Hlab analysis methods part 0: Linear algebra refresher

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4/18/19

Linear Algebra: what is linearity?

Linear algebra is a branch of mathematics that deals with **linear systems**

suppose we have some function **F(x)** (i.e. y = F(x)), a **linear transform** of F(x) holds if:

1) $F(ax) = aF(x)$
2) $F(x_1 + x_2) = F(x_1) + F(x_2)$

In other words, multiplication and addition inside or outside of F(x) is equivalent.

Linear Algebra: what is a linear system

A **linear system** is a series of linear equations (i.e. each equation holds under a **linear transformation** as we just discussed)

$$
\begin{aligned}
y_1 &= x_1 + 2x_3 + 4 \\
y_2 &= 2x_2 - 0.5x_3 + 3 \\
y_3 &= 6x_1 - 5x_3 + 6
\end{aligned}
$$

Instead, we can compactly represent our system of equations and use linear algebra to help us solve it:

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}
$$

Single columns (i.e. [y1,y2,y3]) are called **vectors**, while 2D are **matrices**

Linear Algebra: what is a linear system

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \text{x1 column} & & \text{x3 column} & \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \\ \begin{array}{cccc} & \text{x2 column} & & \text{constants} \end{array} \end{array} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Now we can think in terms of:
columns = inputs
rows = outputs

$$\begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix}$$

$$\times$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix}$$

here we **transpose** the vector **x**

Linear Algebra: what is a linear system

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

y (3x1) **A** (3x4) **x** (4x1)

This is the general form of a linear system

$$y = Ax$$

$$x = yA^{\dagger}$$

and we can solve for x using the **pseudoinverse** of A

Linear Algebra: the dot product

Linear algebra has different forms the multiplication operation.

The **dot product** is defined as:

$$\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + \ldots a_n b_n$$

This gives us a **scalar** for two vectors. So each component in our **y** vector is the result of a **dot product** between the corresponding **row** in our **A** matrix, and the **x** vector.

The dot product gives us many important descriptions of vectors:

1) **vector norm** (i.e. magnitude):

$$||a||^2 = a \cdot a = a_1^2 + a_2^2 + \ldots a_n^2$$

2) **angle between vectors**:

$$v \cdot w = ||v|| \, ||w|| \cos(\theta)$$

3) **orthogonal vectors**

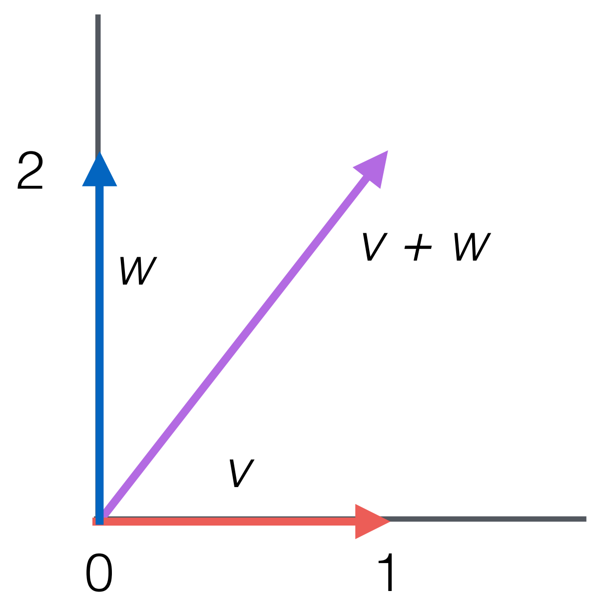
$$v \cdot w = 0$$

Linear Algebra: geometrical interpretation

It’s useful to think about these concepts in terms of the geometry of vectors

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

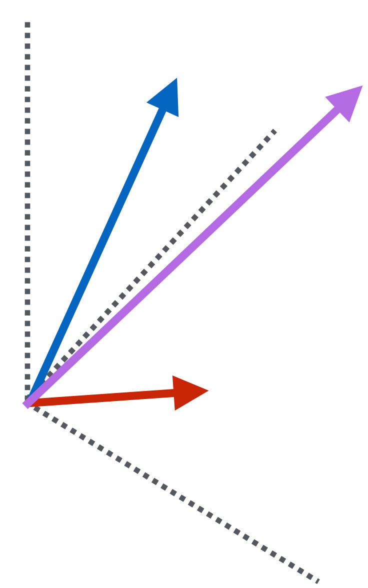
$$w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



$v \bullet w = 0$

|| v ||^2 = 1
|| w ||^2 = 4
|| $v+w$ ||^2 = 5

here both *v* and *w* are 2-dimensional



Linear Algebra: matrix operations

When we perform similar operations between **matrices** instead of vectors, we simply repeat the procedure.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} & \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$$

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$$= \begin{bmatrix} (2+12) & (4+16) \\ (6+16) & (12+32) \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 22 & 44 \end{bmatrix}$$

Note that matrix multiplication does NOT hold under the commutative property