Seminar 6

· Taylor series around x_0 for function f:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

· when $x_0 = 0$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$

↳ Maclaurin series

Ex: $f(x) = e^x$, $f^{(n)}(0) = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

↳ for $x = 1$: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

1) Take $x_0 = 0$

a') $f(x) = \sin x$

$f'(x) = \cos x$, $f'(0) = 1$

$f''(x) = -\sin x$, $f''(0) = 0$

$f^{(3)}(x) = -\cos x$, $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$, $f^{(4)}(0) = 0$

2 cases → $f^{(2n+1)}(x) = (-1)^n \cdot \cos x$

↳ $f^{(2n)}(x) = (-1)^n \cdot \sin x$

$f^{(2n+1)}(0) = (-1)^n$, $f^{(2n)}(0) = 0$

$$\Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\sin(-x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots \Rightarrow \sin(-x) = -\sin x$$

a'') $f(x) = \cos x$

$f'(x) = -\sin x$, $f'(0) = 0$

$f''(x) = -\cos x, \quad f''(0) = -1$

$f^{(3)}(x) = \sin x, \quad f^{(3)}(0) = 0$

$\dots\dots$

2 cases $\to f^{(2n+1)}(x) = (-1)^{n+1} \sin x, \quad f^{(2n+1)}(0) = 0$

$\quad\quad\;\; \to f^{(2n)}(x) = (-1)^n \cos x, \quad f^{(n)}(0) = (-1)^n$

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\cos(-x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \cos x$$

b) Recall that

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \quad (\text{a power series})$$

Radius of convergence R, i.e. the series converges $(\forall)\, x \in \mathbb{R}, |x| < R$

$$S'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}, \quad (\forall)\, |x| < R$$

$$(\sin x)' = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2n+1) \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$$

same way with $(\cos x)' = -\sin x$

c) $\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k}$ — partial sum of the whole Taylor serie $= T_n(x)$

$$f(x) = \sum \frac{f^{(k)}(0)}{k!} \cdot x^k + R_n(x)$$

remainder (the biger the n, the smaller the R)

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}, \quad c \in (0, x)$$

$(c \in (x_0, x)$ - in general case$)$

Prove that $x - \frac{x^3}{3!} < \sin x$, $(\forall)\, x > 0$:

$$\sin x = \underbrace{x - \frac{x^3}{6}}_{T_3(x)} + \underbrace{R_3(x)}_{>0} \; ; \; R_3(x) = \frac{\sin^{(4)}(c)}{4!} \cdot x^4, \; c \in (0, x)$$

(or $c \in (x, 0)$ - in general case)

$$\Rightarrow R_3(x) = \frac{\sin(c)}{4!} \cdot x^4$$

- In this case $x > 0$, $c \in (0, x)$, $R_3(x) > 0$

2') $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots, \; |x| < 1 \quad \longrightarrow$$

$$1 + x + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1} = \frac{1 - x^{n+1}}{1 - x}$$

Take $\underbrace{n \to \infty, |x| < 1}_{x^{n+1} \to 0 \Rightarrow} \Rightarrow \sum x^n = \frac{1}{1-x}$ $\left((\ln(1-x))' = \frac{-1}{1-x} \right)$

Integrate from 0 to t $\Rightarrow \int_0^t \frac{1}{1-x}\, dx = -\ln(1-x)\Big|_0^t = -\ln(1-t)$

$$= \sum_{n=0}^{\infty} \int_0^t x^n dx = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \Rightarrow -\ln(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}, \; (\forall)\, |t| < 1$$

$$\Rightarrow \ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}, \; (\forall)\, |t| < 1$$

What happens when $|t| = 1$?

- $t = 1 \Rightarrow -\sum \frac{1}{n}$ - divergent
- $t = -1 \Rightarrow -\sum \frac{(-1)^n}{n}$ - converges (Leibniz test)

  $\hookrightarrow$ we get $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

· Radius of convergence is $R=1$ because the series converges for $(\forall)\ |t|<1$

· Convergence set $[-1,1)$ because the series converges $\forall\, t\in[-1,1)$

2'') $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-x^n)=\sum_{n=0}^{\infty}(-1)^n x^n$

Integrate term by term from 0 to t (both sides):

$\ln(1+t)=\sum_{n=0}^{\infty}(-1)^n\cdot\int_0^t x^n\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{n+1}$

$\Rightarrow \ln(1+t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot t^n}{n}$

· $t=1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ - converges $=\ln 2$

· $t=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{-1}{n}$ - diverges $[(-1)^{n+1}\cdot(-1)^n=-1]$

5. a) $(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{k}\cdot x^k,\ \alpha\in\mathbb{R}$ → Polinomial series

Recall Newton's binomial formula:

$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k,\ n\in\mathbb{N}$

$\binom{n}{k}=\frac{\cancel{n!}}{\cancel{(n-k)!}\cdot k!}=\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}$

$\binom{\alpha}{k}=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$ (factorial of a real number doesn't make sens!)

· for $\alpha=\frac{1}{2}$: $\sqrt{1+x}=\ldots$

$\alpha=-\frac{1}{2}$: $\frac{1}{\sqrt{1+x}}=\ldots$