Coalgebraic Representation of Büchi Automata

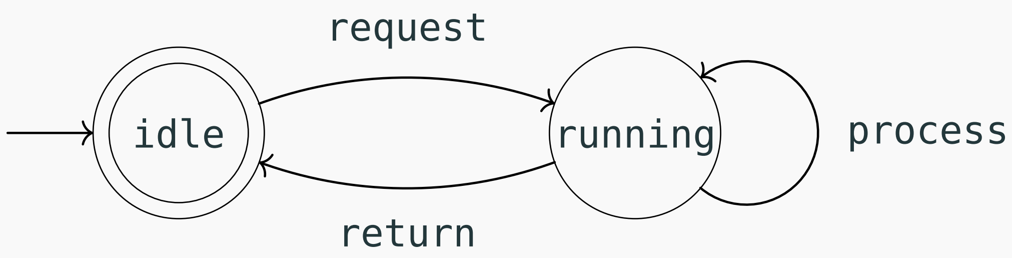
Research Internship Presentation

Jorrit de Boer

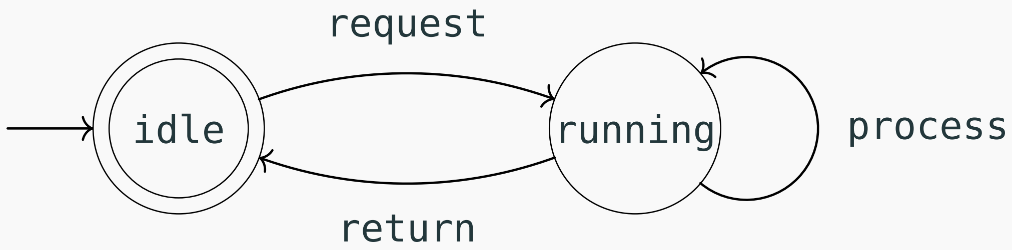
17 January 2025

Büchi Automata

Büchi Automata

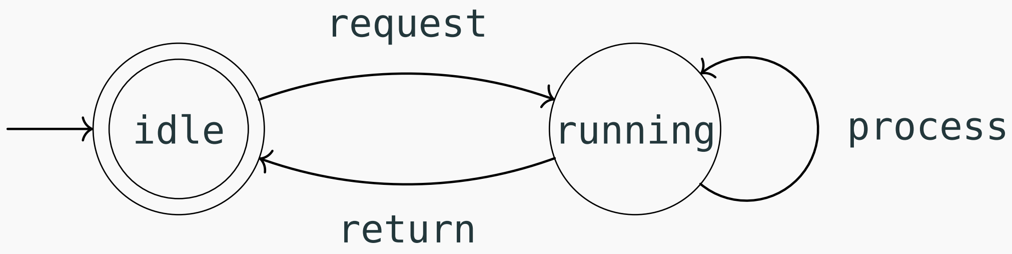


Büchi Automata



Definition: A (nondeterministic) *Büchi Automaton* $A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S: finite set of states
- Σ: alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \to \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.



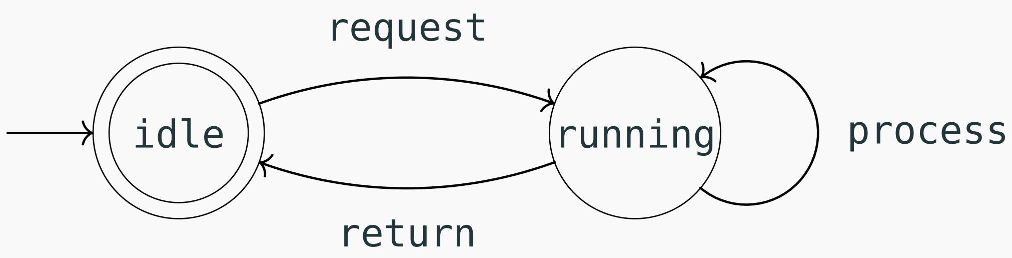
Definition: A (nondeterministic) *Büchi Automaton* $A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S: finite set of states
- Σ: alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \to \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.

A *run* of A on an ω-word $w = \sigma_0 \sigma_1 \ldots \in \Sigma^\omega$ is an infinite sequence of states $s_0, s_1, \ldots \in S^\omega$ such that for all n, $s_{n+1} \in \delta(s_n, \sigma_n)$

A run is *accepted* if it visits F infinitly often.

Büchi Automata



Definition: A (nondeterministic) *Büchi Automaton* $A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S: finite set of states
- Σ: alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \to \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.

A *run* of A on an ω-word $w = \sigma_0 \sigma_1 \ldots \in \Sigma^\omega$ is an infinite sequence of states $s_0, s_1, \ldots \in S^\omega$ such that for all n, $s_{n+1} \in \delta(s_n, \sigma_n)$

A run is *accepted* if it visits F infinitly often.

Accepted language:
$(\texttt{request} \cdot \texttt{process}^* \cdot \texttt{return})^\omega$

Coalgebra

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o \rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta \rangle : S \to 2 \times S^{\Sigma}$ for functor $FS = 2 \times S^{\Sigma}$

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o \rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta \rangle : S \to 2 \times S^{\Sigma}$ for functor $FS = 2 \times S^{\Sigma}$

The final coalgebra for F is $\langle e, d \rangle : 2^{\Sigma^*} \to 2 \times \left(2^{\Sigma^*}\right)^{\Sigma}$. Where

- $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(a) = L_a$ where $L_a(w) = L(aw)$ so $w \in d(L)(a)$ iff $aw \in L$

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o \rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta \rangle : S \to 2 \times S^\Sigma$ for functor $FS = 2 \times S^\Sigma$

The final coalgebra for F is $\langle e, d \rangle : 2^{\Sigma^*} \to 2 \times \left(2^{\Sigma^*}\right)^\Sigma$. Where

- $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(a) = L_a$ where $L_a(w) = L(aw)$ so $w \in d(L)(a)$ iff $aw \in L$

$$\begin{tikzcd}
2 \times S^\Sigma \arrow[r, dashed, "\mathrm{id} \times \mathsf{beh}^\Sigma"] & 2 \times \left(2^{\Sigma^*}\right)^\Sigma \\
S \arrow[u, "\langle o, \delta \rangle"] \arrow[r, dashed, "\mathsf{beh}"'] & 2^{\Sigma^*} \arrow[u, "\langle e, d \rangle"']
\end{tikzcd}$$

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o\rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta\rangle : S \to 2 \times S^\Sigma$ for functor $FS = 2 \times S^\Sigma$

The final coalgebra for F is $\langle e, d\rangle : 2^{\Sigma^*} \to 2 \times \left(2^{\Sigma^*}\right)^\Sigma$. Where

- $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(a) = L_a$ where $L_a(w) = L(aw)$ so $w \in d(L)(a)$ iff $aw \in L$

$$\begin{tikzcd}
2 \times S^\Sigma \arrow[r, dashed, "\mathsf{id} \times \mathsf{beh}^\Sigma"] & 2 \times \left(2^{\Sigma^*}\right)^\Sigma \\
S \arrow[u, "\langle o, \delta\rangle"] \arrow[r, dashed, "\mathsf{beh}"'] & 2^{\Sigma^*} \arrow[u, "\langle e, d\rangle"']
\end{tikzcd}$$

Following the paths through the diagram we obtain:

- $\mathsf{beh}(s)(\varepsilon) = o(s)$, and
- $\mathsf{beh}(s)(\sigma w) = \mathsf{beh}(\delta(s)(\sigma))(w)$,

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o\rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta\rangle : S \to 2 \times S^{\Sigma}$ for functor $FS = 2 \times S^{\Sigma}$

The final coalgebra for F is $\langle e, d\rangle : 2^{\Sigma^*} \to 2 \times \left(2^{\Sigma^*}\right)^{\Sigma}$. Where

- $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(a) = L_a$ where $L_a(w) = L(aw)$ so $w \in d(L)(a)$ iff $aw \in L$

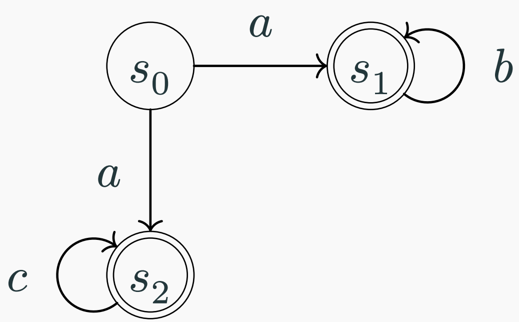
$$\begin{tikzcd}
2 \times S^{\Sigma} \arrow[r, dashed, "\mathrm{id} \times \mathsf{beh}^{\Sigma}"] & 2 \times \left(2^{\Sigma^*}\right)^{\Sigma} \\
S \arrow[u, "\langle o, \delta\rangle"] \arrow[r, dashed, "\mathsf{beh}"'] & 2^{\Sigma^*} \arrow[u, "\langle e, d\rangle"']
\end{tikzcd}$$

Following the paths through the diagram we obtain:

- $\mathsf{beh}(s)(\varepsilon) = o(s)$, and
- $\mathsf{beh}(s)(\sigma w) = \mathsf{beh}(\delta(s)(\sigma))(w)$,

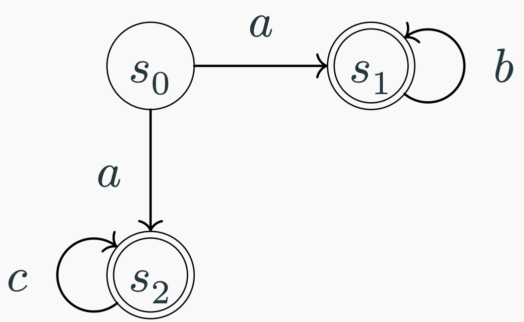
So beh captures exactly the accepted language of the automaton!

Nondeterministic Finite Automata



Might be modeled by coalgebra $c : S \to 2 \times \mathcal{P}(\Sigma \times S)$.

Nondeterministic Finite Automata



Might be modeled by coalgebra $c : S \to 2 \times \mathcal{P}(\Sigma \times S)$.

A final coalgebra $z : Z \to 2 \times \mathcal{P}(\Sigma \times Z)$ cannot exist. Lambek's lemma says z would have to be an isomorphism, which would imply $Z \cong \mathcal{P}(Z)$

Solution by Hasuo et al. 2007

Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**.

Solution by Hasuo et al. 2007

Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**. Concretely:

- $\eta_X : X \to \mathcal{P}(X)$: $\eta_X(x) = \{x\}$
- $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$: $\mu_X(A) = \bigcup_{a \in A} a$.

For $f : X \to Y$, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\mathcal{P}(f)(A) = \{f(a) \mid a \in A\}$

- **objects**: the same as for **Sets**, sets
- **morphisms**: $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}(Y)$ in **Sets**.
  For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{K}\ell(\mathcal{P})$ we define

$$g \odot f := X \xrightarrow{f} \mathcal{P}(Y) \xrightarrow{\mathcal{P}(g)} \mathcal{P}(\mathcal{P}(Z)) \xrightarrow{\mu_Y} \mathcal{P}(Z)$$

Lifted Functor in Kleisli Category

Model NDA $\langle S, \Sigma, \delta, o \rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Lifted Functor in Kleisli Category

Model NDA $\langle S, \Sigma, \delta, o \rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Problem: a map $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}$ in **Sets** so $Ff : FX \to F\mathcal{P}(Y)$

Lifted Functor in Kleisli Category

Model NDA $\langle S, \Sigma, \delta, o \rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Problem: a map $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}$ in **Sets** so $Ff : FX \to F\mathcal{P}(Y)$

We need a natural transformation $\lambda : F\mathcal{P} \Rightarrow \mathcal{P}F$ (distributive law):

$$1 + \Sigma \times (\mathcal{P}(S)) \xrightarrow{\lambda} \mathcal{P}(1 + \Sigma \times X)$$

Lifted Functor in Kleisli Category

Model NDA $\langle S, \Sigma, \delta, o\rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Problem: a map $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}$ in **Sets** so $Ff : FX \to F\mathcal{P}(Y)$

We need a natural transformation $\lambda : F\mathcal{P} \Rightarrow \mathcal{P}F$ (distributive law):

$$1 + \Sigma \times (\mathcal{P}(S)) \xrightarrow{\lambda} \mathcal{P}(1 + \Sigma \times X)$$

- $* \mapsto \{*\}(1 = \{*\})$
- $(\sigma, S) = \{(\sigma, x) | x \in S\}$ for $\sigma \in \Sigma$ and $S \subseteq X$.

For example: $\delta(s)(\sigma) = \{x, y, z\}$ then $(\lambda \circ c)(s) = \{(\sigma, x), (\sigma, y), (\sigma, z)\}$.

Lifted Functor in Kleisli Category

Model NDA $\langle S, \Sigma, \delta, o\rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Problem: a map $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}$ in **Sets** so $Ff : FX \to F\mathcal{P}(Y)$

We need a natural transformation $\lambda : F\mathcal{P} \Rightarrow \mathcal{P}F$ (distributive law):

$$1 + \Sigma \times (\mathcal{P}(S)) \xrightarrow{\lambda} \mathcal{P}(1 + \Sigma \times X)$$

- $* \mapsto \{*\}(1 = \{*\})$
- $(\sigma, S) = \{(\sigma, x) | x \in S\}$ for $\sigma \in \Sigma$ and $S \subseteq X$.

For example: $\delta(s)(\sigma) = \{x, y, z\}$ then $(\lambda \circ c)(s) = \{(\sigma, x), (\sigma, y), (\sigma, z)\}$.

Call $\overline{F}S = FS$ and $\overline{F}f = \lambda \circ \mathcal{P}(f)$ the *lifted functor*

Initial Algebra ⇒ Final Coalgebra

Theorem [Hasuo et al. 2007]: An initial algebra $\alpha : FA \to A$ for the functor F in **Sets** yields the final coalgebra for $\overline{F}$ in $\mathcal{K}\ell(\mathcal{P})$:

$$(\eta_{FA} \circ \alpha^{-1}) : A \to \overline{F}A \ in \ \mathcal{K}\ell(\mathcal{P})$$

Initial Algebra ⇒ Final Coalgebra

Theorem [Hasuo et al. 2007]: An initial algebra $\alpha : FA \to A$ for the functor F in **Sets** yields the final coalgebra for $\overline{F}$ in $\mathcal{K}\ell(\mathcal{P})$:

$$(\eta_{FA} \circ \alpha^{-1}) : A \to \overline{F}A \; in \; \mathcal{K}\ell(\mathcal{P})$$

The initial algebra for $FS = 1 + \Sigma \times S$ is $[\mathsf{nil}, \mathsf{cons}] : 1 + \Sigma \times \Sigma^* \to \Sigma^*$:

- $\mathsf{nil}(*) = \varepsilon$
- $\mathsf{cons}(\sigma, w) = \sigma w$

so we get $(\eta_{1+\Sigma\times S} \circ [\mathsf{nil}, \mathsf{cons}]^{-1}) : \Sigma^* \to 1 + \Sigma \times \Sigma^*$ ($\Sigma^* \to \mathcal{P}(1 + \Sigma \times \Sigma^*)$ in **Sets**)

- $(\eta_{1+\Sigma\times S} \circ [\mathsf{nil}, \mathsf{cons}]^{-1})(\varepsilon) = \{*\}$
- $(\eta_{1+\Sigma\times S} \circ [\mathsf{nil}, \mathsf{cons}]^{-1})(\sigma w) = \{(\sigma, w)\}$

Final Coalgebra Nondeterministic Automaton

$$\begin{tikzcd}
1+\Sigma\times S \arrow[r, dashed, "1+\Sigma\times \mathsf{beh}"] & 1+\Sigma\times\Sigma^* \\
S \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}"'] & \Sigma^* \arrow[u, "\eta_{1+\Sigma\times\Sigma^*}\circ[\mathsf{nil},\mathsf{cons}]^{-1}\ in\ \mathcal{K}\ell(\mathcal{P})."']
\end{tikzcd}$$

Final Coalgebra Nondeterministic Automaton

$$\begin{tikzcd}
1+\Sigma\times S \arrow[r, dashed, "1+\Sigma\times\mathsf{beh}"] & 1+\Sigma\times\Sigma^* \\
S \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}"'] & \Sigma^* \arrow[u, "\eta_{1+\Sigma\times\Sigma^*}\circ[\mathsf{nil},\mathsf{cons}]^{-1}"']
\end{tikzcd} \quad \textit{in } \mathcal{K}\ell(\mathcal{P}).$$

$$\varepsilon \in \mathsf{beh}(s) \iff * \in c(s) \iff \text{state } s \text{ is accepting}$$

$$\sigma w \in \mathsf{beh}(s) \iff (\sigma, w) \in ((\Sigma \times \mathsf{beh}) \circ c)(s) = \{(\sigma, \mathsf{beh}(t)) \mid (\sigma, t) \in c(s)\} \iff \exists t.(t \in \delta(s)(\sigma) \wedge w \in \mathsf{beh}(t)).$$

Possibly Infinite Behavior

Theorem [Jacobs 2004]: A final coalgebra $\xi : Z \to FZ$ yields a *weakly final* coalgebra

$$(\eta_{FZ} \circ \xi) : Z \to \overline{F}(Z) \ in \ \mathcal{K}\ell(\mathcal{P})$$

Possibly Infinite Behavior

Theorem [Jacobs 2004]: A final coalgebra $\xi : Z \to FZ$ yields a *weakly final* coalgebra

$$(\eta_{FZ} \circ \xi) : Z \to \overline{F}(Z) \textit{ in } \mathcal{K}\ell(\mathcal{P})$$

$$\begin{tikzcd}
\overline{F}S \arrow[r, squiggly, "\overline{F}(\mathsf{beh})"] & \overline{F}Z \\
S \arrow[u, "c"] \arrow[r, squiggly, "\mathsf{beh}"'] & Z \arrow[u, "\eta_{FZ} \circ \xi \quad \textit{in } \mathcal{K}\ell(\mathcal{P}),"']
\end{tikzcd}$$

beh is not unique. However, we can take beh^{∞}, the maximal mapping with respect to inclusion.

$\xi : \Sigma^\infty \to 1 + \Sigma \times \Sigma^\infty$ is the final F-coalgebra, defined by $\xi(\varepsilon) = * \in 1$ and $\xi(\sigma w) = (\sigma, w)$ ($\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$).

$\xi : \Sigma^\infty \to 1 + \Sigma \times \Sigma^\infty$ is the final F-coalgebra, defined by $\xi(\varepsilon) = * \in 1$ and $\xi(\sigma w) = (\sigma, w)$ $(\Sigma^\infty = \Sigma^* \cup \Sigma^\omega)$.

$$\begin{tikzcd}
1 + \Sigma \times S \arrow[r, rightsquigarrow, "1 + \Sigma \times \mathsf{beh}_c^\infty"] & 1 + \Sigma \times \Sigma^\infty \\
S \arrow[u, "c"] \arrow[r, rightsquigarrow, "\mathsf{beh}_c^\infty"'] & \Sigma^\infty \arrow[u, "\cong"', "J\xi"]
\end{tikzcd} \qquad \textit{in } \mathcal{K}\ell(\mathcal{P}).$$

$$\varepsilon \in \mathsf{beh}^\infty(s) \iff * \in c(s) \iff \text{state } s \text{ is accepting}$$

$$\sigma w \in \mathsf{beh}^\infty(s) \iff \exists t. \Big(s \xrightarrow{\sigma} t \wedge w \in \mathsf{beh}^\infty(t) \Big).$$

Büchi Automata Coalgebraically, Urabe et al. 2016

Idea: split $S = S_1 \cup S_2$ for S_1 non-accepting and S_2 accepting

Büchi Automata Coalgebraically, Urabe et al. 2016

Idea: split $S = S_1 \cup S_2$ for S_1 non-accepting and S_2 accepting

$$
\begin{tikzcd}
\Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times {[\mathsf{beh}_1, \mathsf{beh}_2]}"] & \Sigma \times \Sigma^\omega \\
S_1 \arrow[u, "c_1"] \arrow[r, rightsquigarrow, "\mathsf{beh}_1"'] \arrow[ur, phantom, "=_{\mu}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times {[\mathsf{beh}_1, \mathsf{beh}_2]}"] & \Sigma \times \Sigma^\omega \\
S_2 \arrow[u, "c_2"] \arrow[r, rightsquigarrow, "\mathsf{beh}_2"'] \arrow[ur, phantom, "=_{\nu}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad \textit{in } \mathcal{K}\ell(\mathcal{P}).
$$

Büchi Automata Coalgebraically, Urabe et al. 2016

Idea: split $S = S_1 \cup S_2$ for S_1 non-accepting and S_2 accepting

$$
\begin{tikzcd}
\Sigma \times S \arrow[r, squiggly, "\Sigma \times [\mathsf{beh}_1, \mathsf{beh}_2]"] & \Sigma \times \Sigma^\omega \\
S_1 \arrow[u, "c_1"] \arrow[r, squiggly, "\mathsf{beh}_1"'] \arrow[ur, phantom, "=_\mu"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\quad
\begin{tikzcd}
\Sigma \times S \arrow[r, squiggly, "\Sigma \times [\mathsf{beh}_1, \mathsf{beh}_2]"] & \Sigma \times \Sigma^\omega \\
S_2 \arrow[u, "c_2"] \arrow[r, squiggly, "\mathsf{beh}_2"'] \arrow[ur, phantom, "=_\nu"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\quad \textit{in } \mathcal{K}\ell(\mathcal{P}).
$$

$$
\begin{aligned}
\mathsf{beh}_1 &\stackrel{\mu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_1 \\
\mathsf{beh}_2 &\stackrel{\nu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_2
\end{aligned}
$$

$$\mathsf{beh}_1 \overset{\mu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_2$$

$$\mathsf{beh}_1 \overset{\mu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\mathsf{beh}_1, \mathsf{beh}_2] \odot c_2$$

Rewrite to:

$$\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

$$\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

Definition: The *solution* to this *equational system* is calculated as follows:

- Intermediate solution $l_1^{(1)} := \mu u_1.f_1(u_1, u_2)$
- $l^{\text{sol}} := \nu u_2.f_2\left(l_1^{(1)}(u_2), u_2\right)$
- $l_1^{\text{sol}} = l_1^{(1)}\left(l_2^{\text{sol}}\right)$

$$\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \restriction S_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \restriction S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

Definition: The *solution* to this *equational system* is calculated as follows:

- Intermediate solution $l_1^{(1)} := \mu u_1.f_1(u_1, u_2)$
- $l^{\mathrm{sol}} := \nu u_2.f_2\left(l_1^{(1)}(u_2), u_2\right)$
- $l_1^{\mathrm{sol}} = l_1^{(1)}\left(l_2^{\mathrm{sol}}\right)$

Concretely:

- $l_1^{(1)} := \mu u_1.\Diamond_\delta([u_1, u_2]) \restriction S_1$
- $l_2^{\mathrm{sol}} := \nu u_2.u_2 \overset{\nu}{=} \Diamond_\delta([\mu u_1.\Diamond_\delta([u_1, u_2]) \restriction S_1, u_2]) \restriction S_2, u_2)$
- $l_1^{\mathrm{sol}} = \mu u_1.\Diamond_\delta\big(\big[u_1, \nu u_2.u_2 \overset{\nu}{=} \Diamond_\delta([\mu u_1'.\Diamond_\delta([u_1', u_2]) \restriction S_1, u_2]) \restriction S_2, u_2\big)\big]\big) \restriction S_1$

Let $A = \langle S, \Sigma, \delta, s_0, F \rangle$ be a Büchi automaton. Take $S_1 = S \setminus F$, $S_2 = F$. Model δ by coalgebras $c_1 : S_1 \to \mathcal{P}(\Sigma \times S)$, $c_2 : S_2 \to \mathcal{P}(\Sigma \times S)$. Take the initial algebra $d : \Sigma^\omega \to \Sigma \times \Sigma^\omega$ defined by $d(\sigma w) = (\sigma, w)$ in **Sets**.

$$\begin{tikzcd} \Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times [\mathsf{beh}_1, \mathsf{beh}_2]"] & \Sigma \times \Sigma^\omega \\ S_1 \arrow[u, "c_1"] \arrow[r, rightsquigarrow, "\mathsf{beh}_1"'] \arrow[ur, phantom, "\overset{}{\underset{\mu}{=}}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"'] \end{tikzcd} \quad \begin{tikzcd} \Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times [\mathsf{beh}_1, \mathsf{beh}_2]"] & \Sigma \times \Sigma^\omega \\ S_2 \arrow[u, "c_2"] \arrow[r, rightsquigarrow, "\mathsf{beh}_2"'] \arrow[ur, phantom, "\underset{\nu}{=}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"'] \end{tikzcd} \quad \text{in } \mathcal{K}\ell(\mathcal{P}).$$

$$\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

Let $A = \langle S, \Sigma, \delta, s_0, F \rangle$ be a Büchi automaton. Take $S_1 = S \setminus F$, $S_2 = F$. Model δ by coalgebras $c_1 : S_1 \to \mathcal{P}(\Sigma \times S)$, $c_2 : S_2 \to \mathcal{P}(\Sigma \times S)$. Take the initial algebra $d : \Sigma^\omega \to \Sigma \times \Sigma^\omega$ defined by $d(\sigma w) = (\sigma, w)$ in **Sets**.

$$
\begin{tikzcd}
\Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times {[\mathsf{beh}_1, \mathsf{beh}_2]}"] & \Sigma \times \Sigma^\omega \\
S_1 \arrow[u, "c_1"] \arrow[r, rightsquigarrow, "\mathsf{beh}_1"'] \arrow[ur, phantom, "\overset{}{\underset{\mu}{=}}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma \times S \arrow[r, rightsquigarrow, "\Sigma \times {[\mathsf{beh}_1, \mathsf{beh}_2]}"] & \Sigma \times \Sigma^\omega \\
S_2 \arrow[u, "c_2"] \arrow[r, rightsquigarrow, "\mathsf{beh}_2"'] \arrow[ur, phantom, "\underset{\nu}{=}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad \textit{in } \mathcal{K}\ell(\mathcal{P}).
$$

$$
\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_1 \qquad\qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_2
$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$
\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.
$$

Theorem [Urabe et al. 2016]: The solutions $\mathsf{beh}_1, \mathsf{beh}_2$ to the system of equations coincide with the accepted language of the Büchi Automaton A.

Proof of Coincidence Result

Problem: system of fixed point equations is convoluted.

Proof of Coincidence Result

Problem: system of fixed point equations is convoluted.

Alternate derivation using game semantics:
Game Semantics Theorem: $s \vDash^T \varphi \iff$ verifier has a winning strategy in $\mathcal{G}(\varphi, T)$

Outline:

- Convert system of equations to modal mu-calculus formula
- Apply game semantics theorem
- Prove: V has a winning strategy in $\mathcal{G}(\varphi, T)$ from state $(x_i, w) \iff w \in \mathsf{beh}(x_i)$

Conclusion

wrap up