Seminar 13

$\mathcal{C}$ - linear code,

$v \in \mathcal{C} \iff H \cdot [v]_E = 0$

$\mathcal{C} = \operatorname{Im} \gamma \leq \mathbb{Z}_2^n$

We can define an equivalent relation:

$(\forall)\, x, x' \in \mathbb{Z}_2^n : x \sim x' \iff \underbrace{x - x'}_{= x + x'} \in \mathcal{C}$

$\hat{x} = x + \mathcal{C} = \{x + y \mid y \in \mathcal{C}\}$

$x \sim x' \iff H \cdot [x - x'] = 0 \iff H \cdot [x] = H \cdot [x']$

For $x \in \mathbb{Z}_2^n$ the vector $H \cdot [x]$ is the syndrome associated to x

The most likely error vector corresp. to a syndrome is called the coset leader

Ex 13.2 Using the parity check matrix $H = \begin{pmatrix} 100 & 101 \\ 010 & 111 \\ 001 & 011 \end{pmatrix}$

and the syndromes and coset leaders

syndrome	000	001	010	011	100	101	110	111
coset leader	000000	001000	010000	000010	100000	000110	000100	000001

decode the following words: 101110, 011000, 001011, 111111, 110011, 010101

To decode - steps:

1. multiply the vector by H to get the syndrome
2. use the table to get the CL
3. correct the vector with the CL (CL + v)
4. extract the message

1. $v_1 = (1\,0\,1\,1\,1\,0)$

$$H\cdot[v_1] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1\\0\\1\\1\\1\\0 \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix}$$

$\Rightarrow$ CL = 000000

$\Rightarrow$ corrected vector = 101110

$\Rightarrow$ message = 110

$v_2 = (0\,1\,1\,0\,0\,0)$

$$H\cdot[v_2] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0\\1\\1\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\1 \end{pmatrix}$$

$\Rightarrow$ CL = 000010

$\Rightarrow$ corrected vector = 011000

$\Rightarrow$ message = 010

$v_3 = (0\,0\,1\,0\,1\,1)$

$$H\cdot[v_3] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0\\0\\1\\0\\1\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1 \end{pmatrix}$$

$\Rightarrow$ CL = 000110

$\Rightarrow$ corrected vector = 001101

$\Rightarrow$ message = 101

$v_4 = (1\,1\,1\,1\,1\,1)$

$$H\cdot[v_4] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1 \end{pmatrix}$$

$\Rightarrow$ CL = 000110

$\Rightarrow$ corrected vector = 111001

$\Rightarrow$ message = 001

$v_5 = (1\,1\,0\,0\,1\,1)$

$$H\cdot[v_5] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\0\\0\\1\\1 \end{pmatrix} = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$$

$\Rightarrow$ CL = 010000

$\Rightarrow$ corrected vector = 100011

$\Rightarrow$ message = 011

$v_6 = (0\,1\,0\,1\,0\,1)$

$$H\cdot[v_6] = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0\\1\\0\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 0\\1\\1 \end{pmatrix}$$

$\Rightarrow$ CL = 000010

$\Rightarrow$ corrected vector = 010111

$\Rightarrow$ message = 111

ex 13.5 Construct a table of coset leaders and syndromes for the (7,4) code with $H = \begin{pmatrix} 100 & 110 \\ 010 & 101 \\ 001 & 011 \end{pmatrix}$

syndrome	000	001	010	011	100	101	110	111
coset leader	0000000	0010000	0100000	0000001	1000000	0000010	0000100	0000100

$A \in \mathcal{M}_{n,n}(\mathbb{Z}), v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{Z}_2)$, $A \cdot v = \sum_{\substack{i=1 \\ v_i = 1}}^{n} c_i$ ↳ columns of A

ex 13.8 Construct a table of coset leaders and syndromes for the (7,3)-code generated by $P = 1 + x^2 + x^3 + x^4 \in \mathbb{Z}_2[x]$

$$H = \begin{pmatrix} 1\,0\,0\,0 & 1\,1\,0 \\ 0\,1\,0\,0 & 0\,1\,1 \\ 0\,0\,1\,0 & 1\,1\,1 \\ 0\,0\,0\,1 & 1\,0\,1 \end{pmatrix}$$

0000	0000000
0001	0001000
0010	0010000
0011	0011000
0100	0100000
0101	0000110
0110	0110000
0111	0000001

1000	10000000
1001	0000011
1010	0001100
1011	0000100
1100	1100000
1101	0001101
1110	0000010
1111	0001010

+ Decode: 1110110

$$\begin{pmatrix} 1\,0\,0\,0 & 1\,1\,0 \\ 0\,1\,0\,0 & 0\,1\,1 \\ 0\,0\,1\,0 & 1\,1\,1 \\ 0\,0\,0\,1 & 1\,0\,1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ CL = 0000100

$\Rightarrow$ corrected vector = 111 0010

$\Rightarrow$ message = 010

ex 13.3 A (7,4) code is defined by the equations:

$u_1 = u_4 + u_5 + u_7$

$u_2 = u_4 + u_6 + u_7$

$u_3 = u_4 + u_5 + u_6$

$$\underbrace{\boxed{u_1}\boxed{u_2}\boxed{u_3}\boxed{u_4}}_{\text{check digits}}\underbrace{\boxed{u_5}\boxed{u_6}\boxed{u_7}}_{\text{message digits}}$$

Write its generator matrix and H. Decode the received words 0000111 and 0001111.

$$\begin{aligned} \text{encoded}(e_1) &= (1111000) \\ \text{encoded}(e_2) &= (1010100) \\ \text{encoded}(e_3) &= (0110010) \\ \text{encoded}(e_4) &= (1100001) \end{aligned} \quad , \quad G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

$$M = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

S	CL
000	0000000
001	0010000
010	0100000
011	0000010
100	1000000
101	0000100
110	0000001
111	0001000

$$H \cdot [v_1]_E = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow CL = 0000000 \Rightarrow C.v = 0000111$$

$$H \cdot [v_2]_E = H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow CL = 0001000 \Rightarrow C.v = 0000111$$