Chapter 1

Introduction

Chapter 2

Theory

2.1 Richard's Equation

The water flow in porous media can be described with the formula of Richard. The equation is here derived. The water flux density vector, $\mathbf{q}$ can be calculated by the Darcys law. For a two-dimensional vertical transect it yields:

$$\mathbf{q} = -\mathbf{K}(\psi)\nabla(\psi + z) \tag{2.1}$$

where $\mathbf{K}(\psi)$ is the hydraulic conductivity tensor, ψ is the potential head. The x-axis is chosen in horizontal direction and the z-axis is positive upwards. The conductivity tensor can be expressed as:

$$\mathbf{K} = \begin{bmatrix} K_{xx} & K_{xz} \\ K_{zx} & K_{zz} \end{bmatrix} \tag{2.2}$$

$$\mathbf{K} = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{zz} \end{bmatrix} \tag{2.3}$$

The mass balance for the system gives

$$\frac{\partial \theta}{\partial t} = -\nabla \cdot \mathbf{q} - \Gamma \tag{2.4}$$

where θ is the water content and Γ is the sink term. The partial differential equation can be developed by combining Darcys law, equation 2.1 and the mass balance, equation 2.4, thus

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (\mathbf{K}(\psi)\nabla(\psi + z)) - \Gamma \tag{2.5}$$

This is known as Richard's equation. For the modelling it is assumed that the soil-water retention is without hysteresis, i.e. there is a unique relation between the matric pressure potential and the water content.

In order to describe Richard equation in ψ the specific water capacity, $C = \partial\theta/\partial\psi$ has applied. To solve Richard's equation it is necessary to specify initial and boundary conditions. The boundary conditions specify a combination of ψ and its derivative on the boundary. In TopFEM it is possible to use different forms of flux (Neumann) and predescribed pressure (Dirichlet) boundary conditions. The problem to be solved for determining the water movement can be summarized to

$$\begin{cases} C \cdot \frac{\partial\psi}{\partial t} = \nabla \cdot (\mathbf{K}(\psi)\nabla(\psi + z)) - \Gamma & \text{in } \Omega \\ \bar{\mathbf{n}} \cdot (\mathbf{K}(\psi)\nabla(\psi + z)) = -q & \text{on } \partial\Omega_1 \\ \psi = \psi_0 & \text{on } \partial\Omega_2 \end{cases} \tag{2.6}$$

where $\bar{\mathbf{n}}$ is the outward unit normal, and q is the size of the size of the outward flow from the domain. ψ_0 is the predescribed pressure at the boundary. $\partial\Omega_1$ and $\partial\Omega_2$ are part of the boundary with Neumann and Dirichlet boundaries, respectively. Each of $\partial\Omega_1$ and $\partial\Omega_2$ are not necessarily one continuous curve piece. For lower boundary condition is a special case of the Neumann boundary conditions often applied. It here assumed that the flow it is only driven by the gravity (gravity boundary condition), i.e. $\partial\psi/\partial x = \partial\psi/\partial y = 0$ which gives

$$q = \bar{\mathbf{n}} \cdot \begin{bmatrix} 0 \\ K_{zz} \end{bmatrix} \tag{2.7}$$

Another often used boundary condition is the seepage boundary condition where we for $\psi > 0$ have specified pressure corresponding to the depth of the overlaying water. For $\psi \leq 0$ have specified flow to be equal to 0, i.e. a Dirichlet boundary condition for $\psi > 0$ and a Neumann boundary condition for $\psi \leq 0$.

Chapter 3

2 Dimensional Model

3.1 FVM Model

Richards ligning, integration over kontrol-volumen (celle) og Gauss-Green divergens teorem:

$$\int_{Q_i} \frac{\partial \theta}{\partial t} d\mathbf{x} = \int_{\partial Q_i} (\mathbf{K}(\psi)\nabla(\psi + z)) \cdot \bar{\mathbf{n}} dl - \int_{Q_i} \Gamma d\mathbf{x} \tag{3.1}$$

Sammenfattes til:

$$|Q_i| \frac{d\theta_i}{dt} = \sum_{j \in \sigma_i} D_{ij}(\boldsymbol{\psi}) + \sum_{j \in \sigma_i} G_{ij}(\boldsymbol{\psi}) + \sum_{j' \in \sigma_i} B_{ij'}(\boldsymbol{\psi}) + S_i(\boldsymbol{\psi}) \tag{3.2}$$

hvor:
$D_{ij}(\boldsymbol{\psi})$ diffusiv transport mellem interne kanter
$G_{ij}(\boldsymbol{\psi})$ gravitation transport mellem interne kanter
$B_{ij'}(\boldsymbol{\psi})$ rand kanter $j' \in \sigma_i'$
$S_i(\boldsymbol{\psi})$ er kildeledet, der bde kan inkludere punkt- og fladekilder i cellen.

$$D_{ij}(\boldsymbol{\psi}) = |e_{ij}|(\mathbf{K}(\boldsymbol{\psi}) \cdot (\nabla\psi)_{ij}) \cdot \bar{\mathbf{n}}_{ij} \tag{3.3}$$

$$G_{ij}(\boldsymbol{\psi}) = |e_{ij}|(\mathbf{K}(\boldsymbol{\psi}) \cdot ([0\ 1]^T)) \cdot \bar{\mathbf{n}}_{ij} \tag{3.4}$$

3.1.1 Rektangulre celler

Taylorudvikling...

$$\psi_E = \psi(x + \delta x^+) = \sum_{k=0}^{m} \frac{1}{k!} \left(\frac{d^k \psi}{dx^k} \right)_f (\delta x^+)^k + R^+ \tag{3.5}$$

hvor m er Taylorudviklingens orden og R^+ er Lagranges restled. Tilsvarende fs

$$\psi_i = \psi(x - \delta x^-) = \sum_{k=0}^{m} \frac{1}{k!} \left(\frac{d^k \psi}{dx^k} \right)_f (-\delta x^-)^k + R^- \tag{3.6}$$

$R^+ - (-1)^{m+1} R^- \approx 0$ Hvis der vlges en frste ordens Taylorudvikling ($m = 1$) fs

$$\left(\frac{d\psi}{dx} \right)_f (\delta x^+ + \delta x^-) \approx \psi_E - \psi_i \tag{3.7}$$

Sfremt der vlges en hjere ordens Taylorudvikling fs:

$$\left(\frac{d\psi}{dx} \right)_f (\delta x^+ + \delta x^-) \approx \psi_E - \psi_i - \epsilon_{Ei} \tag{3.8}$$

hvor korrektionsledet kan beregnes som

$$\epsilon_{Ei} \approx \sum_{k=2}^{m} \frac{1}{k!} \left(\frac{d^n \psi}{dx^n} \right)_f \left[(\delta x^+)^n - (-\delta x^-)^n \right] \tag{3.9}$$

Det kan indses at der opns anden ordens prcision med $m = 1$ sfremt $\delta x^+ = \delta x^-$

$$\sum_{j \in \sigma_i} D_{ij}(\boldsymbol{\psi}) = \tag{3.10}$$

$$\sum_{j \in \sigma_i} D_{ij}(\boldsymbol{\psi}) = \tag{3.11}$$

Dirichlet randbetingelser

Simplest ville det vre..... Her benyttes dog

3.1.2 Trapezoide celler

Lineær rekonstruktion

$$\hat{\psi}(\mathbf{x}, t) = \psi_i(t) + \eta_i(\boldsymbol{\psi}) \cdot (\mathbf{x} - \mathbf{x}_i), \quad \mathbf{x} \in Q_i, \ t > 0 \tag{3.12}$$

Divergens teoremet:
Trekanter:

$$\overline{\nabla \psi} \approx \sum \psi_j \mathbf{n}_j A_j \approx \frac{1}{2|T_i|} \mathbf{R} \left[\psi_\alpha (\mathbf{x}_\beta - \mathbf{x}_\gamma) + \psi_\beta (\mathbf{x}_\gamma - \mathbf{x}_\alpha) + \psi_\gamma (\mathbf{x}_\alpha - \mathbf{x}_\beta) \right] \tag{3.13}$$

Firkanter:

$$\overline{\nabla\psi} \approx \sum \psi_j \mathbf{n}_j A_j \approx \frac{1}{2|Q_i|}\mathbf{R}\left[(\psi_\alpha - \psi_\gamma)(\mathbf{x}_\beta - \mathbf{x}_\delta) + (\psi_\beta - \psi_\delta)(\mathbf{x}_\gamma - \mathbf{x}_\alpha)\right] \tag{3.14}$$

hvor

$$\mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.15}$$

Harmonisk gennemsnit:

$$\frac{1}{K_{ij}} = \frac{1}{2}\left[\frac{1}{K(\psi_i)} + \frac{1}{K(\psi_j)}\right] \tag{3.16}$$

3.1.3 Iterations skema

$$\mathbf{Q}\frac{d\boldsymbol{\theta}}{dt} = \mathbf{E}(\boldsymbol{\psi})\boldsymbol{\psi} + \mathbf{F}(\boldsymbol{\psi}) \tag{3.17}$$

hvor $\mathbf{Q}$ er en diagonalmatrix med $Q(i,i) = |Q_i|$
Backward Euler:

$$\mathbf{Q}\frac{\boldsymbol{\theta}^{n+1,m+1} - \boldsymbol{\theta}^n}{\Delta t} = \mathbf{E}(\boldsymbol{\psi}^{n+1,m})\boldsymbol{\psi}^{n+1,m} + \mathbf{F}(\boldsymbol{\psi}^{n+1,m}) \tag{3.18}$$

massconservative Celia *et al.* (1990)

The ψ-based formulation have the disadvantage that it doesn't conserve the mass and can give erroneous estimate of infiltration depths, Celia *et al.* (1990). In the mixed formulation is the water content at time step $n+1$ and iteration step $m+1$ appromximated by a Taylor expansion:

$$\begin{aligned} \theta^{n+1,m+1} &= \theta^{n+1,m} + \frac{d\theta}{d\psi}\,|^{n+1,m}\,(\psi^{n+1,m+1} - \psi^{n+1,m}) \\ &= \theta^{n+1,m} + C^{n+1,m}(\psi^{n+1,m+1} - \psi^{n+1,m}) \end{aligned} \tag{3.19}$$

The time derivative of θ can then be approximated as:

$$\begin{aligned} \frac{\partial\theta}{\partial t} &\approx \frac{\theta^{n+1,m+1} - \theta_n}{\Delta t} = \frac{\theta^{n+1,m+1} - \theta^{n+1,m}}{\Delta t} + \frac{\theta^{n+1,m} - \theta_n}{\Delta t} \\ &\approx C^{n+1,m}\frac{\psi^{n+1,m+1} - \psi^{n+1,m}}{\Delta t} + \frac{\theta^{n+1,m} - \theta^n}{\Delta t} \end{aligned} \tag{3.20}$$

Hermed fs flgende iterative skema...

$$\begin{aligned} &\left(\frac{1}{\Delta t}\mathbf{QC}(\boldsymbol{\psi}^{n+1,m}) - \mathbf{E}(\boldsymbol{\psi}^{n+1,m})\right)\boldsymbol{\psi}^{n+1,m+1} = \\ &\mathbf{F}(\boldsymbol{\psi}^{n+1,m}) + \frac{1}{\Delta t}\mathbf{QC}(\boldsymbol{\psi}^{n+1,m})\boldsymbol{\psi}^{n+1,m} + \frac{1}{\Delta t}\mathbf{Q}\left(\boldsymbol{\theta}^n - \boldsymbol{\theta}^{n+1,m}\right) \end{aligned} \tag{3.21}$$

hvor $\mathbf{C}$ er en diagonalmatrix med $C(i,i) = C_i$

3.1.4 Lsning af matrixligning

blabre blabre

3.1.5 Hydrauliske modeller

In numerical models for the unsaturated zone the soil-water model by van Genuchten (1980) is widely used:

$$\theta = \begin{cases} \theta_r + \frac{\theta_s - \theta_r}{[1+|\alpha\psi|^n]^m} & \text{for } \psi < 0 \\ \theta_s & \text{for } \psi \geq 0 \end{cases} \tag{3.22}$$

where α, n and m are empirical parameters, θ_s and θ_r are the saturated and the residual water content, respectively. By combination with the hydraulic conductivity model by Mualem (1976) and choosing $m = 1 - 1/n$, the hydraulic conductivity can be calculated as

$$K = K_s S_e^{1/2}[1 - (1 - S_e^{1/m})^m]^2 \tag{3.23}$$

where K_s is the hydraulic conductivity at saturation and S_e is the effective saturation defined as

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} \tag{3.24}$$

The retention model by van Genuchten have been adopted to a large class of soils.

3.1.6 Ridge

For describing the geometry and producing the finite element mesh is the general FEM-code, FEMLAB (1998) used. In the actual case the two-dimensional geometry described using a so called geometry m-file. Of geometrical reasons only the half of a ridge is described. The soil profile is divided into 7 strata or subdomains with different soil properties which are described elsewhere in the paper. The ridge with the different subdomains is plotted in figure XXX. The ridge height can be described with a sine function:

$$f(x) = A\left[1 + sin\left(-\frac{\pi}{2} + 2\frac{\pi x}{W}\right)\right], \quad 0 \leq x \leq W/2 \tag{3.25}$$

where W is the width of the ridge and A is the amplitude of the sine wave which is the same as half of the ridge height. The curve only describes half a ridge that will be used for the modelling.

3.2 Verifikation

Her skal der st lidt Philipssjov

3.2.1 Philips Infiltration Model

Philip (1957b) showed that the infiltration depth as function of time and saturation can be written as a power series in $t^{\frac{1}{2}}$. The coefficients are then functions of soil water content, θ. From the expression for the infiltration depth, as function of water content and time it is relatively easy to derive the cumulative infiltration, also written as a power series in $t^{\frac{1}{2}}$. The assumptions for the theory is an 1-dimensional vertical flow into a homogenous soil semi-infinite soil column, initially with uniform water content. The upper boundary condition is a constantly held water content or pressure (Neumann boundary condition).

Results from infiltration experiments (e.g., Austin og Prendergast 1997; Mbagwu 1995, 1997; Maheswari 1997; Valiantzas *et al.* 2001) are often fitted to a infiltrations equation only consisting of the first 2 terms of the power series solution

$$I = S\sqrt{t} + At \tag{3.26}$$

where S is the often refereed sorptivity as defined in Philip (1969). For some of the experiments, the geometry, initial and boundary condition does not corresponds to the assumptions Philip made for using the power series as a solution to Richards equation. The appliance of the Philip infiltration model is here more of empirical than mechanistic nature.

3.3 Referencer

Liste med referencer:

- (Mollerup 2001)
- Philip (1957b)
- Philip (1955, 1957a)

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