Seminar 7

7.4 $f \in End_{\mathbb{R}}(\mathbb{R}^3)$

$f(x,y,z) = (-y+5z, x, y-5z)$

Determine a basis and the dimensions for Ker f and Im f

$v_1, v_2 \dots v_n \in {}_K V$

$\operatorname{rank}(v_1, v_2, \dots v_n) =$ max of lin. indp vectors among them

If $v_i \in K^n$, then:

$$\operatorname{rank}(v_1, v_2, \dots v_n) = \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ v_n \\ \hline \end{pmatrix} = \operatorname{rank}(v_1 | v_2 | \dots v_n |)$$

$$\operatorname{Ker} f = \{ v \in \mathbb{R}^3 \mid f(v) = 0 \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0 \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid (-y+5z, x, y-5z) = 0 \}$$

$$\begin{cases} -y+5z=0 \\ x=0 \\ y-5z=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=5z \end{cases}$$

$\operatorname{Ker} f = \{ (x,y,z) \in \mathbb{R}^3 \mid x=0, y=5z \}$

$\operatorname{Ker} f = \{ (0,5z,z) \mid z \in \mathbb{R} \} = \{ z(0,5,1) \mid z \in \mathbb{R} \}$

$\operatorname{Ker} f = \langle (\underbrace{0,5,1}) \rangle$

$\dim \operatorname{Ker} = 1$

$\operatorname{Im} f = \{ (-y+5z, x, y-5z) \mid x,y,z \in \mathbb{R} \}$

$\operatorname{Im} f = \{ (0,x,0) + (-y,0,y) + (5z,0,-5z) \mid x,y,z \in \mathbb{R} \}$

$\operatorname{Im} f = \{x(0,1,0) + y(-1,0,1) + z(5,0,-5) \,/\, x,y,z \in \mathbb{R}\}$

$\operatorname{Im} f = \langle (0,1,0), (-1,0,1), (5,0,-5) \rangle$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{vmatrix} = 0 \neq 3 \Rightarrow \uparrow \text{ are not lin. dependent}$$

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow (0,1), (-1,0) - \text{lin. indep.}$$

$\Rightarrow \operatorname{rank}((0,1,0),(-1,0,1),(-5,0,5)) = 2$

$((0,1,0),(-1,0,1),(-5,0,5))$ - basis for $\operatorname{Im} f$

$\dim \operatorname{Im} f = 2$

Def: V - K-v.s, $S, T \leq_K V$

$V = S + T \Leftrightarrow (\forall)\, v \in V \ \exists\, s \in S, \exists\, t \in T$ s.t. $v = s + t$

$V = S \oplus T \Leftrightarrow \forall v \in V \ \exists!\, s \in S, t \in T, v = s + t$

$V = S \oplus T \Leftrightarrow \begin{cases} V = S + T \\ S \cap T = 0 \end{cases}$

For every $S \leq_K V$, $\exists\, T \leq_K V$ s.t. $V = S \oplus T$

How do we find T?

Step 1: Find a basis $(v_1, \dots v_K)$ for S

Step 2: Complete this basis to a basis of V: $(v_1, v_2, \dots v_K, w_{K+1}, \dots w_n)$

Step 3: $T = \langle w_{K+1}, \dots\ w_n \rangle$

7.7 Determine a complement for the following subspaces:

i) $A = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + 2y + 3z = 0\}$ in $\mathbb{R}^3$

ii) $B = \{\ ax + bx^3 \,/\, a,b \in \mathbb{R}\}$ in $\mathbb{R}_3[x] = \{f \in \mathbb{R}[x] \,/\, \deg f \leq 3\}$

i) $A = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + 2y + 3z = 0\} = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = -2y - 3z\}$

$= \{(-2y-3z, y, z) \in \mathbb{R}^3\} = \{(-2y, y, 0) + (-3z, 0, z)\} = \{y(-2,1,0) + z(-3,0,1) \,/\, y,z \in \mathbb{R}\}$

$= \langle (-2,1,0), (-3,0,1) \rangle$

$M = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

$\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3 \neq 0 \Rightarrow \text{rank}(M) = 2 \Rightarrow (-2,1,0), (-3,0,1)$ – lin ind $\Rightarrow ((-2,1,0), (-3,0,1))$ – is a basis for A

$v = (a,b,c) \in \mathbb{R}^3$, $v \in A$

Let $v = (1,5,4) \notin A$

$((-2,1,0), (-3,0,1), (1,5,4))$ – basis for $\mathbb{R}^3$

$B = \langle (1,5,4) \rangle$ compl. for A

ii) $B = \langle (x, x^3) \rangle$

if $ax + bx^3 = 0 \Leftrightarrow a = b = 0$, so x and x^3 are lin indep $\Rightarrow (x, x^3)$ is a basis for B

We add the vector x^2, $x^2 \notin B \Rightarrow x^2, x, x^3$ are lin ind

$\langle x^2, x, x^3 \rangle = \{ax + bx^2 + cx^3 / a, b \in \mathbb{R}\}$

$1 \notin \langle x, x^2, x^3 \rangle$, so $(1, x, x^2, x^3)$ are lin ind since $\dim \mathbb{R}_3[x] = 4$

$\Rightarrow \langle (1, x, x^2, x^3) \rangle$ is a basis for $\mathbb{R}_3[x]$

$C = \langle 1, x^2 \rangle$, $B \oplus C = \mathbb{R}_3[x]$

Thm: (1^{st} dimension theorem):

$f: V \to V'$: linear map

$$\dim V = \underbrace{\dim(\text{Ker } f)}_{=\text{def } f} + \underbrace{\dim(\text{Im } f)}_{=\text{rank}(f)}$$

Thm: (2^{nd} – 11 –):

V: K–VS, $S, T \leq_K V$

$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$

7.10 Determine $\dim(S)$, $\dim(T)$, $\dim(S+T)$, $\dim(S \cap T)$

$S = \langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle$, $T = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \rangle$

$a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ b & b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a+b & a \\ b & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = 0$

$\Rightarrow \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)$ is lin indep + a basis for S

$\dim(S) = 2$

$a \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & a \\ a+b & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = 0$

$\Rightarrow \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right)$ is lin indep + a basis ~~for~~ T

$\dim(T) = 2$

$S+T = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

$a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a+b & a+c \\ b+c+d & b+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{cases} a+b = 0 \Rightarrow b = 0 \Rightarrow d = 0 \\ a+c = 0 \Rightarrow a = 0 \\ \left.\begin{array}{l} b+c+d = 0 \\ b+d = 0 \end{array}\right| \Rightarrow c = 0 \end{cases} \Rightarrow a = b = c = d = 0$

$\Rightarrow \dim(S+T) = 4$

$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 0 \Rightarrow S \cap T = 0 = \{0\}$