Coalgebraic Representation of Büchi Automata

Research Internship Presentation

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Introduction

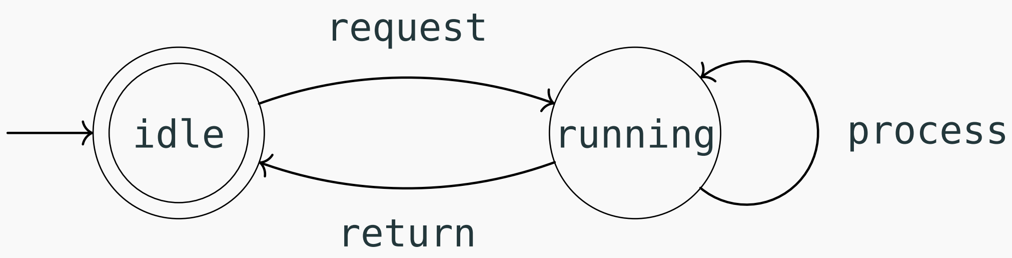
Introduction

Outline:

1. Büchi Automata
2. Coalgebraic Representation Deterministic Finite Automata
3. Coalgebraic Representation Nondeterministic Finite Automata
4. Coalgebraic Representation Possibly Infinite Behavior Nondeterministic Finite Automata
5. Coalgebraic Representation Büchi Automata
6. Outline Derivation using Game Semantics

Büchi Automata

Büchi Automata



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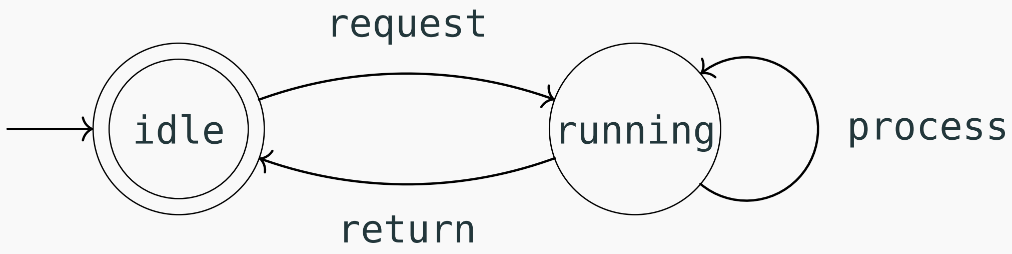

Definition: A (nondeterministic) *Büchi Automaton*
$A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S: finite set of states
- Σ: alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \to \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.

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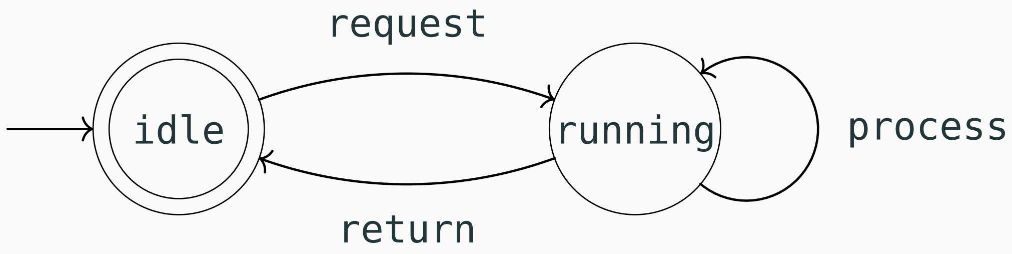

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A *run* of A on an ω-word $w = \sigma_0 \sigma_1 ... \in \Sigma^\omega$ is an infinite sequence of states $s_0, s_1, ... \in S^\omega$ such that for all n, $s_{n+1} \in \delta(s_n, \sigma_n)$

A run is *accepted* if it visits F infinitly often.

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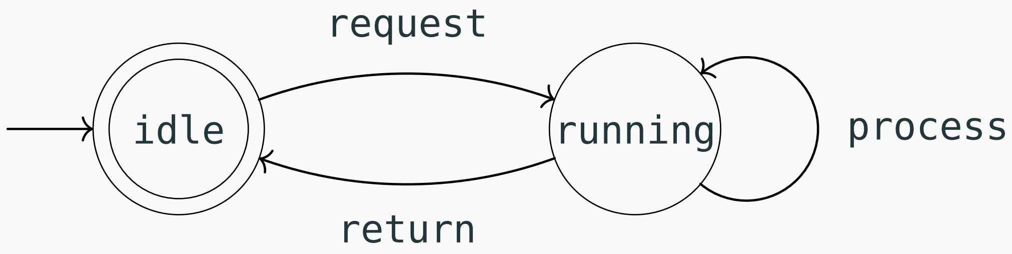

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Accepted language:
$(\texttt{request} \cdot \texttt{process}^* \cdot \texttt{return})^\omega$

Coalgebra

Coalgebraic Representation Deterministic Finite Automata

$\langle S, \Sigma, \delta, o\rangle$ with states S, alphabet Σ, transition function $\delta : S \times \Sigma \to S$, $o : S \to 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta\rangle : S \to 2 \times S^{\Sigma}$ for functor $FS = 2 \times S^{\Sigma}$

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The final coalgebra for F is $\langle e, d\rangle : 2^{\Sigma^*} \to 2 \times \left(2^{\Sigma^*}\right)^{\Sigma}$. Where

- $e(L) = L(\varepsilon)$, i.e. $e(L) = 1$ iff $\varepsilon \in L$
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2 \times S^{\Sigma} \arrow[r, dashed, "\mathrm{id} \times \mathsf{beh}^{\Sigma}"] & 2 \times \left(2^{\Sigma^*}\right)^{\Sigma} \\
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Following the paths through the diagram we obtain:

- $\mathsf{beh}(s)(\varepsilon) = e(\mathsf{beh}(s)) = o(s)$, and
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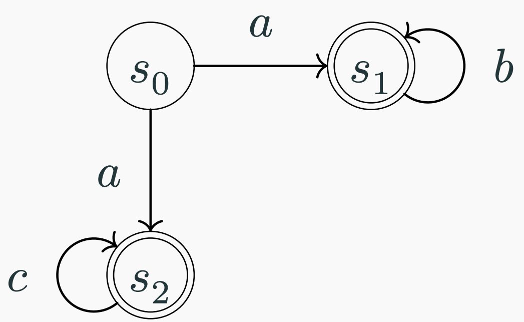
So beh captures exactly the accepted language of the automaton!

Nondeterministic Finite Automata



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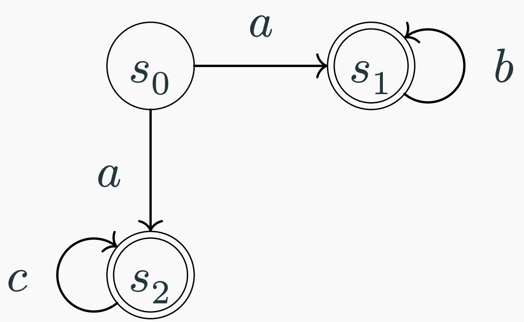

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A final coalgebra $z : Z \to 2 \times \mathcal{P}(\Sigma \times Z)$ cannot exist. Lambek's lemma says z would have to be an isomorphism, which would imply $Z \cong \mathcal{P}(Z)$

Solution by Hasuo, Jacobs, Sokolova 2007

Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**.

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Kleisli Category of the monad $\mathcal{P}$:

A coalgebra $c : S \to \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \to \mathcal{P}(\Sigma \times S)$ in **Sets**. Concretely:

- $\eta_X : X \to \mathcal{P}(X)$: $\eta_X(x) = \{x\}$
- $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$: $\mu_X(A) = \bigcup_{a \in A} a$.

For $f : X \to Y$, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\mathcal{P}(f)(A) = \{f(a) \mid a \in A\}$

- **objects**: the same as for **Sets**, sets
- **morphisms**: $f : X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \to \mathcal{P}(Y)$ in **Sets**.
  For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{K}\ell(\mathcal{P})$ we define

$$g \odot f := X \xrightarrow{f} \mathcal{P}(Y) \xrightarrow{\mathcal{P}(g)} \mathcal{P}(\mathcal{P}(Z)) \xrightarrow{\mu_Y} \mathcal{P}(Z)$$

Initial Algebra $\Rightarrow$ Final Coalgebra

Model NFA $\langle S, \Sigma, \delta, o \rangle$ by coalgebra $c : S \to 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \to \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Lift functor F in **Sets** to $\overline{F}$ in $\mathcal{K}\ell(\mathcal{P})$

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Lift functor F in **Sets** to $\overline{F}$ in $\mathcal{K}\ell(\mathcal{P})$

Theorem [Hasuo, Jacobs, Sokolova 2007]: An initial algebra $\alpha : FA \to A$ for the functor F in **Sets** yields the final coalgebra for $\overline{F}$ in $\mathcal{K}\ell(\mathcal{P})$:

$$(\eta_{FA} \circ \alpha^{-1}) : A \to \overline{F}A \; in \; \mathcal{K}\ell(\mathcal{P})$$

Coalgebraic Representation Nondeterministic Automata

The initial algebra for $FS = 1 + \Sigma \times S$ is $[\mathsf{nil}, \mathsf{cons}] : 1 + \Sigma \times \Sigma^* \to \Sigma^*$:

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$$\begin{tikzcd}
1 + \Sigma \times S \arrow[r, dashed, "1 + \Sigma \times \mathsf{beh}"] & 1 + \Sigma \times \Sigma^* \\
S \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}"'] & \Sigma^* \arrow[u, "\eta_{1+\Sigma\times\Sigma^*} \circ [\mathsf{nil}, \mathsf{cons}]^{-1} \; in \; \mathcal{K}\ell(\mathcal{P})."']
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$$\varepsilon \in \mathsf{beh}(s) \iff * \in c(s) \iff \text{state } s \text{ is accepting}$$

$$\sigma w \in \mathsf{beh}(s) \iff (\sigma, w) \in ((\Sigma \times \mathsf{beh}) \circ c)(s) = \{(\sigma, \mathsf{beh}(t)) \mid (\sigma, t) \in c(s)\} \iff \exists t.(t \in \delta(s)(\sigma) \wedge w \in \mathsf{beh}(t)).$$

Possibly Infinite Behavior

Theorem [Jacobs 2004]: A final coalgebra $\xi : Z \to FZ$ yields a *weakly final* coalgebra

$$(\eta_{FZ} \circ \xi) : Z \to \overline{F}(Z) \ in \ \mathcal{K}\ell(\mathcal{P})$$

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$$\begin{tikzcd}
\overline{F}S \arrow[r, squiggly, "\overline{F}(\mathsf{beh})"] & \overline{F}Z \\
S \arrow[u, "c"] \arrow[r, squiggly, "\mathsf{beh}"'] & Z \arrow[u, "\eta_{FZ} \circ \xi"']
\end{tikzcd} \quad \mathit{in}\ \mathcal{K}\ell(\mathcal{P}),$$

beh is not unique. However, we can take beh^∞, the maximal mapping with respect to inclusion.

Possibly Infinite Behavior

$\xi : \Sigma^\infty \to 1 + \Sigma \times \Sigma^\infty$ is the final F-coalgebra, defined by $\xi(\varepsilon) = * \in 1$ and $\xi(\sigma w) = (\sigma, w)$ ($\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$).

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$$\begin{tikzcd}
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Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

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Definition: The *solution* to this *equational system* is calculated as follows:

- Intermediate solution $l_1^{(1)} := \mu u_1.f_1(u_1, u_2)$
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Concretely:
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- $l_2^{\mathrm{sol}} := \nu u_2.u_2 \Diamond_\delta([\mu u_1.\Diamond_\delta([u_1, u_2]) \upharpoonright S_1, u_2]) \upharpoonright S_2, u_2)$
- $l_1^{\mathrm{sol}} = \mu u_1.\Diamond_\delta\big(\big[u_1, \nu u_2.u_2 \stackrel{\nu}{=} \Diamond_\delta([\mu u_1'.\Diamond_\delta([u_1', u_2]) \upharpoonright S_1, u_2]) \upharpoonright S_2, u_2\big)\big]\big) \upharpoonright S_1$

Let $A = \langle S, \Sigma, \delta, s_0, F \rangle$ be a Büchi automaton. Take $S_1 = S \setminus F$, $S_2 = F$. Model δ by coalgebras $c_1 : S_1 \to \mathcal{P}(\Sigma \times S)$, $c_2 : S_2 \to \mathcal{P}(\Sigma \times S)$. Take the initial algebra $d : \Sigma^\omega \to \Sigma \times \Sigma^\omega$ defined by $d(\sigma w) = (\sigma, w)$ in **Sets**.

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S_1 \arrow[u, "c_1"] \arrow[r, squiggly, "\mathsf{beh}_1"'] \arrow[ur, phantom, "=_{\mu}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma \times S \arrow[r, squiggly, "{\Sigma \times [\mathsf{beh}_1, \mathsf{beh}_2]}"] & \Sigma \times \Sigma^\omega \\
S_2 \arrow[u, "c_2"] \arrow[r, squiggly, "\mathsf{beh}_2"'] \arrow[ur, phantom, "=_{\nu}"] & \Sigma^\omega \arrow[u, "\eta_{\Sigma^\omega} \circ d"']
\end{tikzcd}
\qquad \textit{in } \mathcal{K}\ell(\mathcal{P}).$$

$$\mathsf{beh}_1 \overset{\mu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_1 \qquad \mathsf{beh}_2 \overset{\nu}{=} \Diamond_\delta([\mathsf{beh}_1, \mathsf{beh}_2]) \upharpoonright S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \to (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\mathsf{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \mathsf{beh}(s')\}.$$

Theorem [Urabe, Shimizu, Hasuo 2016]: The solutions $\mathsf{beh}_1, \mathsf{beh}_2$ to the system of equations coincide with the accepted language of the Büchi Automaton A.

Alternate Proof of Coincidence Result

Problem: system of fixed point equations is convoluted.

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Alternate derivation using game semantics:
Game Semantics For Modal Mu-Calculus: $s \vDash^T \varphi \iff$ verifier has a winning strategy in $\mathcal{G}(\varphi, T)$

Outline:

- Convert system of equations to modal mu-calculus formula
- Apply game semantics of modal mu-calculus
- Prove: V has a winning strategy in $\mathcal{G}(\varphi, T)$ from state $(s_i, w) \iff w \in \mathsf{beh}(s_i)$

Proof Outline

Converting formula:

$$l^2_{\mathrm{sol}} = \nu u_2.\Diamond^2_\delta \left[\left(\mu u'_1.\Diamond^1_\delta[u'_1, u_2]\right), u_2\right]$$

$$\overline{\varphi_2} = \nu u_2.(p_2 \wedge \Diamond((\mu u'_1.p_1 \wedge \Diamond((p_1 \wedge u'_1) \vee (p_2 \wedge u_2))) \vee (p_2 \wedge u_2)))$$

Proof Outline

Converting formula:

$$l_{\text{sol}}^2 = \nu u_2.\Diamond_\delta^2\left[\left(\mu u_1'.\Diamond_\delta^1[u_1', u_2]\right), u_2\right]$$

$$\overline{\varphi_2} = \nu u_2.(p_2 \wedge \Diamond((\mu u_1'.p_1 \wedge \Diamond((p_1 \wedge u_1') \vee (p_2 \wedge u_2))) \vee (p_2 \wedge u_2))))$$

Defining Transition System for Büchi Automaton A:

Let $A = (S_1 \cup S_2, \Sigma, \delta)$ be a Büchi automaton. Let Transition System (TS) over the set of propositional variables $\{p_1, p_2\}$, denoted as T_A, as follows:

- States: (s, w) for $s \in S$ and $w \in \Sigma^\omega$
- Transition $(s, \sigma w) \to (s', w)$ for $s, s' \in S$, $\sigma \in \Sigma$, $w \in \Sigma^\omega$, iff $s' \in \delta(s)(\sigma)$
- Labeling function: $\lambda((s, w)) = \{p_i\}$ iff $s \in S_i$

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Game Semantics For Modal Mu-Calculus: $(s, w) \overset{T}{\vDash} \varphi$ iff Verifier has a winning strategy in $\mathcal{G}(\varphi, T_A)$ from state $(\varphi, (s, w))$

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Converting formula:

$$l_{\text{sol}}^2 = \nu u_2 . \Diamond_\delta^2 \left[\left(\mu u_1' . \Diamond_\delta^1 [u_1', u_2] \right), u_2 \right]$$

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Lemma: Verifier has a winning strategy in $\mathcal{G}(\varphi, T_A)$ from state iff the Büchi automaton accepts w from s.

Conclusion

1. Büchi Automata
   - Modeling infinite behavior
2. Coalgebraic Representation Nondeterministic Finite Automata
   - Work in $\mathcal{K}\ell(\mathcal{P})$
   - Initial coalgebra in **Sets** yields final coalgebra in $\mathcal{K}\ell(\mathcal{P})$
3. Coalgebraic Representation Possibly Infinite Behavior Nondeterministic Finite Automata
   - Final coalgebra in **Sets** yields weakly final coalgebra in $\mathcal{K}\ell(\mathcal{P})$
   - This adds infinite behavior
4. Coalgebraic Representation Büchi Automata
   - Split $S = S_1 \cup S_2$
   - Take those traces which are solution to system of fixed point equations
5. Outline Derivation using Game Semantics
   - Use game semantics for modal mu-calculus to obtain more comprehensive proof of coincidence