

# **Hlab analysis methods part 0: Linear algebra refresher**

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# Linear Algebra: what is linearity?

**Linear algebra** is a branch of mathematics that deals with **linear systems**

suppose we have some function **F(x)** (i.e.  $y = F(x)$ ), a **linear transform** of  $F(x)$  holds if:

$$1) F(ax) = aF(x)$$

$$2) F(x_1 + x_2) = F(x_1) + F(x_2)$$

In other words, multiplication and addition inside or outside of  $F(x)$  is equivalent.

# Linear Algebra: what is a linear system

A **linear system** is a series of linear equations (i.e. each equation holds under a **linear transformation** as we just discussed)

$$\begin{aligned}y_1 &= x_1 + 2x_3 + 4 \\y_2 &= 2x_2 - 0.5x_3 + 3 \\y_3 &= 6x_1 - 5x_3 + 6\end{aligned}$$

Instead, we can compactly represent our system of equations and use linear algebra to help us solve it:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Single columns (i.e.  $[y_1, y_2, y_3]$ ) are called **vectors**, while 2D are **matrices**

# Linear Algebra: what is a linear system

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \overset{\text{x1 column}}{1} & 0 & \overset{\text{x3 column}}{2} & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & \underset{\text{x2 column}}{0} & -5 & \underset{\text{constants}}{6} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Now we can think in terms of:  
**columns = inputs**  
**rows = outputs**

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \begin{matrix} \downarrow \\ \times \end{matrix} \begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix}$$

here we **transpose** the vector **x**

# Linear Algebra: what is a linear system

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & -0.5 & 3 \\ 6 & 0 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

**y**                      **A**                      **x**  
(3x1)                      (3x4)                      (4x1)

This is the general  
form of a linear system

$$y = Ax$$

$$x = yA^\dagger$$

and we can solve for x using  
the **pseudoinverse** of A

# Linear Algebra: the dot product

Linear algebra has different forms the multiplication operation.

The **dot product** is defined as:

$$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + \dots a_nb_n$$

This gives us a **scalar** for two vectors. So each component in our **y** vector is the result of a **dot product** between the corresponding **row** in our **A** matrix, and the **x** vector.

The dot product gives us many important descriptions of vectors:

1) **vector norm** (i.e. magnitude):

$$\|a\|^2 = a \cdot a = a_1^2 + a_2^2 + \dots a_n^2$$

2) **angle between vectors**:

$$v \cdot w = \|v\| \|w\| \cos(\theta)$$

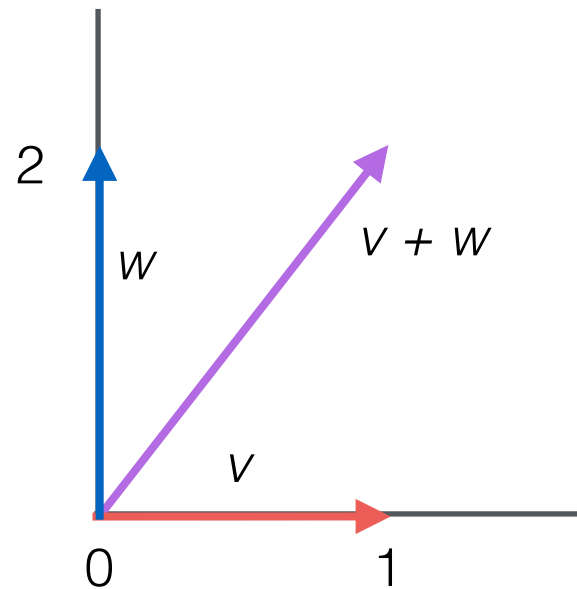
3) **orthogonal vectors**

$$v \cdot w = 0$$

# Linear Algebra: geometrical interpretation

It's useful to think about these concepts in terms of the geometry of vectors

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



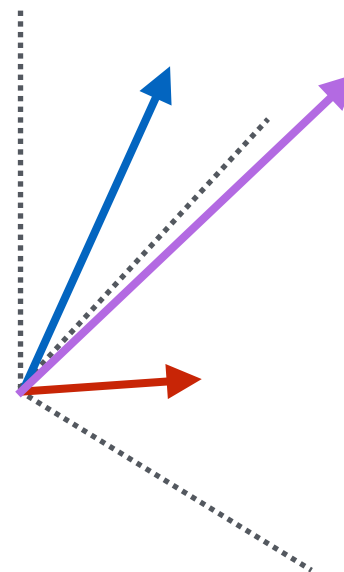
$$v \bullet w = 0$$

$$\|v\|^2 = 1$$

$$\|w\|^2 = 4$$

$$\|v+w\|^2 = 5$$

here both  $v$  and  $w$  are 2-dimensional



# Linear Algebra: matrix operations

When we perform similar operations between **matrices** instead of vectors, we simply repeat the procedure.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} [1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ [3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$$



# Linear Algebra: matrix operations

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# Linear Algebra: matrix operations

When we perform similar operations between **matrices** instead of vectors, we simply repeat the procedure.

The diagram illustrates the process of matrix multiplication for two 2x2 matrices. On the left, the first matrix  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  has its first row  $[1 \ 2]$  circled in blue. The second matrix  $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$  has its first column  $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$  circled in red. Blue arrows point from the circled row of the first matrix to the first row of the resulting matrix's components. Red arrows point from the circled column of the second matrix to the first column of the resulting matrix's components. The resulting matrix is shown as a 2x2 grid of dot products:  $\begin{bmatrix} [1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ [3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$ .

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} [1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ [3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$$

# Linear Algebra: matrix operations

When we perform similar operations between **matrices** instead of vectors, we simply repeat the procedure.

The diagram illustrates the multiplication of two matrices,  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ . The first matrix has its second row  $[3 \ 4]$  circled in blue. The second matrix has its second column  $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$  circled in red. The result is a  $2 \times 2$  matrix where each element is the dot product of a row from the first matrix and a column from the second matrix. The elements are:  $[1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ ,  $[1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ ,  $[3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ , and  $[3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ . Blue arrows point from the blue circle to the two dot products involving the row  $[3 \ 4]$ . Red arrows point from the red circle to the two dot products involving the column  $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ .

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} [1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ [3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$$

# Linear Algebra: matrix operations

When we perform similar operations between **matrices** instead of vectors, we simply repeat the procedure.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} [1 \ 2] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [1 \ 2] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ [3 \ 4] \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} & [3 \ 4] \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} (2+12) & (4+16) \\ (6+16) & (12+32) \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 22 & 44 \end{bmatrix}$$

**Note that matrix multiplication does NOT hold under the commutative property**