Seminar 10

Def: V, V' – K-v.s, $f: V \to V'$ – linear map

$B = (v_1, v_2, \dots v_n)$, $B' = (v_1', v_2', \dots . v_m')$

$$\boxed{[f]_{BB'} = \left([f(v_1)]_{B'} \; [f(v_2)]_{B'} \; \dots \; [f(v_n)]_{B'}\right)} \in M_{m,n}(K)$$

- $(\forall) v \in V : [f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$

10.2 $f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$, $f(x,y,z) = (y, -x)$

$B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$

$B' = (v_1', v_2') = ((1,1), (1,-2))$

$E' = (e_1', e_2') = ((1,0), (0,1))$

Find $[f]_{B,E'}$, $[f]_{BB'}$

* How to write a vector v in a basis $B = (v_1, \dots v_n)$

$v = \alpha_1 v_1 + \dots + \alpha_n v_n$, solve the system, $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

$f(v_1) = f(1,1,0) = (1,-1)$

$f(v_2) = f(0,1,1) = (1,0)$

$f(v_3) = f(1,0,1) = (0,-1)$

$[f(v_1)]_{E'} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $f(v_1) = \alpha_1 e_1' + \alpha_2 e_2'$

$f(v_1) = \alpha_1(1,0) + \alpha_2(0,1) = (1,-1) \Rightarrow (\alpha_1, 0) + (0, \alpha_2) = (1,-1) \Rightarrow (\alpha_1, \alpha_2) = (1,-1) \Rightarrow \begin{matrix} \alpha_1 = 1 \\ \alpha_2 = -1 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$f(v_2) = \alpha_3(1,0) + \alpha_4(0,1) = (1,0) \Rightarrow (\alpha_3, \alpha_4) = (1,0) \Rightarrow \begin{matrix} \alpha_3 = 1 \\ \alpha_4 = 0 \end{matrix} \Rightarrow [f(v_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(v_3) = \alpha_5(1,0) + \alpha_6(0,1) = (0,-1) \Rightarrow (\alpha_5, \alpha_6) = (0,-1) \Rightarrow \begin{matrix} \alpha_5 = 0 \\ \alpha_6 = -1 \end{matrix} \Rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$[f]_{B,E'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

$f(v_1) = (1,-1)$

$f(v_2) = (1,0)$

$f(v_3) = (0,-1)$

$f(v_1) = \alpha_1(1,1) + \alpha_2(1,-2) = (1,-1) \Rightarrow (\alpha_1+\alpha_2, \alpha_1 - 2\alpha_2) = (1,-1)$

$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 1 \mid \cdot 2 \\ \alpha_1 - 2\alpha_2 = -1 \end{cases} \oplus$

$3\alpha_1 = 1 \Rightarrow \alpha_1 = \frac{1}{3} \Rightarrow \alpha_2 = \frac{2}{3} \Rightarrow [f(v_1)]_{B'} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$f(v_2) = \alpha_3(1,1) + \alpha_4(1,-2) = (1,0) \Rightarrow (\alpha_3+\alpha_4, \alpha_3 - 2\alpha_4) = (1,0)$

$\begin{cases} \alpha_3 + \alpha_4 = 1 \mid \cdot 2 \\ \alpha_3 - 2\alpha_4 = 0 \end{cases} \oplus$

$3\alpha_3 = 2 \Rightarrow \alpha_3 = \frac{2}{3} \Rightarrow \alpha_4 = \frac{1}{3} \Rightarrow \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

$f(v_3) = \alpha_5(1,1) + \alpha_6(1,-2) = (0,-1) \Rightarrow (\alpha_5+\alpha_6, \alpha_5 - 2\alpha_6) = (0,-1)$

$\begin{cases} \alpha_5 + \alpha_6 = 0 \mid \cdot 2 \\ \alpha_5 - 2\alpha_6 = -1 \end{cases} \oplus$

$3\alpha_5 = -1 \Rightarrow \alpha_5 = -\frac{1}{3} \Rightarrow \alpha_6 = \frac{1}{3} \Rightarrow \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$

$$[f]_{B,B'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

10.4 $f \in End_{\mathbb{R}}(\mathbb{R}^4)$, $[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}$

i) Show that $v = (1,4,1,-1) \in \operatorname{Ker} f$

$v' = (2,-2,4,2) \in \operatorname{Im} f$

ii) Determine a basis and the dimension of Ker f and Im f

iii) Define f

• $(\forall)\ v \in \mathbb{R}^4 : [f(v)]_E = [f]_E \cdot [v]_E$

i) $\operatorname{Ker} f = \{ v \in \mathbb{R}^4 \mid f(v) = 0 \} = \{ v \in \mathbb{R}^4 \mid [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$

$\operatorname{Im} f = \{ w \in \mathbb{R}^4 \mid \exists\, v \in \mathbb{R}^4 : f(v) = w \} = \{ w \in \mathbb{R}^4 \mid \exists\, v \in \mathbb{R}^4 : [f]_E \cdot [v]_E = [w]_E \}$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 1\cdot 4 + 1\cdot(-3) + 2\cdot(-1) \\ -1\cdot 1 + 1\cdot 4 + 1\cdot 1 + 4\cdot(-1) \\ 2\cdot 1 + 1\cdot 4 + 1\cdot(-5) + 1\cdot(-1) \\ 1\cdot 1 + 2\cdot 4 + (-4)\cdot 1 + 5\cdot(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v \in \operatorname{Ker} f$$

1 2 -4 5 -1

$v' = (2,-2,4,2) \in \text{Im} f \Leftrightarrow \exists u \in \mathbb{R}$ s.t. $f(u) = v'$

Let $u = (a\ b\ c\ d)$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \sim \dots \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 0 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system is compatible $\Rightarrow \exists u = (a, b, c, d) : f(u) = v' \Rightarrow v' \in \text{Im} f$