

Coalgebraic Representation of Büchi Automata

Research Internship Presentation

Jorrit de Boer

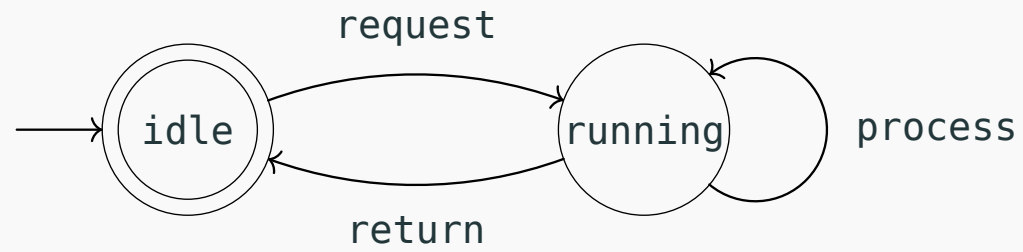
30 January 2025

Outline:

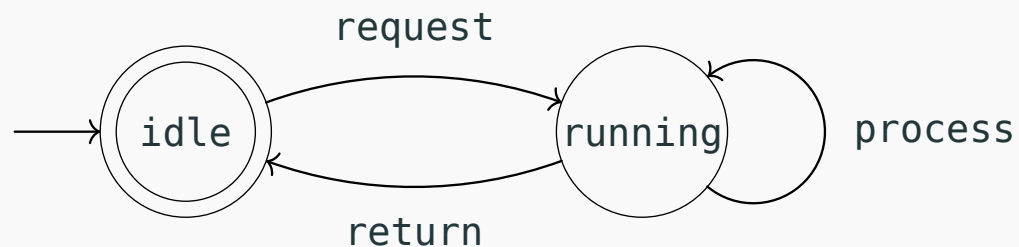
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Büchi Automata

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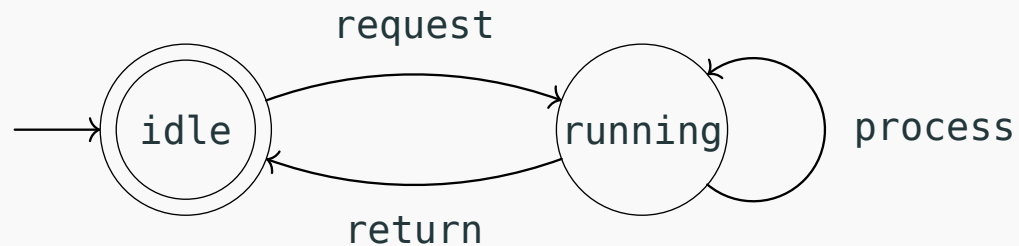
Büchi Automata



Definition: A (nondeterministic) *Büchi Automaton*

$A = \langle S, \Sigma, \delta, s_0, F \rangle$, where:

- S : finite set of states
- Σ : alphabet
- $s_0 \in S$: initial state
- $\delta : S \times \Sigma \rightarrow \mathcal{P}(S)$: transition function
- $F \subseteq S$: set of *final* (or *accepting*) states.

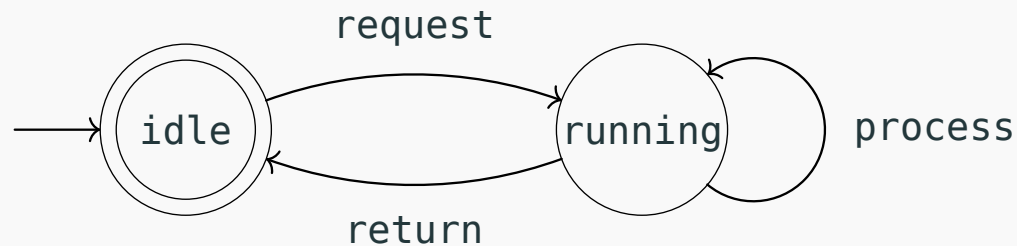


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Word $w \in L(A)$ iff there is a run through the automaton following w that visits F infinitely often.



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Accepted language:

$(\text{request} \cdot \text{process}^* \cdot \text{return})^\omega$

Coalgebra

Coalgebraic Representation Deterministic Finite Automata

$\langle S, \Sigma, \delta, o \rangle$ with states S , alphabet Σ , transition function $\delta : S \times \Sigma \rightarrow S$, $o : S \rightarrow 2$ ($2 = \{0, 1\}$). Can be represented by a coalgebra $\langle o, \delta \rangle : S \rightarrow 2 \times S^\Sigma$ for functor $FS = 2 \times S^\Sigma$

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The final coalgebra for F is $\langle e, d \rangle : 2^{\Sigma^*} \rightarrow 2 \times (2^{\Sigma^*})^\Sigma$. Where

- $e(L) = L(\varepsilon)$, i.e. $e(L) = 1$ iff $\varepsilon \in L$
- $d(L)(\sigma) = L_\sigma$ where $L_\sigma(w) = L(\sigma w)$ so $w \in d(L)(\sigma)$ iff $\sigma w \in L$

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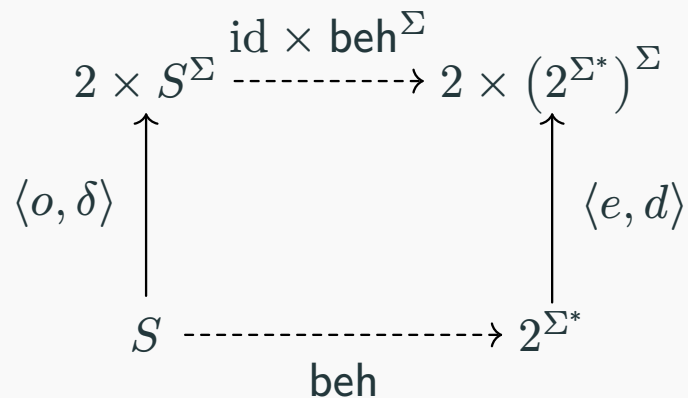
$$\begin{array}{ccc}
 2 \times S^\Sigma & \xrightarrow{\text{id} \times \text{beh}^\Sigma} & 2 \times (2^{\Sigma^*})^\Sigma \\
 \uparrow \langle o, \delta \rangle & & \uparrow \langle e, d \rangle \\
 S & \xrightarrow{\text{beh}} & 2^{\Sigma^*}
 \end{array}$$

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Following the paths through the diagram we obtain:

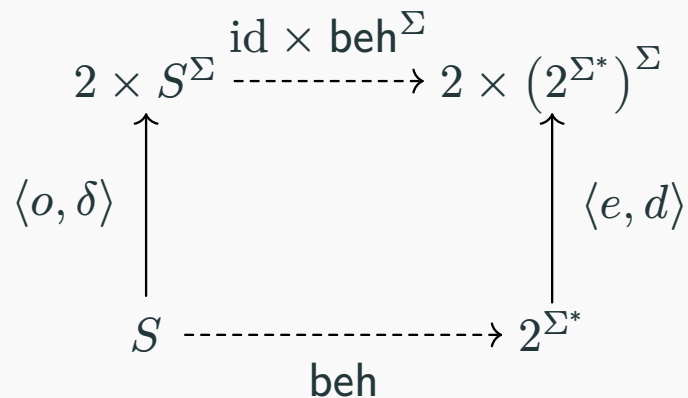
- $\text{beh}(s)(\varepsilon) = e(\text{beh}(s)) = o(s)$, and
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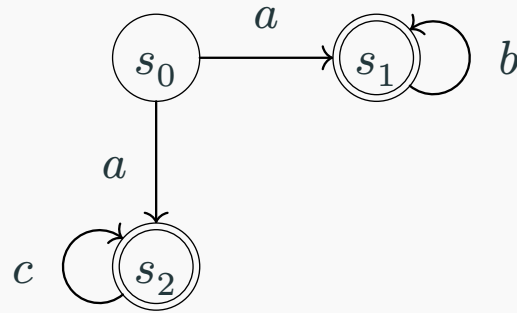


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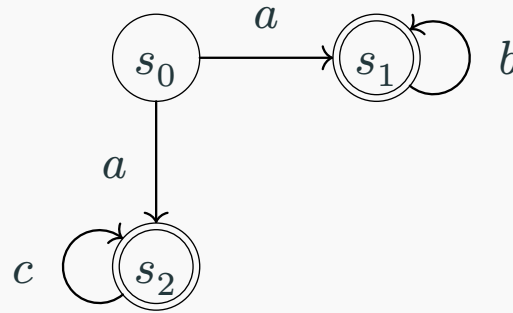
So beh captures exactly the accepted language of the automaton!

Nondeterministic Finite Automata



Might be modeled by coalgebra $c : S \rightarrow 2 \times \mathcal{P}(\Sigma \times S)$.

Nondeterministic Finite Automata



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A final coalgebra $z : Z \rightarrow 2 \times \mathcal{P}(\Sigma \times Z)$ cannot exist. Lambek's lemma says z would have to be an isomorphism, which would imply $Z \cong \mathcal{P}(Z)$

Kleisli Category of the monad \mathcal{P} :

A coalgebra $c : S \rightarrow \Sigma \times S$ in $\mathcal{KL}(\mathcal{P})$ is $c : S \rightarrow \mathcal{P}(\Sigma \times S)$ in **Sets**.

Kleisli Category of the monad \mathcal{P} :

A coalgebra $c : S \rightarrow \Sigma \times S$ in $\mathcal{K}\ell(\mathcal{P})$ is $c : S \rightarrow \mathcal{P}(\Sigma \times S)$ in **Sets**. Concretely:

- **objects**: the same as for **Sets**, sets
- **morphisms**: $f : X \rightarrow Y$ in $\mathcal{K}\ell(\mathcal{P})$ is $f : X \rightarrow \mathcal{P}(Y)$ in **Sets**.

For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in $\mathcal{K}\ell(\mathcal{P})$ we define

$$g \odot f := X \xrightarrow{f} \mathcal{P}(Y) \xrightarrow{\mathcal{P}(g)} \mathcal{P}(\mathcal{P}(Z)) \xrightarrow{\mu_Y} \mathcal{P}(Z)$$

Model NFA $\langle S, \Sigma, \delta, o \rangle$ by coalgebra $c : S \rightarrow 1 + \Sigma \times S$ for the functor $FS = 1 + \Sigma \times S$, which is $c : S \rightarrow \mathcal{P}(1 + \Sigma \times S)$ in **Sets**.

Lift functor F in **Sets** to \overline{F} in $\mathcal{K}\ell(\mathcal{P})$

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Lift functor F in **Sets** to \overline{F} in $\mathcal{K}\ell(\mathcal{P})$

Theorem [Hasuo, Jacobs, Sokolova 2007]: An initial algebra $\alpha : FA \rightarrow A$ for the functor F in **Sets** yields the final coalgebra for \overline{F} in $\mathcal{K}\ell(\mathcal{P})$:

$$(\eta_{FA} \circ \alpha^{-1}) : A \rightarrow \overline{F}A \text{ in } \mathcal{K}\ell(\mathcal{P})$$

Coalgebraic Representation Nondeterministic Automata

The initial algebra for $FS = 1 + \Sigma \times S$ is $[\text{nil}, \text{cons}] : 1 + \Sigma \times \Sigma^* \rightarrow \Sigma^*$:

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$$\begin{array}{ccc}
 1 + \Sigma \times S & \xrightarrow{1 + \Sigma \times \text{beh}} & 1 + \Sigma \times \Sigma^* \\
 \uparrow c & & \uparrow \eta_{1 + \Sigma \times \Sigma^*} \circ [\text{nil}, \text{cons}]^{-1} \text{ in } \mathcal{K}\ell(\mathcal{P}). \\
 S & \xrightarrow{\text{beh}} & \Sigma^*
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$$\begin{array}{ccc}
 & 1 + \Sigma \times \text{beh} & \\
 1 + \Sigma \times S & \xrightarrow{\quad\quad\quad} & 1 + \Sigma \times \Sigma^* \\
 \uparrow c & & \uparrow \eta_{1+\Sigma \times \Sigma^*} \circ [\text{nil}, \text{cons}]^{-1} \text{ in } \mathcal{K}\ell(\mathcal{P}). \\
 S & \xrightarrow{\quad\quad\quad \text{beh} \quad\quad\quad} & \Sigma^*
 \end{array}$$

$$\begin{aligned}
 \varepsilon \in \text{beh}(s) &\iff * \in c(s) \iff \text{state } s \text{ is accepting} \\
 \sigma w \in \text{beh}(s) &\iff (\sigma, w) \in ((\Sigma \times \text{beh}) \circ c)(s) \\
 &= \{(\sigma, \text{beh}(t)) \mid (\sigma, t) \in c(s)\} \\
 &\iff \exists t. (t \in \delta(s)(\sigma) \wedge w \in \text{beh}(t)).
 \end{aligned}$$

Theorem [Jacobs 2004]: A final coalgebra $\xi : Z \rightarrow FZ$ yields a *weakly final* coalgebra

$$(\eta_{FZ} \circ \xi) : Z \rightarrow \overline{F}(Z) \text{ in } \mathcal{K}\ell(\mathcal{P})$$

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$$\begin{array}{ccc} \overline{F}S & \overset{\overline{F}(\text{beh})}{\rightsquigarrow} & \overline{F}Z \\ \uparrow c & & \uparrow \eta_{FZ} \circ \xi \\ S & \overset{\text{beh}}{\rightsquigarrow} & Z \end{array} \text{ in } \mathcal{K}\ell(\mathcal{P}),$$

beh is not unique. However, we can take beh^∞ , the maximal mapping with respect to inclusion.

$\xi : \Sigma^\infty \rightarrow 1 + \Sigma \times \Sigma^\infty$ is the final F -coalgebra, defined by $\xi(\varepsilon) = * \in 1$ and $\xi(\sigma w) = (\sigma, w)$
($\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$).

Possibly Infinite Behavior

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$$\begin{array}{ccc}
 & 1 + \Sigma \times \text{beh}_c^\infty & \\
 1 + \Sigma \times S & \rightsquigarrow & 1 + \Sigma \times \Sigma^\infty \\
 \uparrow c & & \uparrow \cong \eta_{1+\Sigma \times \Sigma^\infty} \circ \xi \text{ in } \mathcal{K}\ell(\mathcal{P}). \\
 S & \rightsquigarrow_{\text{beh}_c^\infty} & \Sigma^\infty
 \end{array}$$

$\varepsilon \in \text{beh}^\infty(s) \iff * \in c(s) \iff \text{state } s \text{ is accepting}$

$\sigma w \in \text{beh}^\infty(s) \iff \exists t. (s \xrightarrow{\sigma} t \wedge w \in \text{beh}^\infty(t)).$

Idea: split $S = S_1 \cup S_2$ for S_1 non-accepting and S_2 accepting

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 \text{=}_{\mu} \\
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 \Sigma \times S \rightsquigarrow \Sigma \times \Sigma^\omega \\
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 \text{=}_{\nu} \\
 S_2 \rightsquigarrow \Sigma^\omega \\
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 \text{=} \nu \\
 S_2 \rightsquigarrow \Sigma^\omega \\
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 \end{array}
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 \end{array}$$

$$\text{beh}_1 \stackrel{\mu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\text{beh}_1, \text{beh}_2] \odot c_1$$

$$\text{beh}_2 \stackrel{\nu}{=} (\eta_{\Sigma^\omega} \circ d)^{-1} \odot \overline{F}[\text{beh}_1, \text{beh}_2] \odot c_2$$

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Rewrite to:

$$\text{beh}_1 \stackrel{\mu}{=} \Diamond_\delta([\text{beh}_1, \text{beh}_2]) \upharpoonright S_1$$

$$\text{beh}_2 \stackrel{\nu}{=} \Diamond_\delta([\text{beh}_1, \text{beh}_2]) \upharpoonright S_2$$

Where $\Diamond_\delta : (\mathcal{P}(\Sigma^\omega))^S \rightarrow (\mathcal{P}(\Sigma^\omega))^S$ is given by

$$\Diamond_\delta(\text{beh})(s) = \{\sigma \cdot w \mid s' \in \delta(s)(\sigma), w \in \text{beh}(s')\}.$$

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The *solution* to this *equational system* is (informally):

- Take the *least* fixed point at $\text{beh}_1 \stackrel{\mu}{=} \Diamond_\delta([\text{beh}_1, \text{beh}_2]) \upharpoonright S_1$
- Take the *greatest* fixed point at $\text{beh}_2 \stackrel{\nu}{=} \Diamond_\delta([\text{beh}_1, \text{beh}_2]) \upharpoonright S_2$

Let $A = \langle S, \Sigma, \delta, s_0, F \rangle$ be a Büchi automaton. Take $S_1 = S \setminus F$, $S_2 = F$. Model δ by coalgebras $c_1 : S_1 \rightarrow \mathcal{P}(\Sigma \times S)$, $c_2 : S_2 \rightarrow \mathcal{P}(\Sigma \times S)$. Take the initial algebra $d : \Sigma^\omega \rightarrow \Sigma \times \Sigma^\omega$ defined by $d(\sigma w) = (\sigma, w)$ in **Sets**.

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 \end{array}
 & \overset{=}{\underset{\mu}{\rightsquigarrow}} &
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 \text{beh}_2
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 \quad \text{in } \mathcal{K}\ell(\mathcal{P}).$$

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Theorem [Urabe, Shimizu, Hasuo 2016]: The solutions $\text{beh}_1, \text{beh}_2$ to the system of equations coincide with the accepted language of the Büchi Automaton A .

Alternate Proof of Coincidence Result

Problem: system of fixed point equations is convoluted.

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Alternate derivation using game semantics:

Game Semantics For Modal Mu-Calculus: $s \models^T \varphi \iff$ verifier has a winning strategy in $\mathcal{G}(\varphi, T)$

Outline:

- Convert system of equations to modal mu-calculus formula
- Apply game semantics of modal mu-calculus
- Prove: V has a winning strategy in $\mathcal{G}(\varphi, T)$ from state $(s_i, w) \iff w \in \text{beh}(s_i)$

Converting formula:

$$l_{\text{sol}}^2 = \nu u_2. \Diamond_{\delta}[(\mu u'_1. \Diamond_{\delta}[u'_1, u_2]), u_2]$$

$$\overline{\varphi_2} = \nu u_2. (p_2 \wedge \Diamond((\mu u'_1. p_1 \wedge \Diamond((p_1 \wedge u'_1) \vee (p_2 \wedge u_2))) \vee (p_2 \wedge u_2))))$$

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Defining Transition System for Büchi Automaton A :

Let $A = (S_1 \cup S_2, \Sigma, \delta)$ be a Büchi automaton. Let Transition System (TS) over the set of propositional variables $\{p_1, p_2\}$, denoted as T_A , as follows:

- States: (s, w) for $s \in S$ and $w \in \Sigma^{\omega}$
- Transition $(s, \sigma w) \rightarrow (s', w)$ for $s, s' \in S, \sigma \in \Sigma, w \in \Sigma^{\omega}$, iff $s' \in \delta(s)(\sigma)$
- Labeling function: $\lambda((s, w)) = \{p_i\}$ iff $s \in S_i$

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- States: (s, w) for $s \in S$ and $w \in \Sigma^{\omega}$
- Transition $(s, \sigma w) \rightarrow (s', w)$ for $s, s' \in S, \sigma \in \Sigma, w \in \Sigma^{\omega}$, iff $s' \in \delta(s)(\sigma)$
- Labeling function: $\lambda((s, w)) = \{p_i\}$ iff $s \in S_i$

Game Semantics For Modal Mu-Calculus: $(s, w) \stackrel{T}{\models} \varphi$ iff Verifier has a winning strategy in $\mathcal{G}(\varphi, T_A)$ from state $(\varphi, (s, w))$

Converting formula:

$$l_{\text{sol}}^2 = \nu u_2. \Diamond_{\delta}[(\mu u'_1. \Diamond_{\delta}[u'_1, u_2]), u_2]$$

$$\overline{\varphi_2} = \nu u_2. (p_2 \wedge \Diamond((\mu u'_1. p_1 \wedge \Diamond((p_1 \wedge u'_1) \vee (p_2 \wedge u_2))) \vee (p_2 \wedge u_2))))$$

Defining Transition System for Büchi Automaton A :

Let $A = (S_1 \cup S_2, \Sigma, \delta)$ be a Büchi automaton. Let Transition System (TS) over the set of propositional variables $\{p_1, p_2\}$, denoted as T_A , as follows:

- States: (s, w) for $s \in S$ and $w \in \Sigma^{\omega}$
- Transition $(s, \sigma w) \rightarrow (s', w)$ for $s, s' \in S, \sigma \in \Sigma, w \in \Sigma^{\omega}$, iff $s' \in \delta(s)(\sigma)$
- Labeling function: $\lambda((s, w)) = \{p_i\}$ iff $s \in S_i$

Game Semantics For Modal Mu-Calculus: $(s, w) \models^T \varphi$ iff Verifier has a winning strategy in $\mathcal{G}(\varphi, T_A)$ from state $(\varphi, (s, w))$

Lemma: Verifier has a winning strategy in $\mathcal{G}(\varphi, T_A)$ from state (s, w) iff the Büchi automaton accepts w from s .

1. Büchi Automata
 - Modeling infinite behavior
2. Coalgebraic Representation Nondeterministic Finite Automata
 - Work in $\mathcal{KL}(\mathcal{P})$
 - Initial coalgebra in **Sets** yields final coalgebra in $\mathcal{KL}(\mathcal{P})$
3. Coalgebraic Representation Possibly Infinite Behavior Nondeterministic Finite Automata
 - Final coalgebra in **Sets** yields weakly final coalgebra in $\mathcal{KL}(\mathcal{P})$
 - This adds infinite behavior
4. Coalgebraic Representation Büchi Automata
 - Split $S = S_1 \cup S_2$
 - Take those traces which are solution to system of fixed point equations
5. Outline Derivation using Game Semantics
 - Use game semantics for modal mu-calculus to obtain more comprehensive proof of coincidence