Curs 4

- Series with positive terms

Ex: Harmonic series : $\sum_{n\geq 1} \frac{1}{n} = +\infty$

$S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$; $|S_{2n} - S_n| > \frac{1}{2}$

$\boxed{\text{Hence, } (S_n) \text{ is not Cauchy} \Rightarrow (S_n) \text{ is not convergent}}$

- Stolz - Cesaro : $\frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{\ln n} \to 1$

ex: $\sum_{n\geq 1} \frac{1}{n^2} \sim \sum_{n\geq 1} \frac{1}{n(n+1)}$

$\boxed{\text{Comparison test : } \frac{x_n}{y_n} \to l > 0 \text{ ; } \sum x_n, \sum y_n \text{ have the same nature}}$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow \sum_{n\geq 1}^{N} \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{N}} - \frac{1}{N+1} = 1 - \frac{1}{N+1}$

Let $N \to \infty \Rightarrow \frac{1}{N+1} \to 0 \Rightarrow \sum_{n\geq 1} \frac{1}{n(n+1)} = 1$

- $\sum_{n\geq 1} \frac{1}{n^p}$, $p > 0$: $p = 1$, Harmonic series, $+\infty$

  $p = 2$, convergence

  $p < 1$, $\sum_{n\geq 1} \frac{1}{n^p}$ - diverges, $\sum_{n\geq 1} \frac{1}{n^p} > \sum_{n\geq 1} = \infty$

Theorem (Cauchy Condensation Test) : Let (x_n), $x_n > 0$ - decreasing sequence. Then $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} 2^n \cdot x_{2^n} = x_1 + 2x_2 + 4x_4 + \ldots$ have the same nature

Proof: Take $S_n = x_1 + x_2 + \dots + x_n$ and $T_n = x_1 + 2x_2 + 4x_4 + \dots + 2^n x_{2^n}$

Let $k \in \mathbb{N}$ such that $2^k \le n \le 2^{k+1} - 1$

$$S_n = x_1 + x_2 + \dots + x_n$$

$$S_n \le x_1 + x_2 + \dots + x_{2^{k+1}-1}$$

$$S_n \le x_1 + \underbrace{(x_2 + x_3)}_{\le 2x_2} + \underbrace{(x_4 + x_5 + x_6 + x_7)}_{\le 4x_4} + \dots + \underbrace{(x_{2^k} + \dots + x_{2^{k+1}-1})}_{\le 2^k \cdot x_{2^k}}$$

Hence, $S_n \le T_k$ (1)

$$S_n = x_1 + x_2 + \dots + x_n$$

$$S_n \ge x_1 + x_2 + x_3 + \dots + x_{2^k}$$

$$S_n \ge x_1 + x_2 + \underbrace{(x_3 + x_4)}_{\ge 2x_4} + \underbrace{(x_5 + x_6 + x_7 + x_8)}_{\ge 4x_8} + \dots + \underbrace{(x_{2^{k-1}+1} + \dots + x_{2^k})}_{2^{k-1} \cdot x_{2^k}}$$

$$S_n \ge x_1 + x_2 + 2x_4 + 4x_8 + \dots + 2^{k-1} \cdot x_{2^k}$$

$$= \frac{x_1}{2} + \frac{1}{2}(x_1 + 2x_2 + 4x_4 + 8x_4 + \dots + 2^k x_{2^k})$$

$$S_n \ge \underbrace{\frac{x_1}{2}}_{>0} + \frac{1}{2} T_k \Rightarrow S_n \ge \frac{1}{2} T_k \quad (2)$$

$(1), (2) \Rightarrow \frac{1}{2} T_k \le S_n \le T_k$

(S_n) bounded $\Rightarrow (T_n)$ bounded

$S_n, T_n \to$ increasing

Example: $\sum \frac{1}{n^p}$ has the same nature as $\sum 2^n \cdot \frac{1}{(2^n)^p} = \sum 2^{n(1-p)}$

$= \sum [2^{(1-p)}]^n$ - geometric series

$$\boxed{\sum q^n, \ |q| < 1 \Rightarrow \text{conv}}$$

$q = 2^{1-p} < 1 \iff 2^{1-p} < 2^0 \iff 1-p<0 \iff p>1 \quad (q<1) \Rightarrow$ CONVERGENT

If $p \leq 1$, $\sum \frac{1}{n^p} \geq \sum \frac{1}{n} = \infty$ DIVERGENT

If $p>1$, $\sum \frac{1}{n^p} \approx \underbrace{\sum (2^{1-p})^n}_{\text{CONVERGENT}}$

Ex: $\sum \frac{1}{n \cdot \ln n}$ "like" $\sum \frac{\cancel{2^n}}{\cancel{2^n} \ln(2^n)} = \sum \frac{1}{n \cdot \ln 2} = \frac{1}{\ln 2} \cdot \sum \frac{1}{n} = +\infty$

Recall: Ratio test: $\frac{x_{n+1}}{x_n} \to l$, if $l<1 \Rightarrow \sum x_n$ converges

Root test: $\sqrt[n]{x_n} \to l$, if $l>1$, $\sum x_n$ diverges

When $l=1$, these tests are inconclusive

Theorem: Let (x_n), $x>0$
(Kummer's Test)

1) If there is (c_n), $c_n>0$ such that:

$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1} \right) > 0 \Rightarrow \sum x_n - \text{conv}$$

2) If there is (c_n), $c_n>0$ with $\sum \frac{1}{c_n} = +\infty$

$$\lim_{n\to\infty} \left(c_n \cdot \frac{x_n}{x_{n+1}} - c_{n+1} \right) < 0 \Rightarrow \sum x_n \text{ diverges}$$

Proof: 1) $\exists\, r>0$ s.t. $c_n \cdot \frac{x_n}{x_{n+1}} - c_{n+1} \geq r$, $(\forall)\, n \geq n_0$

$c_n \cdot x_n - c_{n+1} \cdot x_{n+1} \geq r \cdot x_{n+1}$, $(\forall)\, n \geq n_0$

$c_{n_0} x_{n_0} - \cancel{c_{n_0+1} x_{n_0+1}} \geq r\, x_{n_0+1}$

$+ \dots$

$\cancel{c_n x_n} - c_{n+1} \cdot x_{n+1} \geq r\, x_{n+1}$

$$c_{n_0} \cdot x_{n_0} - c_{n+1} \cdot x_{n+1} \geq r \underbrace{(x_{n_0+1} + \dots + x_{n+1})}_{= S_{n+1} - S_{n_0}}$$

$$c_{n_0}x_{n_0} - c_{n+1}x_{n+1} \geq r(S_{n+1} - S_{n_0})$$

$$S_{n+1} - S_{n_0} \leq \frac{1}{r} c_{n_0}x_{n_0} \Rightarrow S_{n+1} \leq \underbrace{S_{n_0} + \frac{1}{r} c_{n_0} \cdot x_{n_0}}_{\text{constant}} \Rightarrow (S_n)\text{- bounded}$$

$$x_n > 0 \Rightarrow (S_n)\text{- increasing } \left(S_{n+1} - S_n = x_{n+1} > 0\right)$$

(S_n) - increasing and bounded $\Rightarrow$ (S_n) convergent

Proof: 2) $\exists\, n_0 \in \mathbb{N}$ s.t. $c_n \cdot \frac{x_n}{x_{n+1}} - c_{n+1} < 0, (\forall)\, n \geq n_0$

$$c_n \cdot x_n < c_{n+1} \cdot x_{n+1}, (\forall)\, n \geq n_0$$

$$\underbrace{c_{n_0} \cdot x_{n_0}}_{\text{constant}} < c_n \cdot x_n, (\forall)\, n > n_0$$

$$c_{n_0} \cdot x_{n_0} \cdot \frac{1}{c_n} < x_n, \sum \frac{1}{c_n} = \infty \Rightarrow \sum x_n = \infty$$

$$c_n = n, \quad c_n \cdot \frac{x_n}{x_{n+1}} - c_{n+1} = n \frac{x_n}{x_{n+1}} - n - 1 = n\left(\frac{x_n}{x_{n+1}} - 1\right) - 1$$

Theorem (Raabe-Duhamel):

1) If $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) > 1 \to \sum x_n$ - conv.

2) If $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) < 1 \to \sum x_n$ - div.

Ex: $\sum \frac{n!}{a(a+1)(a+2)\cdot\ldots\cdot(a+n)}, a > 0$

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{(n+1)!}^{(n+1)}}{\cancel{a(a+1)\ldots(a+n)}(a+n+1)} \cdot \frac{\cancel{a(a+1)\ldots(a+n)}}{\cancel{n!}} = \frac{n+1}{a+n+1} \to 1$$

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) = n\left(\frac{a+n+1}{n+1} - 1\right) = n\left(\frac{a}{n+1} + \frac{n+1}{n+1} - 1\right) = \frac{n \cdot a}{n+1} \to a$$

if $a > 1 \Rightarrow$ CONVERGE

$a < 1 \Rightarrow$ DIVERGE

Seminar 4

1) a) $\sum_{n\geq 2} \frac{1}{\ln n} > \sum_{n\geq 1} \frac{1}{n} = +\infty$

$\ln n < n \Rightarrow \frac{1}{\ln n} > \frac{1}{n}$

$\boxed{\text{If } \sum x_n \text{ converges, then } x_n \to 0}$

$\to$ Proof: $S_n = x_1 + x_2 + \ldots + x_n$, $\sum x_n$ converges $\Leftrightarrow$ (S_n) converges

$x_{n+1} = S_{n+1} - S_n \to 0$ | But it doesn't go to 0 fast enough

both converge to the limit l

. $\sum \frac{1}{n^2}$ - conv because it goes to 0 fast enough

. $\sum \frac{1}{n^p}$, $p > 1$ so that $\sum x_n$-conv

b) $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

b') $\sum \ln\left(1+\frac{1}{n}\right)$

$\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1 \Rightarrow \ln\left(1+\frac{1}{n}\right)$ is „like" harmonic series

$\Rightarrow \sum \ln\left(1+\frac{1}{n}\right)$ „like" $\sum \frac{1}{n} = +\infty$

$\Rightarrow$ b) $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$ „like" $\frac{\frac{1}{n}}{\frac{1}{n}}$ $= \sum \frac{1}{n^2} \to$ converges (similar cu geometric series)

c) $\sum_{n=1}^{\infty} \frac{1}{n\cdot(\ln n)^p}$ (Use the Cauchy Compensation)

$\Rightarrow \sum_{n=1}^{\infty} x_n$ „like" $\sum_{n=0}^{\infty} 2^n \cdot x_{2^n}$

„like" $\sum_{n=1}^{\infty} \frac{2^n}{2^n(\ldots n)^p}$ $= \frac{1}{(\ln 2)^p} \cdot \sum \frac{2^n}{2^n \cdot n^p} = \frac{1}{(\ln 2)^p} \cdot \sum \frac{1}{n^p}$

$\Rightarrow$ conv if $p>1$
div if $p\leq 1$

When we see series with complicated logarithms we can use the Cauchy compensation
- condition : $(x_n)\searrow$ - decreasing

Absolute convergence

Def: A series x_n is absolutely conv if $\left.\begin{array}{l}\sum x_n \\ \sum |x_n|\end{array}\right\}$ both converge

Leibniz test : If $x_n \searrow 0$, then $\sum (-1)^n x_n$ will be conv

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

$x_n = \dfrac{1}{\sqrt{n(n+1)}} \to 0$, decreasing $\left(\dfrac{x_{n+1}}{x_n} = \dfrac{\sqrt{n\cancel{(n+1)}}}{\sqrt{\cancel{(n+1)}(n+2)}} = \dfrac{\sqrt{n}}{\sqrt{n+2}} < 1 \right)$

$\Rightarrow \displaystyle\sum (-1)^{n+1} \cdot \frac{1}{\sqrt{n(n+1)}}$ is conv

Leibniz test $\to$ series conv

- $\displaystyle\sum |x_n| = \sum \frac{1}{\sqrt{n(n+1)}}$ | we compare it with $\displaystyle\sum \frac{1}{n}$

$\Rightarrow \dfrac{\frac{1}{n}}{\frac{1}{\sqrt{n(n+1)}}} = \dfrac{1}{n} \cdot \dfrac{\sqrt{n(n+1)}}{1}$ | $y_n = \dfrac{1}{\sqrt{n(n+1)}}$, $x_n = \dfrac{1}{n}$

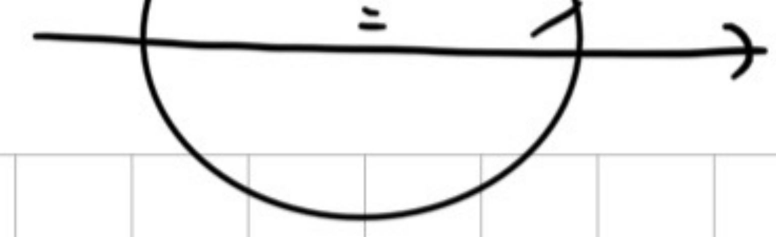
$\dfrac{1}{\sqrt{n(n+1)}} \cdot \dfrac{n}{1} = \sqrt{\dfrac{n^2}{n^2+n}} = \sqrt{1} = 1 \to 1$

$\Rightarrow$ So $\displaystyle\sum \frac{1}{\sqrt{n(n+1)}}$ "like" $\displaystyle\sum \frac{1}{n}$, so $\displaystyle\sum \frac{1}{\sqrt{n(n+1)}}$ is divergent

$\Rightarrow$ So it isn't an absolutely convergent series

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \sin \frac{1}{n} \xRightarrow{\text{Leib}} \sum (-1) \sin \frac{1}{n}$ - conv

$x_n = \sin\frac{1}{n}$ 0



$\sum |x_n| = \sum \sin\left(\frac{1}{n}\right)$ "like" $\sum \frac{1}{n} = +\infty \Rightarrow$ it is <u>not</u> absolutely conv

(is semiconvergent)

<u>Alternating series</u> $(x_n > 0,\ x_1 - x_2 + x_3 - x_4 + \dots)$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges since $\frac{1}{n} \downarrow 0$ (Leibniz)

converges since $\frac{1}{n} \to 0$ (Leibniz)

$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is semi-convergent because $\sum \frac{1}{n} = \infty$

$\sum \frac{(-1)^{n+1}}{n^2}$ is absolutely conv. (we get the same value if we change the order of the elements

$\cdot\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \to \ln(2)$

$= \underbrace{1 - \frac{1}{2}} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}} - \frac{1}{12} + \underbrace{\frac{1}{7} - \frac{1}{14}} - \frac{1}{16} + \dots =$

$= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3}\left(1 - \frac{1}{2}\right) - \frac{1}{8} + \frac{1}{5}\left(1 - \frac{1}{2}\right) - \frac{1}{12} + \frac{1}{7}\left(1 - \frac{1}{2}\right) - \frac{1}{16} + \dots =$

$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \dots =$

$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right) = \frac{1}{2}\ln 2$

3. a) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot 2n}$

Ratio test : $\frac{x_{n+1}}{x_n} = \frac{2n+1}{2n+2} \simeq 1 \to 1$

Raabe - Duhamel Test

- $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$ $\to$ >1 - series conv.
  $\to$ <1 - series div.

$$\lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2-2n-1}{2n+1}\right) = \lim_{n\to\infty} n\cdot\frac{1}{2n+1} = \frac{1}{2} < 1$$

$\Rightarrow$ series $\sum\limits_{n=1}^{\infty} \frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$ - diverges

R-S Test $\Rightarrow \sum \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n} \to \infty$ (diverges)