4 Driven Oscillations

If you've worked though parts 1 - 3 of this manual, you'll have seen lots of oscillatory behavior of your Torsional Oscillator, but all of those oscillations were *transient*. That is, you excited the system, by one means or another, only at the beginning of the motion, and then let the system evolve on its own, without further excitation. Now it's time to consider continually-driven oscillations, in which there's an ongoing, time-dependent, torque applied to the system.

4.0 Applying the simplest drive

These investigations make use of the Helmholtz-coil system as a driver, so you have to give up using it as a velocity sensor. The torques are most easily modeled if you stay in the small-angle regime, to evade some complications you studied in section 2.2. The driven-oscillator investigations are simplest if you use a small amount of v^1-law of magnetic damping. Finally, for reasons not immediately obvious, the simplest drive torques are of very special form: they are *sinusoidal* in time.

You will need some signal generator or other electronic source of sinusoidal voltages, capable of having its frequency conveniently and continuously adjusted in the range 0.1 - 10 Hz. If your generator has a 50-Ω output impedance, it can be directly connected to the Helmholtz coils of the Torsional Oscillator, and it'll drive a current through the coils. An amplitude setting of 5 Volts would drive a current of amplitude 0.1 A though a short circuit, which is about what the coils 'look like' to such a generator. That will be ample to drive oscillations of adequate size. Finally, it's ideal to have a look at the 'drive signal' and the oscillator's 'response' simultaneously. A 2-channel oscilloscope or equivalent is a great way to do so. If you want one channel to display the drive waveform, you can route the generator current first through the coils, and then (via a jumper) through the 1.0-Ω resistor in your Oscillator. Across that resistor you'll get a voltage signal which is a surrogate for the drive current, with a known scale factor.

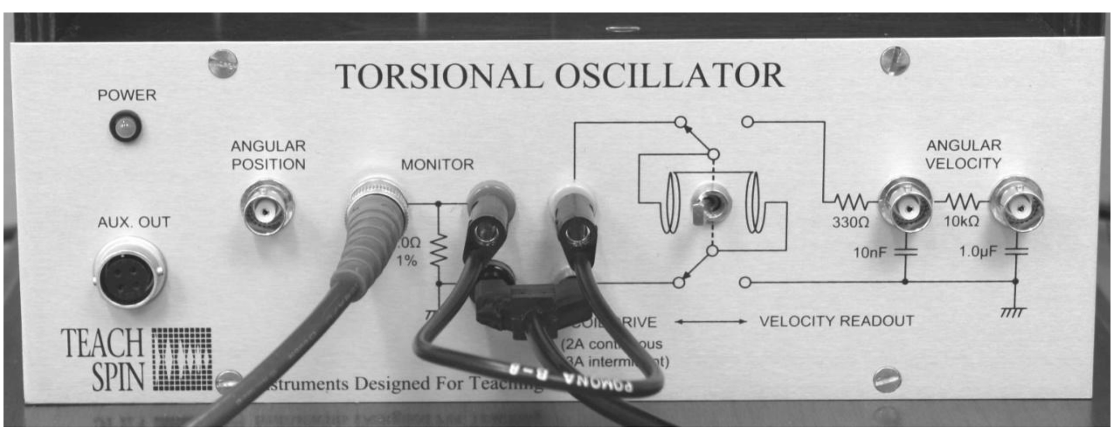

Fig. 4.0: One way to send a generator signal through coils and 1.0 Ω resistor

You will probably want to trigger your 'scope on the drive signal, and now you should see cause and effect, drive and response, quite clearly. Before making any quantitative measurements, note a few qualitative facts:

- after any change in the settings, there's a time delay before you get to a 'steady-state response';
- during that time delay, of 'transient behavior', life is complicated, but the steady-state response is rather simple;
- if the damping is rather larger, then the delay until you reach 'steady state' is rather smaller;
- once you're in steady-state, the response to a sinusoidal drive (of any amplitude, and any frequency) is *also* sinusoidal
- the steady-state sinusoidal response is *not* at the oscillator's 'natural frequency', nor even at its 'damped frequency', but at the *generator's* frequency (and how can you tell this, visually?).

All of these are lessons of enormous generality, and apply across the board, in all sorts of areas and examples of physics, and require only the assumption, or approximation, of a 'linear system'. So learn these qualitative lessons well, as they can give you intuitions of wide utility.

4.1 Resonance in driven oscillations

This section takes up the most 'glamorous' feature of driven oscillations in a damped simple harmonic oscillator -- the phenomenon of *resonance*. To perform these investigations, use the oscillator driven as described in section 4.0, and with a moderate level of magnetic damping in place. You might want to aim for a 'Q' of about 8, and you may want to check section 3.3 for a quick way to estimate the Q of an oscillator. Finally, pick a rather small level of the drive current, setting its amplitude to a value which (applied near zero frequency) would give the rotor a deflection of perhaps 0.05 radians.

Now you're ready to see resonance -- you'll want to monitor the 'drive' and 'response' waveforms in real time. The goal is to make the frequency of the drive waveform the independent variable, holding everything else (particularly, the *amplitude* of the drive) constant. The dependent variable is the amplitude of the steady-state response. Notice that after any change in the drive conditions, you'll have to wait for several Q's of cycles of the drive for the response to settle to steady state -- this wait is a motivation for not making the Q of the system much higher.

Make a plot that shows the amplitude of steady-state response as a function of drive frequency, taking data points at a density suitable to the physics you're investigating. You might take points of higher density in the 'resonant region', and it's your job to characterize not only that region, but also the below-resonance and above-resonance regions.

You should see a plateau, a peak, and a drop-off in various regions of frequency space, and you'll learn to characterize all three. Locate in frequency the peak of the resonance, *and* those points where the response is smaller, by a factor of $\mathbf{1/\sqrt{2} = 0.707}$, than the response at its peak. (That's because the *square* of the oscillator's response is down to *half* its maximum value at these points.)

You might label the resonance by three frequencies, $f_<$, f_m, and $f_>$, for the lower 'half-maximum', the peak, and the higher 'half-maximum' points.

Form the quotient $\mathbf{f_m/(f_> - f_<)}$, which is a dimensionless measure of the relative narrowness of the resonance, and which is expected to be quite nearly equal to the Q of the oscillator. Before changing anything else about the oscillator, disconnect the drive, pump it up to moderate amplitude by hand, and acquire a record of its freely-decaying damped oscillations, since this data gives another, and instrumentally rather distinct, method for finding the Q -- see section 3.3 for details.

Now change the damping to a larger value, perhaps aiming for a Q more like 4 than 8. Repeat the scan over the drive frequency, and plot your data on the same graph you created before. Label the data you obtained under more-damped, vs. less-damped, conditions. You are ready to cure yourself of a very common illusion -- notice that decreasing the damping does *not* really 'narrow the peak' in the usual sense. Said another way, lowering the damping everywhere *raises*, and nowhere *lowers*, the response.

Nevertheless, compute the quotient $\mathbf{f_m/(f_> - f_<)}$ for the data taken at greater damping, and find that this measure of the peak's narrowness has in fact decreased.

What you've done is to survey a resonant peak, and to locate in the traditional ways the peak's center and also its 'width' -- here, adopting the 'full width at half-maximum energy criterion. You've also seen that the peak response of the oscillator occurs near a characteristic frequency, which you should recognize as familiar. Since you know how to use brass quadrants to change the oscillator's properties, you should be ready to confirm that the resonant peak's location can be moved too. You've also seen a second manifestation of the 'Q' of an oscillator, expressed this time in controlling the width of the resonance.

For yet another manifestation of the Q, look at the ratio

(response's amplitude at peak) **/** (response's amplitude near zero frequency) ,

which might be called the 'amplification factor' obtained at resonance. It too should be approximately given by the Q-factor of the oscillator.

4.2 Phase shift in driven oscillations

The data on resonance that you've taken in section 4.1 can all be modeled theoretically, but before going on to that exercise (in section 4.3), you ought here to learn to recognize a less-well-known, but very useful feature that also occurs in the neighborhood of resonance. For taking this data, you might want to use an oscillator with Q set to 4 or even lower -- that's to keep the glamorous 'amplification' phenomenon at resonance from distracting you from this new effect.

The effect you want to look at is the *phase shift* between the drive waveform and the steady-state response waveform. To see the meaning of this, it's best to pick a drive frequency well below resonance, and to get a dual-trace 'scope display of the drive, and the response. You're looking 'in the DC limit' in which you can think of the drive as taking on a succession of independent static values of drive current, and the oscillator responding with a succession of resulting static values of response. In other words, the response waveform ought to be (very nearly) in phase with the drive. (If it appears 180° out of phase, use the 'invert' function on your 'scope to view the in-phase behavior described above.)

You might record the drive and the response in some way that lets you see that there is a systematic, if small, time delay between the drive and the response. You can measure this time delay between a zero-crossing of the drive and the next zero-crossing of the response, and then compute the official value of the phase shift from the quotient

$$\text{(phase shift)} \,/\, (2\pi \text{ radians}) = \text{(time delay)} \,/\, \text{(period of the drive waveform)} .$$

It's also conventional to convert the phase shift to degrees. Of course there are lots of other ways to compute this phase shift, for example from least-squares fits or other transformations of the data waveforms.

Once you've learned to measure phase shift by your favorite method, take the data needed to find the dependence of the phase shift on the drive frequency, and make a plot that shows the phase shift's variation with frequency. The phase shift ought to do something relatively dramatic near a particular frequency, whose value you ought to recognize. That dramatic behavior ought to become *more* dramatic if you lower the damping, i.e., raise the oscillator's Q. For at least one setting of the Q, find the locations in frequency of the 45°, the 90°, and the 135° phase-shift points. That's because the theory claims that the location of the 90° phase-shift point *is* the natural frequency of the oscillator, and that the quotient

$$f_{90}/(f_{135} - f_{45})$$

will give yet another measure of the Q of the oscillator.

Precisely because the phase shift is going rapidly through the value of 90° at resonance (whereas the amplification factor is a maximum there, and therefore locally independent of frequency to first order), the location of the 90° phase-shift point is a very useful operational *definition* of the location of the oscillator's resonant frequency.

4.3 The transfer function of an oscillator

You've now measured the amplitude, and the phase, responses of your Torsional Oscillator, as driven by a sinusoidal torque. These turn out to be two facets of an amazingly general property of any driven 'linear time-invariant' system, called the transfer function. The claim is that such a system, driven by a stimulus or cause

$$d(t) = D \cos [\omega t]$$

will exhibit a steady-state response, r(t), given by

$$r(t) = D\, |T(\omega)| \cos [\omega t - \phi(\omega)]$$

Notice there are a multitude of claims implicit in these forms: in particular, the response is claimed, under this sinusoidal drive, *also* to be sinusoidal, and of the *same* frequency as the drive, and to have a size *linearly proportional* to the size of the drive. Notice that the constant of proportionality is itself a function of frequency, and it's called 'the magnitude of the transfer function'. The entire response of the system, magnitude and phase, can be encapsulated in a single complex-valued function, called the transfer function, given by

$$T(\omega) = |T(\omega)| \exp[i\, \phi(\omega)]$$

This form of the transfer function presupposes taking the real part of complex functions to describe drive and response, and also assumes the physicist's conventional use of time dependence of the form $\exp(-i\omega t)$. At one level, there is no more to be said -- in the previous two sections, you've measured the only two ingredients of this transfer function in your 'amplitude response' and 'phase response' investigations. You could, of course, lay out graphically the 'locus' of points that $T(\omega)$ traces out in the complex plane, as ω varies from near DC to frequencies well above resonance.

At another level, there is a lot more to be done. Specializing to the particular case of the Torsional Oscillator, there are ways to factor units and dimensions out of the transfer function, leaving behind a dimensionless 'core' form that generalizes to all sorts of other systems in physics. Here's one way to do this:

- you've been thinking of torque as the 'input' to the system, but you have things calibrated well enough that instead of the torque, you could think of the current producing the torque, or even the monitor voltage generated by running this current through the 1.0 Ω resistor. Thus $V_{mon}(t)$ is a voltage surrogate for the drive.

- you've also been thinking of the angular-position coordinate as the 'output' of the system, but again you have a transducer calibrated so that instead of the angle, you could think of the output voltage reflecting it, $V_{pos}(t)$, as the response of the system.

- if your Oscillator is 'linear and time-invariant', it follows that $V_{mon}(t)$ as a drive, and $V_{pos}(t)$ as a response, have a transfer function connecting them -- and it'll be dimensionless, as it maps a voltage to a voltage.

- furthermore, this voltage-to-voltage mapping has a well-defined value in the 'DC limit', i.e., as the frequency approaches zero. In practice, this means at a frequency well below the resonant frequency. There the magnitude of the transfer function can be measured by putting an actual DC value of current into the drive coil, reading a steady V_{mon}, and then seeing what steady value of V_{pos} results. (You might have to deal with the 'DC offsets' implicit in a real apparatus.)

- factoring out as a constant this DC-limiting value, you're left with the core of a transfer function which is not only dimensionless, but which has, in the DC limit, a magnitude of one, and zero phase shift. The behavior of this core transfer function away from zero frequency contains all the interesting frequency dependence.

Now you should be ready to work out the *theory* of your oscillator, and to make a <u>prediction</u> for the transfer function. You'll need to introduce various physical parameters, and set up the appropriate differential equation, and learn why it's an inhomogeneous differential equation, and how to form the 'particular solution' that is the whole of the steady-state response. And then you should work out the dimensionless form of the transfer function, and the form of it having DC-limiting value of 1, and show that the core of it depends on only three things: your sole independent variable, the frequency, ω, (or $f = \omega/2\pi$ instead), and just two parameters, constants that characterize your oscillator, namely its 'natural frequency', ω_0, (or $f_0 = \omega_0/2\pi$ instead), and its dimensionless damping coefficient, γ.

Of course, once you have a theoretical prediction of this form, you can find the ω_0 and γ values that best describe your data, by fitting to your amplitude- and phase-response data. Or, you can be more ambitious, and extract ω_0 and γ from the damped-oscillatory-decay data of the sort you modeled in section 3.2. Such data will *also* produce measured values for ω_0 and γ, and these would allow you to make a (zero-free-parameters!) *prediction* of both the magnitude and the phase of the transfer function, before you ever turned on your sinusoidal generator. See if you can make a single pair of numbers, ω_0 and γ, fully describe both the undriven, free-decay data, *and* the magnitude and the phase pieces of the transfer function of the sinusoidally driven oscillator.

4.4 Non-sinusoidal periodic drive

You might wonder about the claim that the transfer function fully describes the response of a system, when after all it seems merely to predict the response to a periodic and sinusoidal drive. What about a drive function that's periodic, but not sinusoidal? (Think of changing your signal generator from sine-wave to triangle-wave excitation.) Or what about a drive function that's not even periodic in time? (Think of some random time series, like a temperature-vs.-time graph, as a signal -- putting it into your Oscillator as a stimulus would surely give *some* response.)

The true power of the claim of linearity shows up in these cases via the wonders of Fourier's theorem. Here's how it works -- just as all molecules are made from a rather short list of atoms, so any and all drive signals can be written as the sum of a weighted list of sinusoids, i.e., they have Fourier-series, or Fourier-integral, representations. Now the big payoff of linearity is the Principle of Superposition:

- the response to a drive which can be written as a sum of functions is given by the *sum* of the responses to the individual terms in the drive function.

And the individual terms in the drive, in this view, are sinusoids, each of a particular frequency. The transfer function, evaluated for magnitude and phase information at that frequency, gives the predicted response to that term in the Fourier sum. Similarly for every term; and the resulting response is just the sum of all those individual responses.

This takes some practice fully to appreciate, but you should start on the empirical side. You might by this time think it's obvious and natural that the response to a sinusoidal drive is a sinusoidal response. But check for yourself that the response to a triangle-wave drive is *not* a triangle-wave response! In particular, pick a moderate-Q system (say, a Q of 5 to 10) and excite it with a triangle-wave drive, with a frequency picked to lie close to the system's natural frequency. What do you see? Or, for more drama, excite the same system with a square wave of the same frequency -- what do you get? More curiously still, excite the system with a square wave of one-*third* the previous choice of frequency -- what do you get?

The answer to all of these questions, certainly in qualitative terms, is best found from a Fourier-series point of view. The method works in quantitative detail, too, if you care to work that out. But it's indispensable to have in your mental tool-kit the 'frequency-domain view', to complement the 'time-domain view' you've found natural so far. What *is* a triangle wave, in the frequency-domain view? What are its terms, and what can you say qualitatively about the response to those terms, taken individually? Which will dominate? What shape will that response have? You have before you the tools that will make it natural to think in this new way.

4.5 Time- and frequency-domain views

If you have played with non-sinusoidal drive of your Torsional Oscillator, and have begun to learn all the insight that can be gained using a frequency-domain point of view, you might be willing to give up the time-domain point of view you first found natural. Here's an exercise to show that the two viewpoints are actually complementary, and co-exist fruitfully, in making the behavior of a system understandable.

Set up your Torsional Oscillator as a moderate-Q system, perhaps using a Q of order 10. Again, excite it with a non-sinusoidal but periodic drive function -- you might try a 'square wave', chosen because of its considerable harmonic content (to use a frequency-domain vocabulary). This time, look not at the detailed waveform-in-time of the response, but simply measure the rms value of the response function. (In practice, this means exciting the system, waiting for steady-state conditions to obtain, recording the response function over one or more full cycles of the drive, and then forming the root-mean-square value of that response.) Now the question is -- keeping the drive's size constant, and varying its frequency, how does that rms value of the response vary with that choice of drive frequency?

In taking the data, you will want to pick excitation frequencies thoughtfully. Your intuition should suggest you need to pay attention to the three regimes of below-, near-, and above-resonance. But in this case, you'll find you need to take extra data where the drive frequency lies near *sub*-multiples of the system's resonant frequency. A frequency-domain view will help you understand why -- and why the 1/3, 1/5, . . sub-multiples repay more effort than the 1/2, 1/4, . . . sub-multiples.

After you get the idea of what will turn out to be a complicated graph, you should be able to 'narrate' the result using frequency-domain ideas. But just before you convert entirely to this viewpoint, you might look past the rms value to the response waveform itself, viewing it in the time domain again. Since your complicated graph taught you there was a lot of structure in the low-frequency regime, set the drive to a rather low frequency, and see what occurs that's so special near those <u>odd</u> sub-multiples of the resonant frequency. Go lower still in frequency, and suddenly you'll see things from another viewpoint, one best described as a series of excitations of the *step response* of your Oscillator. That puts you back in a time-domain vocabulary, and teaches you the value of having *both* time- and frequency-domain viewpoints as part of your mental equipment.

4.6 Transient behaviors

You've been looking at the behavior of a driven oscillator for some time now, and you have fixated (or been led to fixate) on the steady-state response. There are good reasons for this -- in particular, that response is independent of those pesky initial conditions of the system. But there's lots of good physics in the 'transient behavior' of the system, all the response that happens before the system settles to its steady-state.

In studying such behavior mathematically, you have perhaps been told to solve the problem in two parts. First, you find the general solution to the un-driven problem: the homogeneous differential equation. Then, you find the particular solution to the driven problem -- the *in*homogeneous differential equation. Then you write the whole solution as the sum of these two parts, and finally you apply initial conditions to that entire sum, to fix the last of the unknown constants. Whew! But the implication is that the solution, for a good while after the launch from initial conditions, is the sum of two parts, which oscillate at possibly different frequencies. The particular solution oscillates only at the drive frequency, and it has constant amplitude. But the general solution, which is important until it decays away, will oscillate at the 'damped frequency'

$$\omega_d = \omega_0 \sqrt{(1 - \gamma^2)} \ .$$

So the total solution will be the sum of two terms of distinct frequency -- and that sum should therefore display 'beat phenomena'.

To move from the mathematical to the physical world is now easy for you. Adjust your Torsional Oscillator for rather low damping, so that the Q might be of order 20 to 40. Stop the rotor movement by disconnecting the drive signal and damping it carefully by hand. You'll want a way to connect the drive suddenly at your t=0 point. Choose a drive frequency which lies in the vicinity of the system's 'natural frequency', and choose quite a small amplitude -- you'll be driving a high-Q system near resonance, and the steady-state response is subject to a large 'amplification factor' because of resonance.

Start by setting the drive frequency right at your best estimate of the system's damped frequency, and record the whole history of the Oscillator's response. You should see an oscillation of growing amplitude -- growing linearly at first, and later leveling off at a steady-state amplitude. Once you've understood that, both physically and mathematically, try the same experiment, only this time choosing a drive frequency that is (say) 5% *off* from the system's natural frequency. You should see short-term behavior that is *nearly identical* to the previous case, and then moderate-term behavior which shows dramatic 'beats', before you finally get to the steady-state behavior. What sets the period of these 'beats'? What does it take for them to have maximal amplitude? Repeat with a drive frequency that's 2%, or 1%, or -2%, etc. off from the natural frequency. In each case, look at the short-, moderate-, and long-term behavior of the system to see if your mental vocabulary contains the concepts needed to explain what's changing, and what's not. And don't forget to tear your attention away, from time to time, from your data-recording system, to have a look at your Oscillator's copper disc, to see it undergo the complicated motion that a driven system can (transiently) possess.

4.7 Projects

Here is a collection of projects generally unified under the theme of driven oscillators.

4.7.1 The oscillator as a low-pass filter

If you've completed section 4.3, you know how to compute (in magnitude and phase) the theoretical transfer function for your Torsional Oscillator. While the high-Q version of the transfer function displays glamorous features around the resonant frequency, there is merit in plotting the transfer function, in magnitude and phase, in some rather low-Q situations. In fact, certain choices of Q, or the damping

$$\gamma = 1/(2Q) ,$$

give responses famous enough to have been given names.

Work out an analytic expression for the *magnitude* of the transfer function, and plot it for a variety of rather low Q values (i.e., for some rather *large* γ values). You should notice that the choice

$$\gamma = 1/\sqrt{2}$$

has a property that renders it special. In particular, you get a transfer function whose magnitude is 'maximally flat' in the low-frequency regime. Of course it shares features with the transfer function for any γ-value: it has a value going to 1 in the low-frequency limit, and it has a drop-off of the form $(f_0/f)^2$ in the high-frequency limit. The $\gamma = 1/\sqrt{2}$ value gives the special feature of low frequency response of 'maximal flatness', and this choice is named 'Butterworth response'. Your system's behavior can be viewed as a low-pass filter, in that low frequencies at the drive input are 'passed' to the output, while high frequencies are (somewhat) blocked from appearing at the output.

Next, work out an analytic expression for the *phase shift* of the transfer function, and plot it too for a variety of rather low Q values (i.e. for some rather large γ values). You should notice that the choice

$$\gamma = (\sqrt{3})/2$$

has a property that renders it special. In particular, you get a transfer function whose phase shift is 'maximally linear' in the low-frequency regime. This turns out to render its 'group delay' for a pulsed waveform as frequency-independent as possible, and this choice is called 'Bessel response'.

Note that neither of these responses is critically damped, since both require the choice $\gamma < 1$. Since both Butterworth and Bessel response are *under*-damped, it follows that their *step* responses will show some overshoot. You can work this out analytically, and display it graphically too. But as you learn how to recognize the effects of varying the damping, you have the opportunity to set up your actual Oscillator to a chosen γ-value, and to get it to the point of displaying actual Butterworth, or a Bessel, filter response.

Once you have it set up in your chosen way, it's up to you what to do with it -- you can send in sinusoids, and confirm the filtering action via the amplitude response; or you can send in step-functions, and look for the overshoot in the time-domain response; or you can find a way to make 'wave packets' and measure their phase, or their group, delay in passing through.

This is just an introduction to the well-developed theory of filter design, and you've now worked through 'two-pole filters', which is the most you can do with a single oscillator as a filter element. Can you think of a way to turn your Oscillator into a narrow-band *band-pass* filter?

4.7.2 Life beyond linearity

Everything thus far in section 4 has had you using the magnetic dampers, whose v^1-law of damping offers the remarkable mathematical property of linearity. One of the many consequences of linearity is that scaling the input will scale the output in the same proportions. Within limits, your Oscillator, under magnetic damping, shares this property. In this section, you'll get to see some evidence of *departures* from this property of linearity, under magnetic and other damping laws.

You can see the consequences of linearity in some of the simplest properties of the Oscillator. The simplest of all is to drive the system with a sinusoid, of frequency chosen to be on-resonance, and with amplitude to be adjusted. Then you need only look at the output, which (under the assumption of linearity) will also be a sinusoid, with some output amplitude. A plot of output amplitude as a function of input amplitude is a test of linearity as a system property -- in particular, a 'linear system' will give a straight line for this plot.

Try setting up your Oscillator with moderate Q, of perhaps 5 or so, and use one or another criterion to find the resonant frequency and operate there. Try a wide variety of choices for the amplitude of the drive, and for each choice, wait for steady-state operation, and record the amplitudes of input and output, i.e., drive and response, waveforms. Make a plot, and find a departure from a straight line. Why? Look back to section 2.3 and realize that your torque drive gives torque that is proportional to the drive current, but with a proportionality 'constant' that is actually constant only for small-amplitude motion of your oscillator. Nevertheless, you should find a small-enough-drive regime in which you *do* see a straight-line plot, as system linearity predicts.

Now change to either the v^0- or v^2-law of damping, and life will get more complicated. For these cases of damping, it's not even obvious how to *define* the 'resonant frequency', since the location of maximum response can actually vary with choice of drive amplitude. But you can make *some* sensible choice that will give some version of resonance in some regime of drive, and then at that fixed frequency, you can again plot amplitude of response vs. amplitude of drive. You will see very different plots than those displayed by your former data! In particular, for the v^0-law of sliding friction, there ought to be a threshold level of drive, below which you get no response at all -- why? And what happens above this threshold? For the v^2-law approximated by fluid friction, you will get another quite distinct plot, with approximate power-law dependence. Can you predict why, and what exponent you expect in the power law? You might use an 'energy budget' that applies to steady-state operation: averaged over a full cycle, the work done by the drive and the work done by friction have to be *equal* in steady-state operation.

4.7.3 Intermodulation in non-linear systems

You've become used to the idea that if you drive a system sinusoidally at frequency f, out will come a response which will *also* be sinusoidal, and of frequency f, too. This is not automatically a property of all driven systems! Or, consider another property of linear systems: if the input is a superposition of two sinusoids, at frequencies f_1 and f_2, out will come another superposition of sinusoids, containing (only) the same two frequencies, f_1 and f_2. In this case, of course, the two sinusoidal components might well have experienced different magnitude response, and different phase shifts, but still there are only two frequencies represented in the output.

One of the most characteristic features of *non*-linear response is 'intermodulation', and the 'two-tone' input waveform containing frequencies f_1 and f_2, is just the way to diagnose such systems. The test is to get a frequency-domain view of the output waveform, and in particular to look for any frequencies other than f_1 and f_2 to appear. Under conditions of even rather weak non-linearity, you should expect to see 'intermodulation distortion', with the appearance of frequencies including $f_1 + f_2$, $| f_1 - f_2 |$, and other small-whole-number combinations too.

You might test this first on your Oscillator working as a rather low-Q system under magnetic damping. The resonant peak will be broad, so you can pick two frequencies that both lie not too far from the peak. Give them enough separation such that the difference, $| f_1 - f_2 |$, will not be too small, and choose a low level of drive (to evade the known non-linearity implicit in large-angle operation of your torque drive). Now you'll need to wait for steady-state operation (i.e., for all the transients related to initial conditions to die away), and you'll need to get a spectral view of the output via the Fourier transform. You're looking for just two peaks to appear, and in particular, you can check to what degree (i.e., by how many decibels, on the traditional logarithmic scale) the expected 'intermodulation tones' are suppressed.

Now change to a higher level of drive, where you do expect some degree of non-linearity, and see if you can detect it by this test. Once you've confirmed by these two 'control group' experiments that you can get proper negative, and positive, results of the test, it's time to change to a different law of damping, and see if intermodulation distortion is present, as theory predicts it should be. As you raise the level of drive, you might see *lots* of frequencies, of the form

$$|m\, f_1 \pm n\, f_2|$$

for integers, *m* and *n*, appearing in your Fourier spectrum. Along the way, you'll learn that for high spectral resolution in your frequency-domain view, you need to take records of output waveform that are of long duration in the time domain.

4.7.4 The Kramers-Kronig relationships

You've learned about transfer functions: that the complex-valued transfer function conveys information on both the amplitude response, and the phase-shift response, of the system it describes. Now you might imagine that you can build a system with a tailored-to-choice magnitude-response function, and also with a tailored-to-choice phase-response function. For example, you might want to build a filter that:

- has amplitude response of one in a chosen band of frequencies (but zero response elsewhere), and further
- has no pesky phase shifts at all.

But such specifications are not realizable for generic systems! In particular, there are very powerful statements that can be made about any system that is linear and *causal* -- that is, for which effect can only follow (not precede) the cause. For a causal system, in fact, the transfer function is so tightly constrained that knowing the magnitude response fully <u>fixes</u> the form of the phase-shift response, and conversely too.

The name for these tight connections is the 'Kramers-Kronig relationships', and in advanced texts you can find, in closed form, the expressions that give the phase response (if the amplitude response is known), or conversely give the amplitude response (if the phase response is known). You will learn a great deal about the mathematics of complex variables if you follow the standard derivations, and a good deal about your oscillator system if you start to ask just how you would use the Kramers-Kronig relationships operationally.

For starters, you might work with a paper system, i.e., a torsional oscillator entirely described by a simple model of a second-order differential equation. You'll get an exact and closed-form result for the complex-valued transfer function, and you can temporarily pretend you know only the magnitude of that function. Get a bit more realistic, and suppose you knew the amplitude-response function at only some finite list of frequency values. Can you use this finite set of information to execute the actual integral that gives, according to the K-K relations, the phase response? In this paper example, you can of course check those results against the predicted phase response of the system.

Life gets harder still with actual experimental data. Here, you not only are limited to some finite list of values in a finite range of frequencies, but you are also limited to some finite level of precision in your ability to measure the amplitude response at any of them. So it's actually a rather difficult problem to execute the K-K program in practice, and you can look into the research literature for suggestions. But practicality aside, there's so much *romance* in the far-reaching consequences of a stipulation so apparently obvious as causality.

5 Advanced Topics

This section of the manual contains some advanced-project topics that are not so naturally associated with the sections you've seen so far. Some of them also involve the use of some additional electronic equipment. Any of them could take you deep into a research topic, and all of them are covered in these notes in rather less detail than you've seen in previous sections.

5.0 Overview of topics

Here are the 'abstracts' for the advanced topics -- more complete descriptions follow:

1. **Drive by noise waveforms**: You're used to treating electronic noise as a nuisance, or an outright enemy -- but here's a chance to see its usefulness, and learn the basis of 'Fourier transform spectroscopy' at the same time.

2. **Feedback and its effects**: What happens when you take the angular-position signal, process it according to some recipe, and 'feed it back' into the drive coil of your Oscillator? You can change the properties of the oscillator by that feedback, varying either the natural frequency or the damping.

3. **Building a 'torsional clock':** If you've worked on feedback in section 5.2, and seen that you could change the damping of the Oscillator, you might have wondered if you can change the damping all the way down to *zero* -- and what happens then?

4. **Aligning the magnet-in-coil differently**: Everything you've done thus far has had the magnets on the rotor, at its equilibrium position, come out perpendicular to the coils' axis. How can you change this to the *parallel* configuration, and what's the usefulness of this new geometry?

5. **The quartic oscillator:** The elastic behavior of the torsion fiber has given you a torque linear in displacement, which corresponds to an elastic potential-energy function that's quadratic in displacement. How can you change this to a *quartic* potential, and what is the result?

6. **Parametric drive of an oscillator:** You're used to driving an oscillator at its own 'natural frequency' to get resonant behavior. Under what circumstances can a drive at *double* the natural frequency nevertheless pump up an oscillator?

7. **Coupled oscillators:** If you have the luxury of two Torsional Oscillators, you can couple them magnetically, and see the surprising phenomena that result. There are plenty of qualitative observations, and quantitative measurements, that will enable you to learn the very generally applicable lore of coupled oscillators.

5.1 Drive by noise waveforms

In sections 4.1 and 4.2, you learned how to measure the amplitude, and the phase, response of your driven oscillating system. In both cases, the independent variable of your plots was your choice of the frequency of sinusoidal drive you put into your system. In both cases, you took the data by putting in one frequency at a time, and perhaps you got bored by the time you had measured the tenth or twentieth point. Here's a way to get *all those points at once*, and lots more points besides. The method is applicable only to linear systems, but it's wonderfully efficient where it works.

The secret is to take advantage of linearity in a serious way. Since the system's response to a sum-of-inputs is just the sum of the responses to the inputs applied individually, the idea is to subject the input of the system to a waveform that contains *all* frequencies at once. You can even ask for all frequencies to be 'equally represented', and the name for one version of that waveform is 'white noise' ('white' by analogy to white light, containing as it does all the frequencies of visible light). In particular, for this investigation you need some sort of white-noise generator, in place of your usual signal generator limited to sine-square-triangle waves.

Of course you don't really want <u>all</u> frequencies, since you don't need to send gamma rays into your system. In fact, for these experiments on your Torsional Oscillator, all you need is decent coverage of the 0-10 Hz range, since all the interesting response of your oscillator appears in that range. There are lots of ways to get noise that is effectively white over such a range, and you can pick any one of them.

Pick a strength of the input signal which is reasonable -- one method is to measure the rms value of the white noise, and make it comparable to that of the sinusoidal drives you've used before. Set the Q of your system to order 5, such that the magnitude response of the transfer function ought to have a well-defined, but not too narrow, peak. Finally, in this experiment you'll need (possibly simultaneously-acquired) time records of both input and output waveforms, and you'll need to get frequency-domain views of both waveforms.

The spectrum of your <u>input</u> waveform ought to confirm that (over some frequency range) your signal is indeed 'white'. Or, if it's not, you will have a measure of how much power it contains at each of the frequency points at which you diagnose it. The time-domain record of your system's <u>output</u> will look like *garbage* -- in fact, you're putting noise in, so the best you can hope for is noise out. But this will be amazingly informative 'garbage', if you get a frequency-domain view of the output. In particular, if the input spectrum is white, the output's spectrum will display, all in one go, the transfer function of the oscillator! If you can get the complex values of the Fourier coefficients of both the input and the output waveforms, acquired simultaneously, you can go on to get the phase shift of each frequency component, and similarly get the phase-shift plot not point-by-point, but for all frequencies in one investigation.

This very powerful effect, of exciting a system with all frequencies at once, and then separating the frequencies after the fact, is powerfully put to use in Fourier-transform spectroscopy and lots of other places. You can learn a great deal of transferable knowledge about these Fourier-based techniques even on so simple a system as your oscillator.

5.2 Feedback and its effects

In this section you'll see some of the modifications you can make in the behavior of the Oscillator by external electronic changes, instead of internal mechanical changes. The general technique is feedback, in which a signal is taken out of the mechanical system, electronically modified, and fed back into the system. To perform these experiments, you'll need some external 'bread-boarding' analog-electronics capabilities.

Start by getting the angular-position signal, $V_{pos}(t)$, out to an electronic arena in which you can modify, or amplify, it. In this same arena, you want electronics that can drive the Helmholtz coils, not with some signal generator's output, but with a signal derived from $V_{pos}(t)$ itself. Possibly the best way to drive those coils is to build a voltage-to-current converter, which has its output current pass through the Helmholtz coils. Model your circuit according to

$$i_{coil} = V_{V\text{-to-}i} / R$$

where $V_{V\text{-to-}i}$ is the input voltage to your coil-driving circuit, and R is some quantity (of dimensions resistance) characterizing your voltage-to-current converter.

Now write the differential equation describing the motion of the oscillator, under conditions of some generic function, $V_{V\text{-to-}i}(t)$. But go on to assume, not a generic value of $V_{V\text{-to-}i}$, but instead a value *proportional* to $V_{pos}(t)$, the angular-position output of your oscillator. You get to choose, by actual electronic implementation on your breadboard, the *sign* and the value of this proportionality constant. And since $V_{pos}(t)$ is itself proportional to $\theta(t)$ with a known constant of proportionality, you should now be able to get a homogeneous differential equation, i.e., one involving $\theta(t)$ and its derivatives, but no unknown or external function. Your system is now electro-mechanical in character, but it's autonomous, with no connection to the outside world, acting under 'proportional feedback'.

Show that your differential equation corresponds to that of a modified simple harmonic oscillator. In particular, the 'natural frequency', ω_0, will have been changed, by an amount which you can calculate with no unknown parameters. Because of the change in ω_0, there will also be changes in the damping parameter, γ, also by a calculable amount. By your choices of circuit connections, you can make this new natural frequency larger than, or smaller than, the former value. Which direction of change has the effect of making the oscillator more, or less, damped? Can you get data confirming both effects?

Your voltage-to-current converter, driving the coils, will have some upper limit to the size of currents it can deliver, and this will limit the size of oscillations your system can undergo and still fit your model. But within these small-oscillation limits, you can still measure the frequency and damping of small oscillations, and gain confidence in your model.

The modification of the behavior of systems in general, and instruments in particular, by external electronic circuits and feedback is a very useful general technique. Suppose you wanted to achieve the effect of a much smaller torsion constant κ for your fiber, so as to give your torsion apparatus some extra sensitivity to small torques -- can you achieve that with feedback?

5.3 Building a 'torsional clock'

In the previous section, you've seen some of the uses of feedback in modifying the mechanical behavior of your Oscillator by the use of external electronics, and feedback. Here's another exercise along that line, with even more glamorous results.

Once again, you will use the position signal, $V_{pos}(t)$, coming out of your Oscillator as the input to some external electronic breadboard, and once again you will use a voltage-to-current converter, or the equivalent, as a way to send currents into the drive coils of your Oscillator. What's different, compared to the previous section, is the interposition of an extra stage of processing between $V_{pos}(t)$ and the voltage, $V_{V\text{-to-i}}$, that you send into your voltage-to-current converter. The stage of processing that will generate the dramatic new results is to put either an electronic differentiator, or an electronic integrator, in this intermediate position.

Note that you can build analog-electronic realizations of both these circuits, with very nearly ideal performance from the integrator, and adequate low-frequency behavior from the differentiator. Note that you can build the circuits so that they'll behave according to well-defined models -- that is to say, such that they introduce no unknown constants. Notice too that you still have control of the *sign* of the whole feedback term, either by circuit changes, or just by interchanging the two leads at the grey-banana coil connections to the drive coil.

Before you build or test these circuits, try working out the differential equation that will describe the behavior of your electro-mechanical system. With the *differentiator*, you should retain a second-order differential equation, and you should be able to see that in this case, the electronic modifications change the decay constant of the system. Note that you can make the system decay faster, and offer the advantage of faster 'settling time', or decay slower. A system decaying slowly enough will never settle -- so, show on paper that you can reduce the net damping from whatever former damping you had, all the way to zero. What ought to happen then?

Similar analysis of the circuit with the *integrator* will give a third-order differential equation, still linear and solvable in principle. Rather than wade through the details, find out under what conditions this equation has an undamped sinusoid as a solution -- that's a lot easier, mathematically, and it'll lead you to a choice of parameters that you can actually build.

The payoff is rather exciting -- you will have an electro-mechanical system which is autonomous, in which the external electronics can be arranged precisely to make up for the mechanical damping that was originally present. Such a system ought to oscillate without damping, and in fact will oscillate at its own natural frequency as a 'clock'. If you measure the period of oscillation electronically (say, with a digital counter), you'll find you have remarkable short-term stability, and amazing sensitivity to small changes in the period caused (for example) by the addition of tiny masses to the rotor.

5.4 Aligning the magnet-in-coil differently

In every investigation thus far, the equilibrium position of the magnets on the rotor shaft has been *perpendicular* to the axis of the Helmholtz coils. This makes the torque of the coils on the rotor a simple function (at least for small angles of deflection), and also allows the coils-in-magnet to serve as a velocity transducer. But in this and the next two sections, that arrangement is changed, to a situation in which the magnets' moment is aligned to lie *along* the coils' axis. The usefulness of this arrangement is found in the next two sections; this section tells you how to achieve this re-alignment.

The basic idea is to loosen the rotor shaft from attachment to the torsion fiber using the wire clamps at the top and bottom of the rotor shaft, to turn the whole rotor by 90°, and then re-tighten those clamps. Here are the details:

- slide the rotor set-up tool (typically stored atop the upper support of the Helmholtz coils) forward, until it's under the rotor;

- use the tensioning knob at the top of the instrument's case to slacken the torsion fiber somewhat, until the rotor rests on the rotor set-up tool;

- loosen the wire-clamping screws at the top and bottom of the rotor (these are the socket-head screws whose axes lie in the *horizontal* plane) by a turn or less;

- slightly loosen the rotor-mounting screws in the wire clamps (these are the socket-head screws whose axes are *vertical*), which will allow the clamps to lose their grip on the fiber;

- rotate the whole rotor structure by 90° (either way) until the permanent magnets' flat faces lie perpendicular (and their magnetic-moment vectors lie *parallel*) to the axis of symmetry of the Helmholtz coils;

- re-tighten the wire-clamping screws on the wire clamps, symmetrically;

- re-tighten the rotor-clamping screws on the wire clamps;

- (if desired) remove all four screws that hold the copper rotor disc to the rotor shaft, and (counter)rotate that disc by (-)90°, so that it's restored to its original orientation (this permits the familiar use of the 'radian protractor scale'), and then re-fasten it with the four screws;

- (if desired) remove all four screws (down near the bottom of the rotor shaft) that hold the rotating central plate of the angular-position transducer, and (counter)rotate that plate by (-) 90°, so that it's restored to its original orientation (this permits the familiar use of the angular position transducer), and then re-fasten it with the four screws;

- re-tension the fiber, slide the rotor set-up tool out of the way, and check to see if you've achieved the desired orientation of the magnets' axes -- if not, iterate the above procedure. (Small deficiencies in orientation can be corrected using the fiber angular adjuster at the top of the torsion fiber.)

You will see in the next section that there are rather direct checks to see if you have indeed achieved the parallel orientation of magnetic moments, and coil axis, that you seek. If you can achieve alignment to 'eyeball accuracy' by the procedure above, the use of those diagnostics will allow you to achieve optimal alignment.

5.5 The quartic oscillator

The previous section has taught you how to achieve a novel orientation of the rotor magnets' axes along the axis of the Helmholtz coils of your Oscillator, and this section will show you how to diagnose that alignment, and to test it systematically, and finally to apply it to some glamorous new kinds of oscillation.

If you have, mounted on the rotor at elastic equilibrium, a magnet system with axis exactly aligned along the Helmholtz coils' axis, then adding a modest current in the coils will not change the location of the equilibrium position. But clearly, if those axes differ, then the equilibrium position due to elastic and to magnetic interactions acting jointly will be a compromise between minimizing elastic energy and minimizing magnetic energy. The consequences will be most obvious with the use of a small current in the coils, chosen to have a direction that would (by itself) create a *de*-stabilizing effect on the rotor.

If you see a change in the equilibrium location of the rotor with modest sizes of this sort of current, then you have a signal which is diagnostic of imperfect alignment. You can then use the angular adjuster at the top of the torsion fiber to fine-tune the zero position of the rotor for elastic-only interactions, until you achieve the alignment desired.

To get some quantitative indication of the effects of the combined elastic-plus-magnetic interactions, you can choose a sign and magnitude of the current in the coils, and then hand-excite a small oscillation about the equilibrium position of the rotor. If you plot the square of the angular frequency of oscillation, ω^2, of this motion, against the current, i, you send through the Helmholtz coils, then you should get a linear dependence. (Why?) Make a model of the potential energy of this system (elastic plus magnetic), and use it, or a torque equation, not only to understand why such a linear dependence is observed, but also to understand its slope and intercept in terms of other parameters of your electro-mechanical system.

In particular, for a certain value of current, i_b, (with a sign that makes for *de*-stabilizing magnetic interactions) you can extrapolate to find where your small-oscillation frequency, ω, goes to zero. This is the point at which the quadratic and positive *elastic* contribution to the system's potential energy is 'balanced out' by a *magnetic* and negative contribution. In fact, you should draw models, as a function of the rotor's deflection angle θ, and the current in the coil i, of the total potential-energy function of the system, and show that

- for currents in the vicinity of i=0, the potential energy has a quadratic minimum;
- for one sign of currents (taken to be positive), that minimum becomes 'deeper' with current;

- for currents of the *other* sign, headed down to i_b, the potential-energy function gets 'softer';
- for a current of exactly (-)i_b, the quadratic term to the total (elastic plus magnetic) potential energy *vanishes*;
- at that current, the next surviving term is quartic (θ^4) in angle, and *restoring* in character;
- and beyond (-)i_b, the potential's minimum *bifurcates*, leading to two stable equilibria.

You will note that it doesn't pay to go far beyond (-)i_b in current, since you can very easily stray into a regime where the whole rotor will suddenly slew through a large angle, and oscillate with large amplitude about an angular position 180° away from your original position, where the magnets are favorably aligned with the magnetic field of the coils. Even short of that point, you will still discover that in the vicinity of (-)i_b, the system as a whole becomes amazingly sensitive to small displacements (and to the 'memory' effect of hysteresis in the fiber).

Nevertheless, you should be able to be confident about the value of the 'balancing current', and should be able to obtain stable oscillations about the potential-energy minimum you get using that current. But they will be oscillations of a 'quartic oscillator', whose small-displacement behavior is dominated by a potential of the form

$$U(\theta) = q\,\theta^4 ,$$

for some positive (and predictable!) constant, q. Motion in such a potential does indeed have a restoring force, but it's *cubic*, rather than linear, in the displacement. It follows that this motion will *not* be sinusoidal, and will not be isochronous either -- in particular, the period of small oscillations will depend, rather dramatically, on the amplitude of the oscillations.

It's a relatively complicated task to predict the period of those oscillations, even in a pure-quartic potential -- and for amplitudes of any finite size, your potential is not exactly quartic. But the potential is entirely predictable, if you assume a pure-quadratic elastic contribution, a magnetic contribution of the expected form, and operation right at the 'balancing current'. In a well-understood potential, both the motion, and the period of the motion, of the system ought to be entirely predictable by one or another numerical means.

If you can operate *just beyond* the balancing current, you'll have the chance to see motion in a double-well potential. In principle, you could measure the frequency of small oscillations in each of the two wells, each of which ought to be (locally) quadratic. In practice, hysteresis effects will complicate your understanding of this situation. But it's still worth thinking about this situation, since a double-well potential with a small 'barrier' between the two wells is just the sort of system in which dramatic effects like <u>tunneling</u> are predicted to occur quantum-mechanically.

5.6 Parametric drive of an oscillator

Here's another curious capability of your Torsional Oscillator, placed here in the manual since it requires the magnets-along-coil-axis geometry set up in section 5.4 and exploited in section 5.5. You've modeled the interactions of the magnets, so oriented, in the previous section, when you created a static magnetic field using DC currents in the drive coils. Now you're ready to try exciting the oscillator by using *AC* currents in the coils.

Everything in your preparation has led you to expect that this effect will be resonant when the frequency of the drive current matches the 'natural frequency' of the Oscillator. In fact, let's review what happens with ordinary drive of the oscillator, as in section 4.0. For any <u>frequency</u> of the drive, you get *some* response of the oscillator, but you get the most response if you're 'at resonance'. For any <u>amplitude</u> of the drive, you get response of the oscillator, and the strength of the steady-state response is linearly proportional to the strength of the drive. If you start with a quiescent oscillator, and drive 'on resonance', the response grows linearly in time at first, before leveling off toward the steady-state amplitude.

Now here's the contrast. In the 'parametric drive' that you're about to undertake, you will indeed get the biggest response if you're 'on resonance', but here, 'on resonance' will require a drive <u>frequency</u> *double* that of the 'natural frequency' of the oscillator. If you're off resonance a bit, you'll get less response, but if you're off by *too much*, you'll get no steady-state response at all. On the matter of drive <u>amplitude</u>, you'll find there's a *threshold* amplitude, below which oscillations will only decay; and above threshold, you'll find oscillations do grow, but exponentially in time. Only exactly at threshold will you get what you thought was normal: steady oscillations of stable amplitude.

Here's one way to understand some of these effects. If you have a model for the magnets-in-coils as a velocity transducer, from section 1.4, you can adapt it to the new geometry you're using here. Then theoretically, and experimentally, you ought to be able to show that small oscillations of the rotor about its equilibrium position, at 'natural frequency' ω_0, will induce emfs in the coil which are of frequency $2\omega_0$. If you were to connect an external resistor to the coils, there would also be currents of frequency $2\omega_0$, in phase with the emf -- and there'd also be dissipation, in the resistor, of the mechanical energy of the oscillator. Now imagine removing the resistor, and using an external generator to drive currents in the coil, still of frequency $2\omega_0$, but which are 180° *opposite* in phase to those just discussed. Because of this flip in phase, these currents must be of a character so as to *add* energy to the mechanical oscillation. This would be called 'on-resonance parametric drive' of the oscillator -- and note that the right *phase* of the generator is required for it to pump up the energy of the oscillator.

The theory of 'parametric drive' is rather involved, so it might be best to discover all these facts empirically first -- that way, the theory will mean something to you when you wade into its mathematics. So set up the Torsional Oscillator using magnetic damping to achieve a not-vastly-high Q, of about 20 to 40, and use the methods of your choice to find a decent estimate of the Q. It turns out that a fractional tolerance of order 1/Q will tell you how far in <u>frequency</u> from the resonance condition you can afford to be. If, in your geometry, you've measured that 'balancing current', i_b, introduced in section 5.5, it will turn out that the threshold <u>amplitude</u> required of the drive current is of order (1/Q) i_b. In order to see what's going on, and achieve the relatively fussy conditions required for parametric excitation, you might want to use a real-time XY-display on an oscilloscope, with $V_{pos}(t)$ on the horizontal axis, and the drive waveform (or its surrogate) on the vertical axis. You may also want to hand-excite the oscillator at a modest amplitude, on the order of 0.1 radian, so as to have some signals to see.

Your display will show a certain kind of 'Lissajous figure' which is stable when you've reached the required condition for the frequency of excitation. You will eventually find a condition in which you can be persuaded that the angular-position is growing in amplitude, and growing exponentially too (though probably with a slow growth rate). If you think you've achieved parametric drive, here's a really convincing test that exposes the phase sensitivity of the method. Have a reversing switch in place, so that you can suddenly reverse the connections to the drive coil. Then, when you think you're in the exponential-growth mode, *flip* that switch, so that suddenly you're driving with the same frequency and amplitude, but reversed-in-phase. Your oscillator should now go into exponential *decay* of its amplitude, and at a decay rate *faster* than that due to the damping magnets alone.

In practice, exponential growth in the oscillator can't go on too long, since the interaction between the coil and the magnets on the rotor is only simple in the small-angle approximation. Similarly, exponential decay induced by reversed-phase parametric drive will not go down to zero, but will eventually break into exponential growth of a fresh oscillating mode that *is* in the right phase relationship with the drive. But short of these limiting cases, you should be able to get time records that do show characteristic growth and decay rates, and you should be able to quantify these rates. In fact, since this system lacks anything corresponding to steady-state amplitude, such growth (or decay) rates are your chief observable, the dependent variable you can measure. Your *in*dependent variables are the frequency and amplitude of the drive waveform. You should be able to confirm the claims above:

- as to frequency, you should (at any fixed amplitude above threshold) find there's only a *finite* range around the 'resonant', i.e., 2-to-1, frequency at which you can get growth in oscillation, and this range increases with amplitude, but decreases with the Q of the oscillator you're driving;

- as to amplitude, specializing to the 'resonant', i.e., 2-to-1 condition, you should be able to find the *threshold* amplitude, and show that it is proportional to (1/Q) i_b as claimed above.

Finally, there are details as to the phase shift that will exist between the drive and the response. You might see this empirically first, since it's hard to think theoretically about the meaning of a phase shift between two signals that differ by a factor of two in frequency. But you should confirm that when you are in the exponential-growth mode, the oscillator is running not at its natural frequency, but at half the drive frequency, and that you can detune the drive a bit from your target (double the natural frequency of the oscillator) at the cost of this phase shift. In fact, the oscillator will be phase-locked to half the drive frequency, with a phase error that is zero right at 'resonance', and finite and stable in time when you're a bit away from 'resonance'.

This section has not begun to introduce even the equation of motion that describes your oscillator under these conditions, but rather leaves it to you to work out. You will find that it is *not* a second-order differential equation with constant coefficients, inhomogeneous because it's driven by a function, as formerly. Instead it's a second-order differential equation with *non*-constant coefficients, and homogeneous in character. In fact, you can think of it as the equation of an *un*driven oscillator, except that one of the parameters of the system has become a time-dependent function. (Hence the term 'parametric excitation'.) If you search the appropriate references, you will be amazed at the mathematical depth at which all of this can be understood, using artful approximations in an analytic treatment. Alternatively, you can find the mathematical literature on the Mathieu Equation. You'll also be amazed if you look up the exotic applications of parametric drive, parametric excitation, and parametric oscillation that turn up in curious branches of science and technology.

5.7 Coupled oscillators

There are lots of cases in physics which can be modeled as two simple harmonic oscillators that are more or less *coupled* to each other. The phenomena that emerge from this coupling are of widespread interest in classical and quantum physics. If you have two Torsional Oscillators, you're ready to be able to investigate these phenomena, both qualitatively and quantitatively.

To perform this investigation, you need to have the rotor magnets' moments aligned perpendicularly to the Helmholtz coils' axis, as they are shipped (and as they are used everywhere except in sections 5.4 - 5.6). You'll need to know how to 'tune' the frequency of oscillation of undamped oscillators, after the fashion of section 1.3, by adding masses to the rotors. You have brass quadrants for coarse adjustment, steel balls for finer adjustments, and the possibility of adding steel balls of smaller diameters for smaller adjustments still. You'll want your two oscillators to have natural frequencies that match to better than 1%, and at this level there are issues that arise, such as the location and orientation of the oscillator -- recall section 2.7.3.

To start this investigation, you want to reduce the damping of both oscillators to a bare minimum, and you'll need to place two Oscillators side-by-side on a tabletop, with their bases nearly touching -- label them and their locations by #1 and #2. You want to match two frequencies: that of oscillator #1 at location #1 (measured when oscillator #2 is distant), and that of oscillator #2 at location #2 (measured when oscillator #1 is distant). The reason for doing the measurements this oddball way is that the oscillators interact magnetically when they're adjacent! In fact, before going on, you should work out the magnetic energy of interaction for two dipoles, $\boldsymbol{\mu}_1$ at angular orientation θ_1, and $\boldsymbol{\mu}_2$ at angular orientation θ_2, when their centers are separated by vector $\mathbf{r}$. You can treat the two dipoles as point-like to adequate accuracy, and after getting the exact $U(\theta_1, \theta_2)$ function, you should expand it for small (θ_1, θ_2) values. Now write the total potential energy of the system of two oscillators, and show that it has the form

$$U(\theta_1, \theta_2) = (1/2)\, \kappa_1\, \theta_1^2 + (1/2)\, \kappa_2\, \theta_2^2 + c\, \theta_1\, \theta_2\ ,$$

which is just the form of coupling that leads to the simplest possible interaction between two oscillators. Notice that your modeling of the magnetic moments gives the coupling constant c as a predictable number, and you can even predict how it should change with the separation of the two oscillators.

Now for some qualitative observations. Embark on a series of observations, each of which starts with oscillator #1's rotor held away from equilibrium, and oscillator #2 hand-damped to quiescence. Now release #1's rotor and observe. You should see oscillator #1 start oscillating, and oscillator #2 start to 'awaken from its sleep'. You will see a slow cycle of behavior that reduces the energy of #1 to a minimum and raises that of #2 to a maximum, followed by a reverse flow of energy back into #1. You can fine-tune the natural frequency of either oscillator until you achieve the desired matching condition: at this condition, the energy of #1 in its slow cycle will not only reach a minimum, but that minimum's value will be *zero*. This is quite a remarkable phenomenon to watch! You might not be surprised to see a more energetic oscillator give energy to another of smaller energy, but in the late stages of the first energy exchange, you're seeing an oscillator with little energy give up *even that little energy which it has* to another oscillator of greater energy! It's a bit like seeing water flowing uphill spontaneously.

You can easily measure the 'recurrence time' in this system of coupled oscillators -- that's the time from having all the energy in oscillator #1, until it's all back in #1 again -- and if you work out the theory of coupled oscillators, you'll find that this recurrence time can be predicted in terms of the coupling constant *c*. In fact, this method probably offers you the most precise way of measuring the constant.

There are lots of other quantitative observations you can make, including novel kinds of 'phase plane' plots, such as the locus of the $[\theta_1(t), \theta_2(t)]$ points with time. But real progress comes from understanding this system in terms of 'normal modes'. If you have tuned your system to the matching condition discussed above, the normal modes are quite simple and can easily be excited by hand. One of them is the 'symmetric mode', the other the 'anti-symmetric mode', and in both modes, the amplitudes of the two oscillators' motions will be equal in magnitude. What's novel is that the two normal modes have *different frequencies*, and the difference in these frequencies can also be predicted from the value of the coupling constant *c*.

Yet another check of the coupling constant can be made in an entirely static measurement. From the form of the potential-energy function $U(\theta_1, \theta_2)$ above, you can find the 'global minimum'; but you can also show that if θ_1 is hand-turned to a small static value (such as $\theta_1 = 0.1$ radian), then the system will have lowest energy if θ_2 *also* departs from zero, and by an amount calculable in terms of the constant *c*. The effect on θ_2's position is not large, but readily detectable using the high sensitivity of the angular-position transducers.

There is another clever trick you can use to understand and even 'purify' the normal modes. As you've set up the Oscillators thus far, they both have minimal damping. And as you've used the Oscillators thus far, you've had the drive coils either un-connected, or used as velocity transducers. But now bring out the coil connections directly (via the grey-banana terminals) and simply connect one coil to the other, via two wires. In one of the two normal modes, the emf's separately generated in the two coils will be 'pointing in opposite senses', so the two emf's will cancel each other out, leading to a negligible current flowing in the circuit you have established. In the other normal mode, the two emf's will be 'pointing in the same sense', and will thus drive a non-zero current through the two coils. Clearly, this mode will have a larger rate of energy loss, due to Joule heating. The result is that the two normal modes still exist, but they show up with quite different damping constants! And just by interchanging the two leads connecting the coils, you can interchange which mode is high-loss, and which is low-loss.

Now you'll need less skill to get a pure normal mode. You could excite the system *any way you like*, say by a hold-and-release of only one of the two rotors. That will set up a superposition of the two normal modes, with the energy-exchange consequences you saw earlier. But if you connect the coils, the energy in one of those two modes will be much more rapidly depleted than the other, thus soon leaving the low-loss mode in 'purified' form. Now you can disconnect the wires, leaving the system in the normal mode you've created.

(If you know the physics of the neutral-kaon system, you can translate this language of excitation and normal-mode loss rates, into the vocabulary of K_1 and K_2 mesons, and K_L and K_S eigenstates.)

Thus far you've aimed for, and exploited, a 'tuning' of the system such that the two oscillators, if isolated from each other, would display equal frequencies. But you can also fine-tune the rotational inertia of either oscillator, by small and precisely-calculable amounts, and explore the 'spectroscopy' of the system. For each setting of 'tuning', you can measure the frequency of both normal modes -- by this point, you will have a good method for exciting 'pure versions' of both normal modes in turn. You could label the 'isolated frequencies' of the two oscillators as ω_1 and ω_2, and the frequencies of the two normal modes as ω_a and ω_b. Then you should show theoretically that the combinations

$$\omega_a^2 + \omega_b^2 \,, \text{ and } (\omega_a^2 - \omega_b^2)^2 \,,$$

are predicted to be particularly simple functions of the system's parameters. In fact, if you graph each of these combinations as a function of 1/I, where I is the rotational inertia of the rotor you are varying, you will be able to extract a set of parameters that describe the system neatly. Finally, you should make a plot of how ω_a^2 and ω_b^2 vary as a function of 1/I, to show a beautiful 'avoided crossing' of the sort that shows up in related coupled-oscillator systems in quantum mechanics.

6 Specifications

Here's a collection of information about parts in the Torsional Oscillator.

6.0 Surfaces and care of the Torsional Oscillator

This section of the manual tells you how the surfaces of the various parts of your Torsional Oscillator are finished, and suggests what care they'll need accordingly.

- the wooden box and base are finished with polyurethane, and will only need dusting; the coating is resistant to moisture, and reasonably hard.
- the aluminum pieces are mostly hard-anodized, some 'clear' and some black in color. This makes them highly resistant to scratching. By exception, the clamps that hold the torsion fiber, and the card guides that hold the edges of the position transducer, are purposely left as uncoated aluminum.
- the torsion fibers are hard-drawn low-carbon steel 'music wire', and are phosphate-treated to leave behind a slight surface film. This film should be left on the fibers, to keep them rust-resistant. Liquid moisture should be kept off the fibers, since they are not rust-proof in wet conditions.
- the copper rotor disc is high-conductivity copper, and it would rapidly tarnish in air, except that it's been 'hard powder coated' and thus ought to retain its color indefinitely. The coating could be scratched with tools, but certainly won't be damaged by touching.
- the brass quadrants and brass weights and hang-downs are unfinished as-machined brass. They will retain their color and finish in indoor environments, but will eventually tarnish under fingerprints. The effect is purely cosmetic, and will scarcely change the mass significantly. Commercial brass polish could be used to restore a shiny finish.
- the steel balls are 'chrome steel', typical of most ball bearings. They are supplied with an oily surface film, and will stay rust-free if this is left on. Removal of the film, and exposure to very moist air, would eventually cause surface rusting of the steel; this can be removed, if desired, with steel wool.
- the permanent magnets are nickel-plated, and ought to remain corrosion-free indefinitely. In the magnetic dampers, the 'magnet arcs' joining the pairs of magnets are steel, also nickel-plated.
- the front panel is brass, with a brushed and protected front surface that should neither corrode nor require maintenance.

Finally, various machine screws apparently sunk into the wooden box are in fact fastening into threaded metal inserts, themselves press-fit into the plywood. Thus such screws could be removed and re-inserted without fear. If such a screw were to be grossly over-tightened, the metal insert might slip relative to the plywood -- so take reasonable care if you should have occasion to tighten such screws.

6.1 Masses and sizes of relevant parts

This section gathers into one place a number of relevant dimensions and masses of various parts of the Torsional Oscillator. Masses are given in grams, and dimensions are quoted (for cultural reasons) in inches, where 1 inch = 1" ≡ 0.0254 m exactly.

- the copper rotor disc is an annulus of maximal outer diameter of 4.95", an inner diameter of 1.02", and total mass of 962 ± 2 g. The diameter of the circle of holes receiving the brass quadrants' dowel pins is 2.720" ± 0.005".

- the rotating part of the angular-position sensor has an outer diameter of 4.74", an inner diameter of 1.02", and a total mass of 37 ± 1 g. It's made of standard printed-circuit board material, 1/16"-thick epoxy-reinforced fiberglass (FR4), with copper electrodes.

- the brass quadrants, as supplied with stainless-steel dowel pins in place, and as mounted on the copper rotor disc, describe arcs with an outer diameter of 3.72", an inner diameter of 1.72", and have a total mass of 214.5 ± 0.5 g each.

- the steel balls are chromium-steel bearing balls of diameter, 1.0000", with amazingly tight tolerances on diameter and roundness. They have a mass of 66.8 g each. Placed in the conical depressions atop either the copper rotor disc or the brass quadrants, their centers lie on a circle of diameter 2.720" ± 0.005".

- the rotor shaft is made of aluminum, and has maximal outer diameter of 1.50", typical outer diameter of 1.00", and a 0.38"-diameter hole through most of the length of its axis. The aluminum part, without magnets or mounting screws, has a mass of approximately 283 g.

- the magnets on the rotor shaft are nickel-plated NdFeB discs, each with a diameter of 1.00" and a thickness of 0.25". The stack of four magnets is separated at its center by a rib, 0.24" thick that is a part of the rotor shaft. The mass of the four magnets together is 97 ± 1 g.

- the torsion fibers supplied with the apparatus are music wire conforming to ASTM A228, with very tight control over the diameter. The nominal diameters are 0.029", 0.039", 0.047", and 0.055", and these should be reliable in value, and in roundness, to ±0.0005". The fibers are supplied with nominal lengths of 30", and the measured masses of the four fibers are 2.53 g, 4.62 g, 6.56 g, and 9.12 g respectively.

- each 'air paddle' is made of a 20" piece of aluminum tubing, of outer diameter 0.250" and wall thickness of 0.014", and each tube has a measured mass of about 9.2 g. The paddle itself is constructed of foil-covered foam, nominally 6" by 4.5" in size, of a measured mass of 8.4 g. Of the tubing's length, 3" is immersed in the foam.