Seminar 9

9.3 Compute the rank of the matrix by using elementary transformation (Gauss)

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}, \ \alpha, \beta \in \mathbb{R}$$

Let $A = \begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \xrightarrow{\alpha_1 \Leftrightarrow \alpha_2} \begin{pmatrix} \boxed{1} & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \xrightarrow[\alpha_3 = \alpha_3 - 2\alpha_1]{\alpha_2 = \alpha_2 - \beta\alpha_1}$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \xrightarrow{\alpha_2 \Leftrightarrow \alpha_3} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$$

if $\alpha = 0$: $\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \xrightarrow{\alpha_2 \Leftrightarrow \alpha_3} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & \boxed{1} & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$

$\Rightarrow$ rank $A = 3$

if $\alpha \neq 0$: $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \xrightarrow{\alpha_2 = \frac{1}{2}\alpha_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -\frac{2}{\alpha} & \frac{1}{\alpha} \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

$$\xrightarrow{\alpha_3 = \alpha_3 - (1-\alpha\beta)\alpha_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -\frac{2}{\alpha} & \frac{1}{\alpha} \\ 0 & 0 & 3-3\beta-\frac{2}{\alpha}(1-\alpha\beta) & 4-3\beta-\frac{1}{\alpha}(1-\alpha\beta) \end{pmatrix}$$

$$\begin{cases} 3-3\beta+\dfrac{2}{\alpha}(1-\alpha\beta)=0 \mid \cdot\alpha \\ 4-3\beta-\dfrac{1}{\alpha}(1-\alpha\beta)=0 \mid \cdot\alpha \end{cases} \Leftrightarrow \begin{cases} 3\alpha-3\alpha\beta+2-2\alpha\beta=0 \\ 4\alpha-3\alpha\beta-1+\alpha\beta=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3\alpha+2-5\alpha\beta=0 \\ 4\alpha-1-2\alpha\beta=0 \end{cases}$$

Let $\gamma=\alpha\beta$

$$\Rightarrow \begin{cases} 3\alpha+2-5\gamma=0 \mid \cdot 2 \\ 4\alpha-1-2\gamma=0 \mid \cdot 5 \end{cases} \ominus$$

$$6\alpha-20\alpha+4+5-\cancel{10\gamma}+\cancel{10\gamma}=-14\alpha+9=0$$

$$\Rightarrow \alpha=\frac{9}{14}$$

$$\Rightarrow \gamma=\frac{11}{14} \Rightarrow \beta=\frac{11}{9}$$

$$\begin{cases} \text{rank } A=2 \Leftrightarrow \alpha=\dfrac{9}{14},\ \beta=\dfrac{11}{9}, \\ \text{rank } A=3 \text{ otherwise} \end{cases}$$

9.5 Compute the inverse (by Gauss) of the matrix:

$$A=\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}, \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$A=\left(\begin{array}{ccc|ccc} \boxed{1} & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3=L_3-3L_1}{\overset{L_2=L_2-2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_2=L_2:(-5)}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3=L_3+2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \overset{L_3=L_3\cdot 5}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \boxed{1} & 9 & -12 & 5 \end{array}\right) \overset{L_1=L_1-2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & \boxed{1} & 0 & -5 & 37 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_1=L_1-4L_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

Verification:

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{good}$$

9.9 $S = \langle (1,0,4), (2,1,0), (1,1,-4) \rangle$

$T = \langle (-3,-2,4), (5,2,4), (-2,0,-8) \rangle$

Find a basis for S, T, $S+T$ and $\dim S$, $\dim T$, $\dim S+T$, $\dim S \cap T$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 = L_3 - L_1}{\overset{L_2 = L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & \boxed{1} & -8 \\ 0 & 1 & -3 \end{pmatrix} \overset{L_3 = L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis of S $((1,0,4), (0,1,-8))$