Seminar 11

Def: V: K-vs, $\mathcal{B}=(v_1,\dots v_n)$, $\mathcal{B}'=(v_1',\dots v_n')$ - bases for V

$\underbrace{T_{\mathcal{B},\mathcal{B}'}}_{\text{base change matrix from } \mathcal{B} \text{ to } \mathcal{B}'} = [id_V]_{\mathcal{B}',\mathcal{B}} = ([v_1']_{\mathcal{B}} \dots [v_n']_{\mathcal{B}})$

- $\forall v\in V$: $[v]_{\mathcal{B}} = [id_V]_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}'}$
- $\forall f,g: V\to V'$, $(\forall)\alpha,\beta\in K$: $[\alpha f+\beta g]_{\mathcal{B},\mathcal{B}'} = \alpha[f]_{\mathcal{B},\mathcal{B}'} + \beta[g]_{\mathcal{B},\mathcal{B}'}$
- $\forall f: V\to V'$, $g: V'\to V''$
  $\mathcal{B},\mathcal{B}',\mathcal{B}''$ - bases of V, V', V'' $\Rightarrow [g\circ f]_{\mathcal{B},\mathcal{B}''} = [g]_{\mathcal{B}',\mathcal{B}''}\cdot[f]_{\mathcal{B},\mathcal{B}'}$
- $\forall f: V\to V'$, $\mathcal{B}_1, \mathcal{B}_2$ bases of V
  $\mathcal{B}_1', \mathcal{B}_2'$ bases of V'

$[f]_{\mathcal{B}_1,\mathcal{B}_2} = [id]_{\mathcal{B}_1',\mathcal{B}_2'}\cdot[f]_{\mathcal{B}_1,\mathcal{B}_1'}\cdot[id]_{\mathcal{B}_2,\mathcal{B}_1} = T_{\mathcal{B}_2',\mathcal{B}_1'}\cdot[f]_{\mathcal{B}_1,\mathcal{B}_1'}\cdot T_{\mathcal{B}_1,\mathcal{B}_2}$

! In general $[f]_{\mathcal{B},\mathcal{B}'} \neq [f]^{-1}_{\mathcal{B}',\mathcal{B}}$. However $[id]_{\mathcal{B}',\mathcal{B}} = [id]^{-1}_{\mathcal{B},\mathcal{B}'}$
$(T_{\mathcal{B},\mathcal{B}'} = T^{-1}_{\mathcal{B}',\mathcal{B}})$

11.2 $\left.\begin{array}{l}\mathcal{B}=(v_1,v_2)=((1,2),(1,3))\\ \mathcal{B}'=(v_1',v_2')=((1,0),(2,1))\end{array}\right\}$ bases of $\mathbb{R}^2$

$f,g\in End_{\mathbb{R}}(\mathbb{R}^2)$

$[f]_{\mathcal{B}} = \begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$, $[g]_{\mathcal{B}'} = \begin{pmatrix}-7 & -13\\ 5 & 7\end{pmatrix}$

Find $[2f]_{\mathcal{B}}$, $[f+g]_{\mathcal{B}}$, $[f\circ g]_{\mathcal{B}'}$

$[2f]_{\mathcal{B}} = 2\cdot\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix} = \begin{pmatrix}2 & 4\\ -2 & -2\end{pmatrix}$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$$[id]_{B',B} = ([v_1']_B \;\; [v_2']_B)$$

$$\alpha v_1 + \beta v_2 = \alpha(1,2) + \beta(1,3) = (\alpha+\beta, 2\alpha+3\beta)$$

$$v_1': \begin{cases} \alpha_1+\beta_1 = 1 & | \cdot 2 \\ 2\alpha_1+3\beta_1 = 0 & | \cdot (-1) \end{cases} \Rightarrow \begin{cases} 2\alpha_1+2\beta_1 = 2 \\ -2\alpha_1-3\beta_1 = 0 \end{cases} \oplus$$

$$-\beta_1 = 2 \Rightarrow \beta_1 = -2 \Rightarrow \alpha_1 = 3 \Rightarrow [v_1']_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$v_2': \begin{cases} \alpha_2+\beta_2 = 2 & | \cdot 2 \\ 2\alpha_2+3\beta_2 = 1 & \cdot (-1) \end{cases} \Rightarrow \begin{cases} 2\alpha_2+2\beta_2 = 4 \\ -2\alpha_2-3\beta_2 = -1 \end{cases} \oplus$$

$$-\beta_2 = 3 \Rightarrow \beta_2 = -3,\ \alpha = 5 \Rightarrow [v_2']_B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[id]_{B,B'} = ([v_1]_{B'} \; [v_2]_{B'}) = [id]_{B',B}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -22 & 13 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ -22 & 13 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ -23 & 12 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B,B'} \cdot [g]_{B',B}$$

$$f_{B,B'} = [id]_{B,B'} \cdot [f]_B = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[g]_{B',B} = [id]_{B',B} \cdot [g]_{B'} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix}$$

$$\Rightarrow [f \circ g]_{B'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Def: $f \in End_K(V)$

$\lambda \in K$ - eigen value if $\exists \underline{v \in V \setminus \{0\}}$ s.t. $f(v) = \lambda v$

eigen vector coresp to λ

$S(\lambda) = \{v \in V \mid f(v) = \lambda v\}$

$\hookrightarrow$ the eigen space coresp to λ

Dummy's guide for eigen stuff =)))

$f \in End_K(V)$ or $A \in M_n(K)$

Step 0: choose B - basis of V and write $A = [f]_B$

Step 1: Calculate $P_A(x) = \det(A - xI_n)$

Step 2: Find the roots of P_A. They are the eigen values

Step 3: To find the eigen vectors corresponding to an eigen value λ we solve the system $AX = \lambda X$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$\Updownarrow$$

$$(A - \lambda I_n) X = 0$$

The solution of this system from the eigen space: $S(\lambda)$

11.5 $A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$

Step 1: $P_A(x) = \begin{vmatrix} 3-x & 1 & 0 \\ -4 & -1-x & 0 \\ -4 & -8 & -2-x \end{vmatrix} = (-2-x) \begin{vmatrix} 3-x & 1 \\ -4 & -1-x \end{vmatrix} =$

$= (-2-x) \cdot (-3 - 3x + x + x^2 + 4) = (-2-x)(x^2 - 2x + 1) = -(x+2)(x-1)^2 \Rightarrow$

Step 2:

$\Rightarrow$ the eigen values $= \{-2, 1\}$ $\{\lambda_1 = -2, \lambda_2 = 1\}$

Step 3 : we find $S(\lambda_1) = S(-2)$

$$\begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -4 & -8 & 0 & 0 \end{array}\right) \begin{array}{l} L_2 \Leftrightarrow L_2 + \frac{4}{5} L_1 \\ L_3 \Leftrightarrow L_3 + \frac{4}{5} L_3 \end{array} \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 0 & \frac{9}{5} & 0 & 0 \\ 0 & -\frac{36}{5} & 0 & 0 \end{array}\right) \quad L_3 \Leftrightarrow L_3 + 4L_2$$

$$\left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 0 & \frac{9}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} 5x + y = 0 \\ \frac{9}{5} y = 0 \end{cases} \Rightarrow x = 0,\ y = 0,\ z \in \mathbb{R}$$

$$S(-2) = \{(0, 0, z) \mid z \in \mathbb{R}\} = \langle (0, 0, 1) \rangle$$