Coalgebraic Representation of Büchi Automata

Research Internship Presentation

Jorrit de Boer

17 January 2025

Büchi Automata

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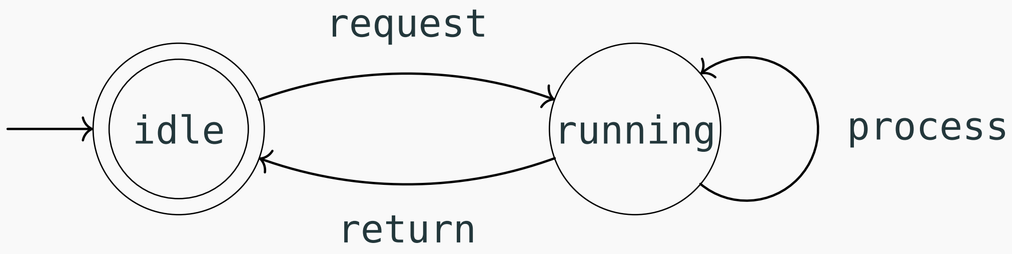

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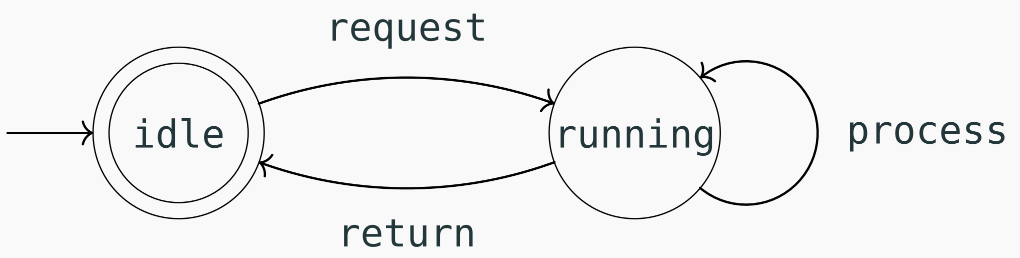

Definition: A (nondeterministic) *Büchi Automaton* $A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S: finite set of states
- Σ: alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \to \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.

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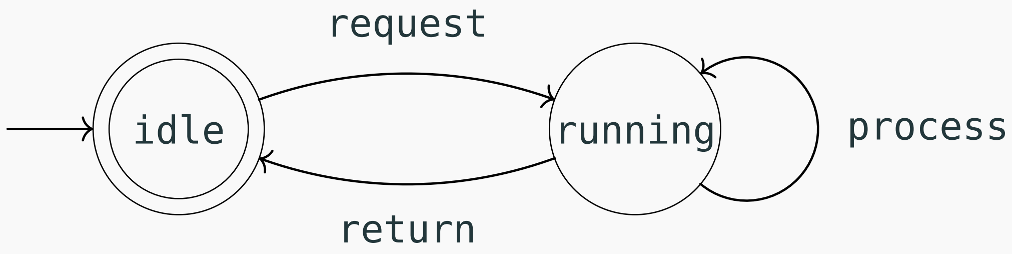

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A *run* of A on an ω-word $w = \sigma_0 \sigma_1 \ldots \in \Sigma^\omega$ is an infinite sequence of states $s_0, s_1, \ldots \in S^\omega$ such that for all n, $s_{n+1} \in \delta(s_n, \sigma_n)$

A run is *accepted* if it visits F infinitly often.

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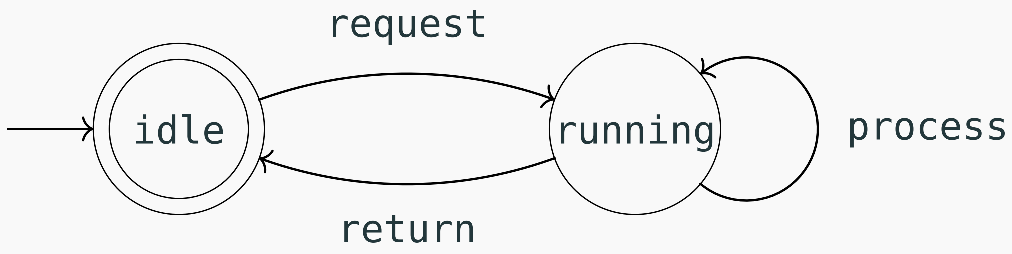

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Accepted language:
$(\mathtt{request} \cdot \mathtt{process}^* \cdot \mathtt{return})^\omega$

Coalgebra

Final Coalgebra Deterministic Finite Automata

$\langle S, \Sigma, \delta, o \rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta \rangle : S \to 2 \times S^{\Sigma}$ for functor $FS = 2 \times S^{\Sigma}$

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- $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(a) = L_a$ where $L_a(w) = L(aw)$ so $w \in d(L)(a)$ iff $aw \in L$

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2 \times S^\Sigma \arrow[r, dashed, "\mathrm{id} \times \mathsf{beh}^\Sigma"] & 2 \times \left(2^{\Sigma^*}\right)^\Sigma \\
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Following the paths through the diagram we obtain:

- $\mathsf{beh}(s)(\varepsilon) = o(s)$, and
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- $\mathsf{beh}(s)(\varepsilon) = o(s)$, and
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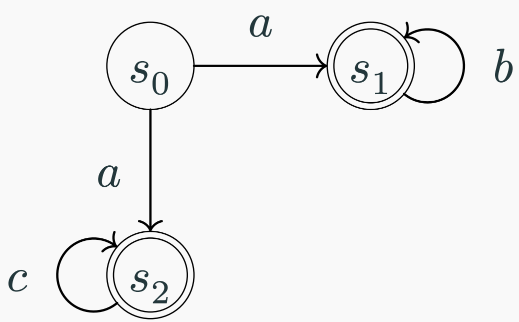
So beh captures exactly the accepted language of the automaton!

Nondeterministic Finite Automata



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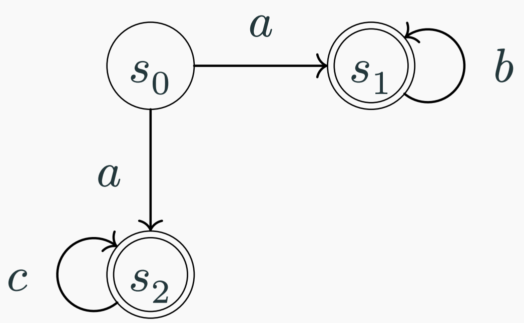

Might be modeled by coalgebra $c : S \to 2 \times \mathcal{P}(\Sigma \times S)$.

A final coalgebra $z : Z \to 2 \times \mathcal{P}(\Sigma \times Z)$ cannot exist. Lambek's lemma says z would have to be an isomorphism, which would imply $Z \cong \mathcal{P}(Z)$

Solution by Hasuo et al. 2007

Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**.

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Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**. Concretely:

- $\eta_X : X \to \mathcal{P}(X)$: $\eta_X(x) = \{x\}$
- $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$: $\mu_X(A) = \bigcup_{a \in A} a$.

For $f : X \to Y$, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\mathcal{P}(f)(A) = \{f(a) \mid a \in A\}$

- **objects**: the same as for **Sets**, sets
- **morphisms**: $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}(Y)$ in **Sets**.
  For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{K}\ell(\mathcal{P})$ we define

$$g \odot f := X \xrightarrow{f} \mathcal{P}(Y) \xrightarrow{\mathcal{P}(g)} \mathcal{P}(\mathcal{P}(Z)) \xrightarrow{\mu_Y} \mathcal{P}(Z)$$

Solution by Hasuo et al. 2007

Distributive law?

Initial algebra in **Sets** is final coalgebra in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{tikzcd}
1+\Sigma\times S \arrow[r, dashed, "1+\Sigma\times\mathsf{beh}"] & 1+\Sigma\times\Sigma^* \\
S \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}"'] & \Sigma^* \arrow[u, "\cong"', "\eta_{1+\Sigma\times\Sigma^*}\circ[\mathsf{nil},\mathsf{cons}]^{-1}\ \textit{in}\ \mathcal{K}\ell(\mathcal{P})."]
\end{tikzcd}$$

$$\varepsilon \in \mathsf{beh}(s) \iff * \in c(s) \iff \text{state } s \text{ is accepting}$$

$$\sigma w \in \mathsf{beh}(s) \iff \exists t.\Big(s \xrightarrow{\sigma} t \wedge w \in \mathsf{beh}(t)\Big).$$

Which are the right traces!

Possibly Infinite Behavior

Büchi Automata final Coalg

Using Game Semantics to derive

outline this? steps and saying these things are possible?

Conclusion

wrap up