left[R(r)\Theta(\theta)\Psi(z)\right] = 0.$$

In cylindrical coordinates the Laplacian operator is

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

Applying this to the product function in the Helmholtz equation gives

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \left[R(r)\Theta(\theta)\Psi(z)\right]}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \left[R(r)\Theta(\theta)\Psi(z)\right]}{\partial \theta^2} + \frac{\partial^2 \left[R(r)\Theta(\theta)\Psi(z)\right]}{\partial z^2} + k^2 \left[R(r)\Theta(\theta)\Psi(z)\right] = 0.$$

The spatial derivatives only act on the functions of their respective spatial variable giving

$$\frac{\Theta(\theta)\Psi(z)}{r}\frac{d}{dr} r \frac{dR(r)}{dr} + \frac{R(r)\Psi(z)}{r^2}\frac{d^2\Theta(\theta)}{d\theta^2} + R(r)\Theta(\theta)\frac{d^2\Psi(z)}{dz^2} + k^2\left[R(r)\Theta(\theta)\Psi(z)\right] = 0.$$

Dividing through by $R(r)\Theta(\theta)\Psi(z)$ gives

$$\frac{1}{rR(r)}\frac{d}{dr} r \frac{dR(r)}{dr} + \frac{1}{r^2\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} + \frac{1}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2} + k^2 = 0.$$

Multiplying through by r^2 gives

$$\frac{r}{R(r)}\frac{d}{dr} r \frac{dR(r)}{dr} + \frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} + \frac{r^2}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2} + r^2k^2 = 0.$$

Moving the term dependent on θ to the *rhs* gives

$$\frac{r}{R(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+\frac{r^2}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}+r^2k^2=-\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2}.$$

The equality must hold at given values of r and z for all values of θ, and for a given value of θ at all combinations of r and z. The only way this could possibly be the case is if both sides are equal to the same constant. Although the separation constant can be given any symbol, with experience and hindsight it has been found convenient to name it m^2. So we have

$$\frac{r}{R(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+\frac{r^2}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}+r^2k^2=-\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2}=m^2.$$

For the term containing the θ variable this gives

$$-\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2}=m^2.$$

With a small rearrangement this is

$$\boxed{\frac{d^2\Theta(\theta)}{d\theta^2}+m^2\Theta(\theta)=0.} \tag{9}$$

Separating the longitudinal variable

At this point, the θ variable has been successfully separated from the other two spatial variables. The next task is to separate the r variable from the z variable.

Setting the terms containing the r and z variables equal to the separation constant m^2 gives

$$\frac{r}{R(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+\frac{r^2}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}+r^2k^2=m^2.$$

Bringing the m^2 term to the *lhs* and the term dependent on the z variable to the *rhs* gives

$$\frac{r}{R(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+r^2k^2-m^2=-\frac{r^2}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}.$$

Dividing through by r^2 gives

$$\frac{1}{rR(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+k^2-\frac{m^2}{r^2}=-\frac{1}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}.$$

The equality must hold at given values of r for all values of z, and for a given value of z at all values of r. The only way this could possibly be the case is if both sides are equal to the same constant. Although the separation constant can be given any symbol, with experience and hindsight it has been found convenient to name it l^2. So we have

$$\frac{1}{rR(r)}\frac{d}{dr}r\frac{dR(r)}{dr}+k^2-\frac{m^2}{r^2}=-\frac{1}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2}=l^2.$$

For the term containing the z variable the separation gives

$$-\frac{1}{\Psi(z)}\frac{d^2\Psi(z)}{dz^2} = l^2.$$

With a small amount of rearrangement this is

$$\boxed{\frac{d^2\Psi(z)}{dz^2} + l^2\Psi(z) = 0.} \tag{10}$$

Separating the radial variable

For the term containing the r variable the separation gives

$$\frac{1}{rR(r)}\frac{d}{dr}r\frac{dR(r)}{dr} + k^2 - \frac{m^2}{r^2} = l^2.$$

Moving the n^2 term to the lhs gives

$$\frac{1}{rR(r)}\frac{d}{dr}r\frac{dR(r)}{dr} + k^2 - l^2 - \frac{m^2}{r^2} = 0.$$

Multiplying through by $rR(r)$ gives

$$\frac{d}{dr}r\frac{dR(r)}{dr} + \left(k^2 - l^2 - \frac{m^2}{r^2}\right) rR(r) = 0.$$

Carrying out the first r derivative using the product rule gives

$$r\frac{d^2R(r)}{dr^2} + \frac{dR(r)}{dr} + \left(k^2 - l^2 - \frac{m^2}{r^2}\right) rR(r) = 0.$$

Dividing through by r gives

$$\boxed{\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - l^2 - \frac{m^2}{r^2}\right) R(r) = 0.} \tag{11}$$

This is the well known Bessel equation.

Separation summary

By separation of variables we have arrived at four second order ordinary differential equations. Collecting them:

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - l^2 - \frac{m^2}{r^2}\right) R(r) = 0, \tag{12}$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + m^2\Theta(\theta) = 0, \tag{13}$$

$$\frac{d^2\Psi(z)}{dz^2} + l^2\Psi(z) = 0, \tag{14}$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0. \tag{15}$$

Boundary Conditions

The four equations contain three separation constants, k, m, and l, with the angular frequency defined in terms of one of them, $\omega \equiv ck$. At the moment there are no restrictions on the values the separation constants can take, but restrictions will appear once the boundary conditions are imposed.

Boundary condition at the "floor"

Working with the equation for $\Psi(z)$ first,

$$\frac{d^2\Psi(z)}{dz^2} + l^2\Psi(z) = 0.$$

As a second order equation it has two independent solutions. One possible choice is the pair of functions $A\cos(lz)$ and $B\sin(lz)$. Any linear combination of these is also a solution. Therefore, the most general solution is

$$\Psi_l(z) = A_l\cos(lz) + B_l\sin(lz). \tag{16}$$

At the moment, there are no restrictions on the coefficients A_l and B_l, nor on the separation constant l.

But two of the boundary condition affect $\Psi_l(z)$. The gradient of the pressure normal to the floor and ceiling of the cylindrical cavity must vanish. The relevant gradient is the directional derivative in the z direction. Expressed as a derivative of the product function, the z component of the gradient is

$$\frac{\partial p(r,\theta,z)}{\partial z} = \frac{\partial\left[R(r)\Theta(\theta)\Psi(z)\right]}{\partial z} = R(r)\Theta(\theta)\frac{d\Psi(z)}{dz}.$$

The boundary condition at the floor, $z = 0$, of the cavity says this must equal zero at the floor:

$$R(r)\Theta(\theta)\frac{d\Psi(z)}{dz}\bigg|_{z=0} = 0.$$

At first glance it might seem that this would be satisfied if any of the three factors in the product were zero—true enough. But, the $R(r)$ and $\Theta(\theta)$ functions would have to be zero for all values of r and θ, respectively; in other words, one or the other of these functions would have be the "zero function," in which case the gauge pressure would be zero, $p = 0$, everywhere. This would be an uninteresting solution; it is the one for which there are no sound waves at all in the cavity. The only interesting way this boundary condition is satisfied is by requiring

$$\frac{d\Psi_l(z)}{dz}\bigg|_{z=0} = 0. \tag{17}$$

To apply this boundary condition the derivative of $\Psi(z)$ with respect to z must be taken. It is

$$\frac{d\Psi_l(z)}{dz} = -A_l l\sin(lz) + B_l l\cos(lz).$$

This must equal zero at $z = 0$, giving

$$\left.\frac{d\Psi_l(z)}{dz}\right|_{z=0} = 0 = -A_l l\sin(l(0)) + B_l l\cos(l(0)) = -A_l l(0) + B_l l(1) = B_l l,$$

in which we have used $\sin(l(0)) = 0$, and $\cos(l(0)) = 1$. This boundary condition is satisfied by requiring the coefficient B to be zero,

$$B_l = 0.$$

Substituting this result back into the general solution gives

$$\Psi_l(z) = A_l\cos(lz) + (0)\sin(lz) = A_l\cos(lz).$$

Boundary condition at the "ceiling"

There is also a boundary condition on the derivative of $\Psi_l(z)$ at the other end of the cylinder. Having applied the first boundary condition already, the derivative of $\Psi_l(z)$ is now

$$\frac{d\Psi_l(z)}{dz} = -A_l l\sin(lz).$$

This must equal zero at $z = b$.

$$\left.\frac{d\Psi_l(z)}{dz}\right|_{z=b} = 0 = -A_l l\sin(lb). \tag{18}$$

The only interesting way to satisfy this condition is by requiring

$$\sin(lb) = 0.$$

The zeros of the sine function are for arguments that are a multiple of π. This imposes a restriction on the allowed values of the separation constant, l. It must be that

$$lb = n\pi \qquad n = 0, \pm1, \pm2\ldots,$$

so that

$$l = \frac{n\pi}{b} \qquad n = 0, \pm1, \pm2\ldots. \tag{19}$$

Substituting this result into the general expression for $\Psi(z)$ gives

$$\Psi_n(z) = A_n\cos\left(\frac{n\pi z}{b}\right) \qquad n = 0, \pm1, \pm2\ldots, \tag{20}$$

where, in this form, it is now more natural to index the solutions, $\Psi(z)$, and the coefficients, A, with the integer index, n, instead of l.

Periodic boundary condition

The equation for $\Theta(\theta)$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + m^2\Theta(\theta) = 0,$$

has the same form as the one for $\Psi(z)$, so the form of the general solution can be represented in the same way:

$$\Theta_m(\theta) = \alpha_m \cos(m\theta) + \beta_m \sin(m\theta) = C_m \sin(m\theta - \phi_m),$$

with no restrictions on the separation constant, m, at the moment. (The second form follows from basic trigonometric identities with $C_m = \sqrt{\alpha_m^2 + \beta_m^2}$ and $\tan\phi_m = \beta_m/\alpha_m$.)

$\Theta(\theta)$ must satisfy a boundary condition, but it is different than the one imposed on $\Psi(z)$. It is required that the function for the pressure in the cylindrical cavity, $p(R, \theta, z, t)$, be single valued. Therefore, for the angular dependence of p, contained in the function $\Theta(\theta)$, a periodic boundary condition is imposed. This means the value of Θ at a given angle must have the same value when evaluated at that angle plus any integer multiple of 2π. This periodic boundary condition is satisfied if the separation constant m is restricted to integer values,

$$m = 0, \pm 1, \pm 2 \ldots. \tag{21}$$

With this condition, the solution for $\Theta(\theta)$ is

$$\Theta_m(\theta) = C_m \sin(m\theta - \phi_m) \qquad m = 0, \pm 1, \pm 2 \ldots, \tag{22}$$

with the unknown constants C_m and ϕ_m determined by initial conditions.

Boundary condition at the curved wall

The equation for $R(r)$ is

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - l^2 - \frac{m^2}{r^2}\right) R(r) = 0.$$

This is the well known Bessel equation. If you research this equation, it is also common to see it in the form for which the above is multiplied through by r^2:

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \left(\left(k^2 - l^2\right) r^2 - m^2\right) R(r) = 0.$$

To put this in an even more "textbook" form, define a new variable

$$\rho^2 \equiv \left(k^2 - l^2\right) r^2. \tag{23}$$

Substituting gives

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \left(\rho^2 - m^2\right) R(r) = 0.$$

As it now stands, this differential equation is awkward because, even though ρ and r are related, it has two variables in it. To put it in canonical form the goal is to transform the equation so that all occurrences of r are properly transformed to ρ.

The transformation is completed by considering the terms containing derivatives. Start with the term containing the first derivative:

$$r\frac{dR}{dr} = r\frac{dR}{d\rho}\frac{d\rho}{dr} = \frac{\rho}{\sqrt{k^2-l^2}}\frac{dR}{d\rho}\frac{d\left(\sqrt{k^2-l^2}r\right)}{dr} = \frac{\rho}{\sqrt{k^2-l^2}}\frac{dR}{d\rho}\sqrt{k^2-l^2} = \rho\frac{dR}{d\rho}.$$

The transformation of the first term in the differential equation proceeds in much the same way, and it would be found that

$$r^2\frac{d^2R}{dr^2} = \rho^2\frac{d^2R}{d\rho^2}.$$

This completes the required transformations. Substituting back into the differential equation gives

$$\rho^2\frac{d^2R(\rho)}{d\rho^2} + \rho\frac{dR(\rho)}{d\rho} + \left(\rho^2 - m^2\right)R(\rho) = 0. \tag{24}$$

This is the Bessel equation in canonical form. As a second order differential equation it has two independent solutions. One solution is the Bessel function of the first kind, $J_m(\rho)$. The value of the index m is called the “order” of the Bessel function. For example, $J_2(\rho)$, is the “second order” Bessel function of the first kind. In general, the Bessel function can have any order; the order is not restricted to integers. But because of the boundary conditions already imposed, in this problem, the index m, which appeared as a separation constant, has been restricted to integer values, so the Bessel functions in the developing solution will all be integer order. The integer order Bessel functions of the first kind form an important subset of Bessel functions, in part because they appear regularly in problems like this. Many plotting programs and math packages (such as Mathematica and Matlab) have the integer order Bessel functions “built-in.” They are really no different than other special built-in functions such as sine and cosine. Figure 1 plots the first three integer order Bessel functions for $0 \leq \rho \leq 20$.

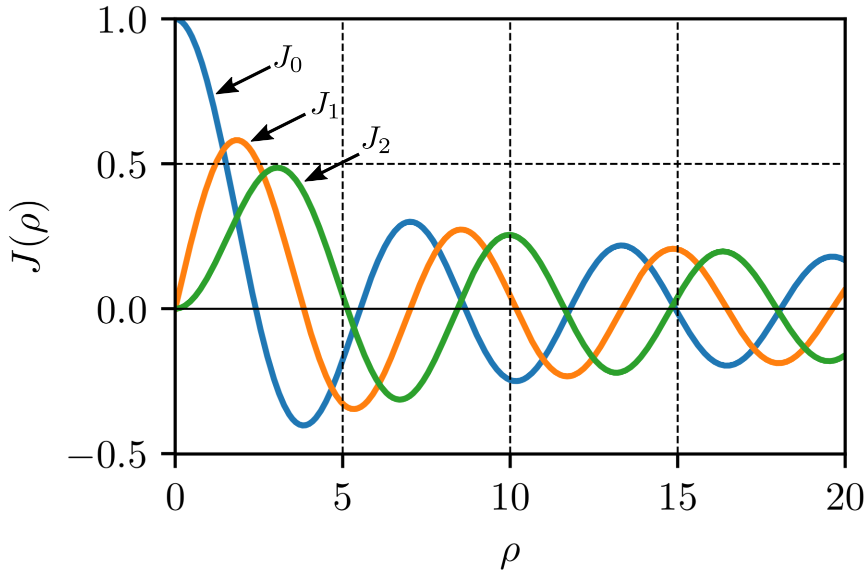

Figure 1: The first three integer order Bessel functions, J_m, of the first kind.

If m in the Bessel equation is restricted to integer values, then the second independent solution is called the Bessel function of the second kind, $Y_m(\rho)$. Figure 2 plots the first three integer order Bessel functions of the second kind for $0 \leq \rho \leq 20$.

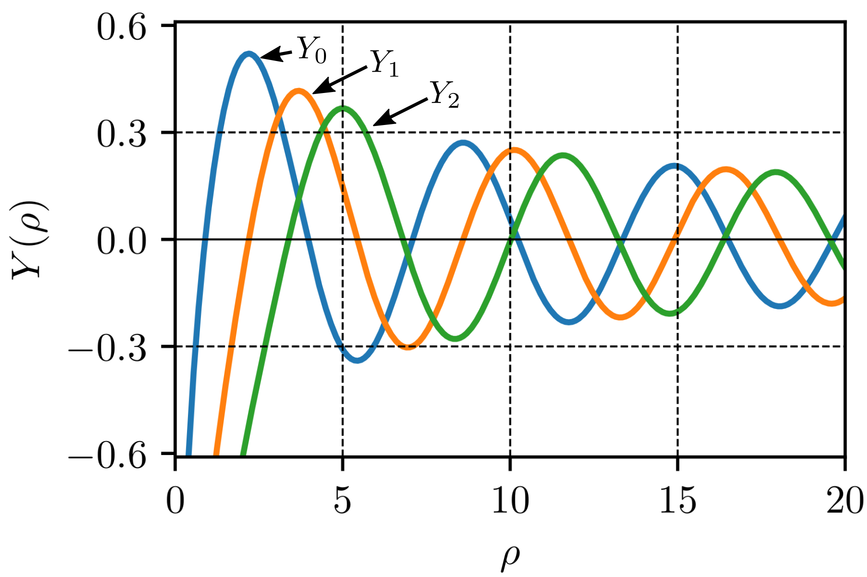

Figure 2: The first three Bessel functions, Y_m, of the second kind.

Transforming back from ρ to the radial variable r, the two independent solutions are expressed as

$$J_m(\sqrt{k^2 - l^2}\, r) \qquad \text{and} \qquad Y_m(\sqrt{k^2 - l^2}\, r).$$

Any linear combination of these two independent solutions is also a solution of the Bessel equation. Therefore, the most general solution is

$$R_m(r) = D_m J_m(\sqrt{k^2 - l^2}\, r) + E_m J_m(\sqrt{k^2 - l^2}\, r), \tag{25}$$

in which D_m and E_m are constant coefficients, yet to be determined.

As Fig. 2 hints, all Bessel functions of the second kind go to negative infinity as r goes to zero. In our problem, the presence of these functions in the solution would be unphysical, as it would give a negative infinite pressure along the longitudinal axis of the cylindrical cavity. Therefore, it must be the case that $E_m = 0$ for all m. For the cylindrical cavity, solutions are restricted to Bessel functions of the first kind, which are all finite at $r = 0$. This gives

$$R_m(r) = D_m J_m(\sqrt{k^2 - l^2}\, r). \tag{26}$$

There is a boundary condition that affects $R(r)$. The gradient of the pressure normal to every internal surface must vanish, include the curved side surface. The relevant gradient for this case is the directional derivative in the r direction. Expressed as a derivative of the product function, the r component of the gradient is

$$\frac{\partial p(r,\theta,z)}{\partial r} = \frac{\partial\,[R(r)\Theta(\theta)\Psi(z)]}{\partial r} = \Theta(\theta)\Psi(z)\frac{dR(r)}{dr}.$$

The boundary condition says this must equal zero at the curved wall of the cavity, $r = a$:

$$\Theta(\theta)\Psi(z)\frac{dR(r)}{dr}\bigg|_{r=a} = 0.$$

As with this same boundary condition applied to the floor and ceiling of the cavity, the only interesting way to satisfy this condition is as a restriction on $R(r)$. Therefore, it is required that

$$\frac{dR_m(r)}{dr}\bigg|_{r=a} = 0.$$

Applying this to the Bessel function solutions, it becomes a requirement that

$$\frac{dJ_m(\sqrt{k^2 - l^2}\, r)}{dr}\bigg|_{r=a} = 0.$$

To make the notation simpler, a prime will be used to denote the first derivative of J with respect to r. The boundary condition expressed more compactly is thus

$$J_m'(\sqrt{k^2 - l^2}\, a) = 0. \tag{27}$$

To summarize, the radial part of a product function solution to the wave equation must be an integer order Bessel function of the first kind, and the derivative of said Bessel function with respect to the radial variable, r, when evaluated at $r = a$, must be zero. As we will elaborate below, this condition is not met for every sound wave frequency, only for a discrete set.

There are an infinite number of integer order Bessel functions, and each has a derivative. The derivative of an integer order Bessel function, J'_m, has an infinite number of zeros. Figure 3 plots the first three integer order derivative function for $0 \leq \rho \leq 20$, and highlights some of the zeros.

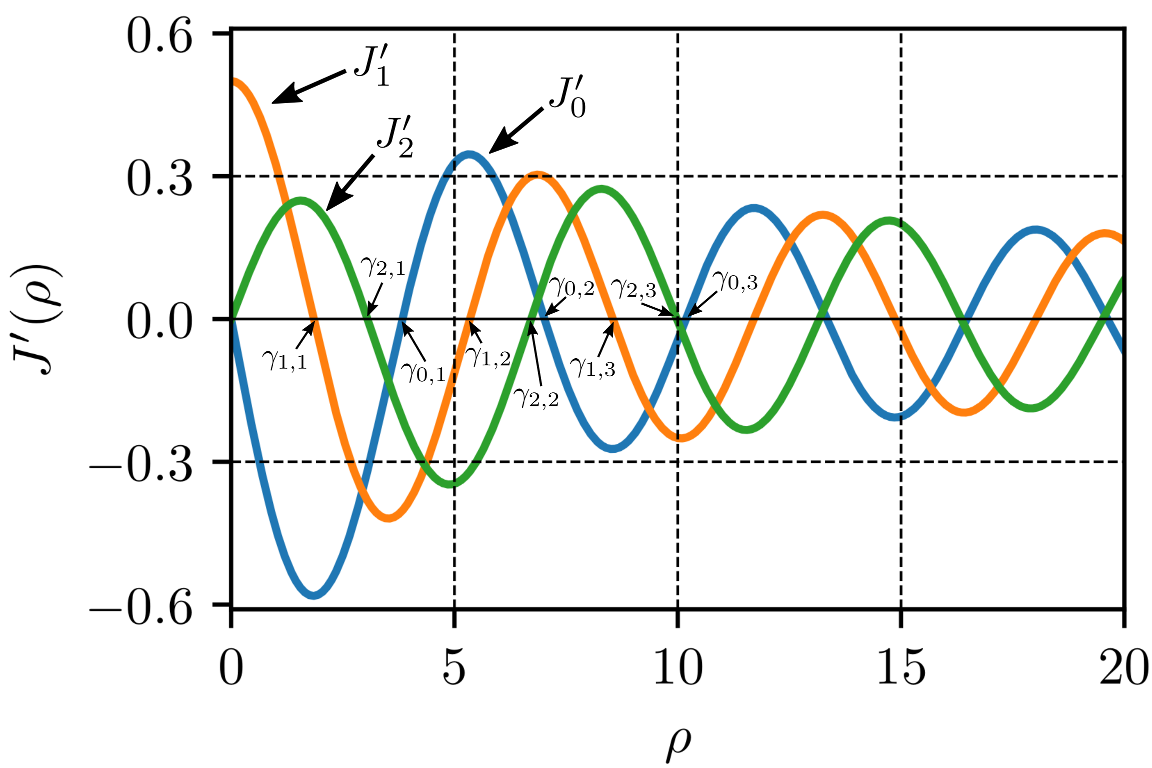

Figure 3: The derivatives, J'_m, of the first three integer order Bessel functions of the first kind. Some of the zeros, $\gamma_{m,s}$ are labeled.

"The zeros" of this function are those values of the argument (ρ in the graph) that give $J'_m = 0$. Since there are an infinite number discrete zeros for each of an infinite number of integer order J'_m, it is best to identify each zero with two integer indices: m for the order of J'_m, which starts counting from $m = 0$; and s to mark which zero for a given order. The index s starts its count from $s = 1$. Let's use the symbol $\gamma_{m,s}$ to name the zeros of J'_m. For example, $\gamma_{2,3}$, is the third zero of the derivative of the second order integer-order Bessel function of the first kind, J'_3. An arrow points to this zero on the graph above. You can read off from the horizontal scale that the argument that gives this third zero is approximately

$\rho = 10$; that is, $\gamma_{2,3} \approx 10$. (Make sure that makes sense. We are using the symbol $\gamma_{m,s}$ for those special values of ρ—the argument of J'_m—for which it evaluates to zero.) The zeros of J'_m can be found tabulated on the web at various math sites, such as the Wolfram Mathworld page

`https://mathworld.wolfram.com/BesselFunctionZeros.html`

There you would find, to five significant figures, $\gamma_{2,3} = 9.9695$. The zeros can also be generated in many math packages, such as with Python's Scipy library.

At times we have been using the symbol ρ generically, as a single letter symbol for the argument of J_m and J'_m. But the development of our solution of the wave equation, through the method of separation of variables, has driven us to something more specific. Because of the form of the Bessel equation, resulting from the separation of variables, we have been led to express the argument of J_m and J'_m, for this problem, as $\sqrt{k^2 - l^2}r$, in which r is the radial variable and k and l are separation constants. Therefore, to say that the derivative of the integer-order Bessel function of the first kind must be zero at $r = a$ to satisfy a boundary condition is the same as saying it must be the case that

$$\boxed{\sqrt{k^2 - l^2}a = \gamma_{m,s}.} \tag{28}$$

This is a significant result. It will soon lead directly to an equation for the frequency spectrum of normal modes. As we will show by analyzing the time dependent factor, $T(t)$, of the product function, k is directly related to the wave frequency. As a result, only very specific frequencies, the normal mode frequencies, will satisfy the above equation for a given zero, $\gamma_{m,s}$; thus, the spectrum.

Time dependence

The time dependent factor, $T(t)$, of the proposed product function, $p(r, \theta, z, t) = R(r)\Theta(\theta)\Psi(z)T(t)$, satisfies the following second order differential equation

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0.$$

One way of expressing the general solution is

$$T(t) = E\sin(\omega t + \delta), \tag{29}$$

in which E and δ are constants to be evaluated from initial conditions.

Given this solution, we see that, as suggested earlier, it is correct to associate the parameter ω with the wave's angular frequency (radians per unit time). It is related to the frequency (cycles per unit time) in the usual way,

$$\omega = 2\pi f,$$

and it was defined earlier, by a relationship with the separation constant, k, and the speed of sound, c, as

$$\omega \equiv ck,$$

by which k can now be interpreted as a wave number. We can now express k in term of f, the frequency, by

$$k = \frac{\omega}{c} = \frac{2\pi f}{c}.$$

The normal mode frequencies

We are now in a position to put everything together and develop an equation for the normal mode frequencies—the frequency spectrum of supported modes in the cylindrical cavity.

From the boundary condition on the function $R(r)$ it was found that the following condition must be satisfied

$$\sqrt{k^2 - l^2}a = \gamma_{m,s}.$$

Solving this for k gives

$$k = \sqrt{\left(\frac{\gamma_{m,s}}{a}\right)^2 + l^2}.$$

In terms of the frequency this is

$$\frac{2\pi f}{c} = \sqrt{\left(\frac{\gamma_{m,s}}{a}\right)^2 + l^2}.$$

Solving for f gives

$$f = \frac{c}{2\pi}\sqrt{\left(\frac{\gamma_{m,s}}{a}\right)^2 + l^2}.$$

Recall that the the separation constant, l, was restricted by the boundary conditions to the values

$$l = \frac{n\pi}{b} \qquad n = 0, \pm 1, \pm 2 \ldots,$$

in which b is the height of the cylinder. Substituting this gives

$$f = \frac{c}{2\pi}\sqrt{\left(\frac{\gamma_{m,s}}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}.$$

In this expression m is restricted to integer values starting from 0, s is restricted to integer values starting from 1, and n is restricted to integer values starting from 0. Any combination of the three gives an allowed normal mode frequency. Therefore, the spectrum of normal mode frequencies is identified by three indices, and is

$$\boxed{f_{n,m,s} = \frac{c}{2\pi}\sqrt{\left(\frac{\gamma_{m,s}}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}.} \tag{30}$$

These should be calculated, tabulated, and then compared to the observed spectrum of frequencies.

The general solution

It was proposed that some solutions of the wave equation would take the form of a product function

$$p(r, \theta, z, t) = R(r)\Theta(\theta)\Psi(z)T(t).$$

An expression for each factor was found. Summarized here, with indices on the constant coefficients and phase angles suppressed, they are:

$$R(r) = DJ_m(\sqrt{k^2 - l^2}\, r) \tag{31}$$

$$\Theta(\theta) = C \sin(m\theta - \phi) \tag{32}$$

$$\Psi(z) = A \cos\left(\frac{n\pi z}{b}\right) \tag{33}$$

$$T(t) = E \sin(\omega t - \delta) \tag{34}$$

When the product function is written out, the constant coefficient pre-factors (D, C, A, and E) will form a single pre-factor that will be labeled with the single letter P. To write the solution in terms of the normal mode frequencies these substitutions will also be made:

$$k \to \frac{2\pi f_{n,m,s}}{c} \qquad \omega \to 2\pi f_{n,m,s}.$$

In addition, the following substitution will be used:

$$l \to \frac{\pi n}{b}.$$

There is a distinct product function for each normal mode frequency. All are possible solutions of the wave equation which means all are supported by the cylindrical cavity. The general solution to the wave equation is a linear superposition of all normal mode product functions. At any given moment, the "amount" that a particular normal mode contributes to the actual wave in the cavity, and the mode's corresponding phase angles ϕ and δ, depend on the initial conditions, the driving source if there is one (such as a speaker), and how rapidly the mode damps out. Therefore, in the superposition, there is a distinct coefficient, P, and phase angles, ϕ, and δ, for each normal mode. As a result, all of these factors need to be indexed. The general solution can be written:

$$p(r, \theta, z, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} P_{n,m,s} J_m \left(\sqrt{\left(\frac{2\pi f_{n,m,s}}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\, r \right) \sin(m\theta - \phi_{n,m,s}) \cos\left(\frac{n\pi z}{b}\right) \sin(2\pi f_{n,m,s} t - \delta_{n,m,s}). \tag{35}$$

It would be interesting to plot $p(r, \theta, z, t)$ in various ways, for single modes, or for a superposition of a small collection of modes. A two dimensional "surface" plot of the pressure across a transverse cross-section, possibly animated, would be especially enlightening. Give it a try!

The end.