Curs 3

Recall from lecture 2:

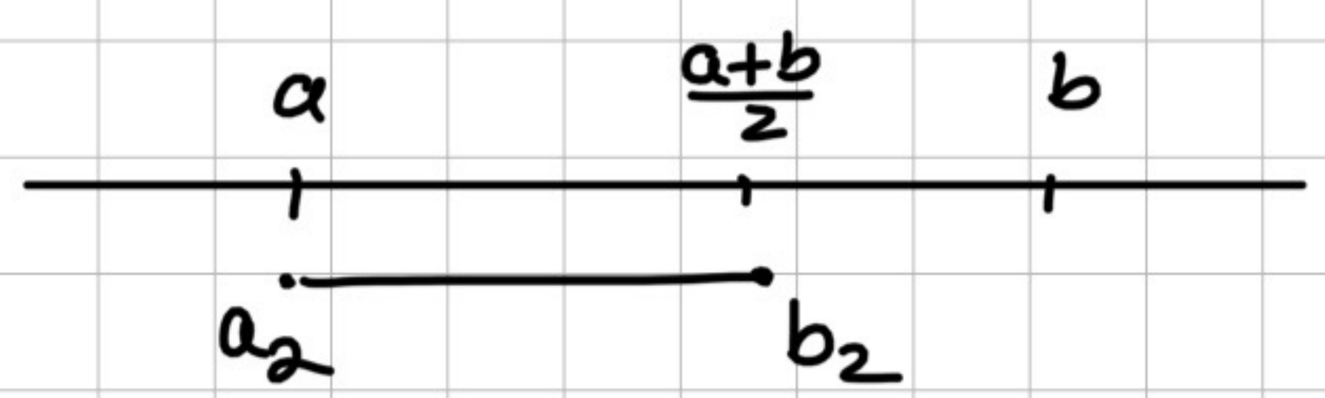
- a sequence (x_n) converges to a limit $l \in \mathbb{R}$ if $(\forall)\varepsilon>0, \exists$ $n_\varepsilon \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon, (\forall) n \geq n_\varepsilon$

- a bounded + monotone sequence converges

Thm (BOLZANO-WEIERSTRASS): Any bounded sequence has a convergent subsequence.

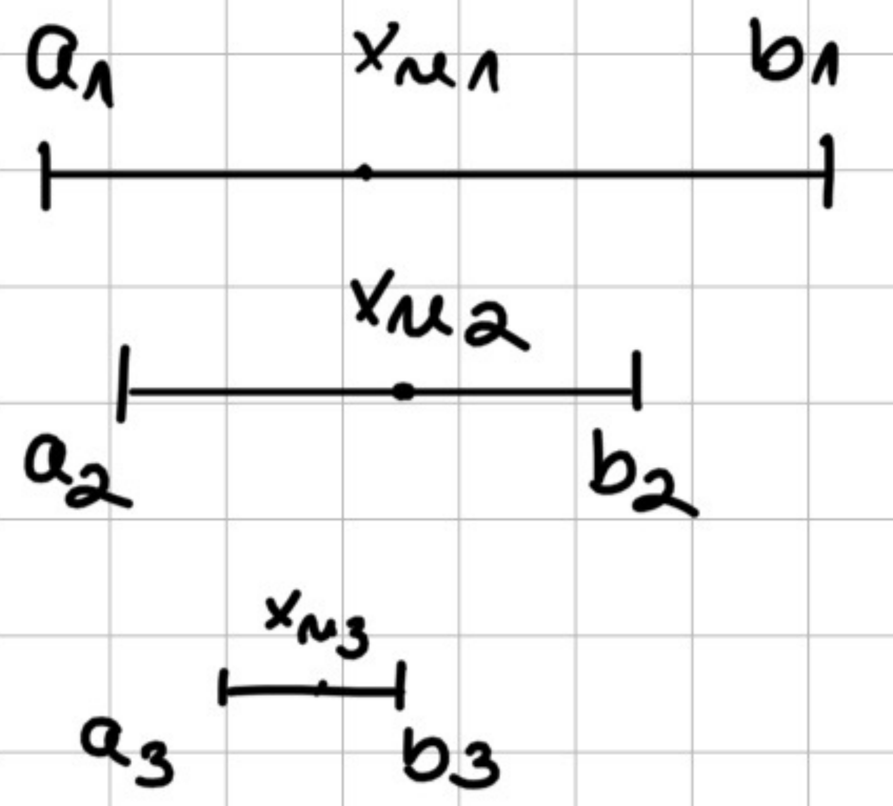
→ Proof: the sequence is $(x_{n_k})_{k\geq 1}$

(x_n) bounded $\Rightarrow \exists\, a, b \in \mathbb{R}$ s.t. $a \leq x_n \leq b, n \in \mathbb{N}$



$a_1 = a$
$b_1 = b$

Idea: $[a_1, b_1] \to$ bisect (split in 2). Take the half that contains an infinite number of terms from (x_n): call it $[a_2, b_2]$



- Intervals $[a_n, b_n]$ ...
  $b_n - a_n \to 0$
- Each interval $[a_n, b_n]$ contains an infinite number of terms from (x_n)

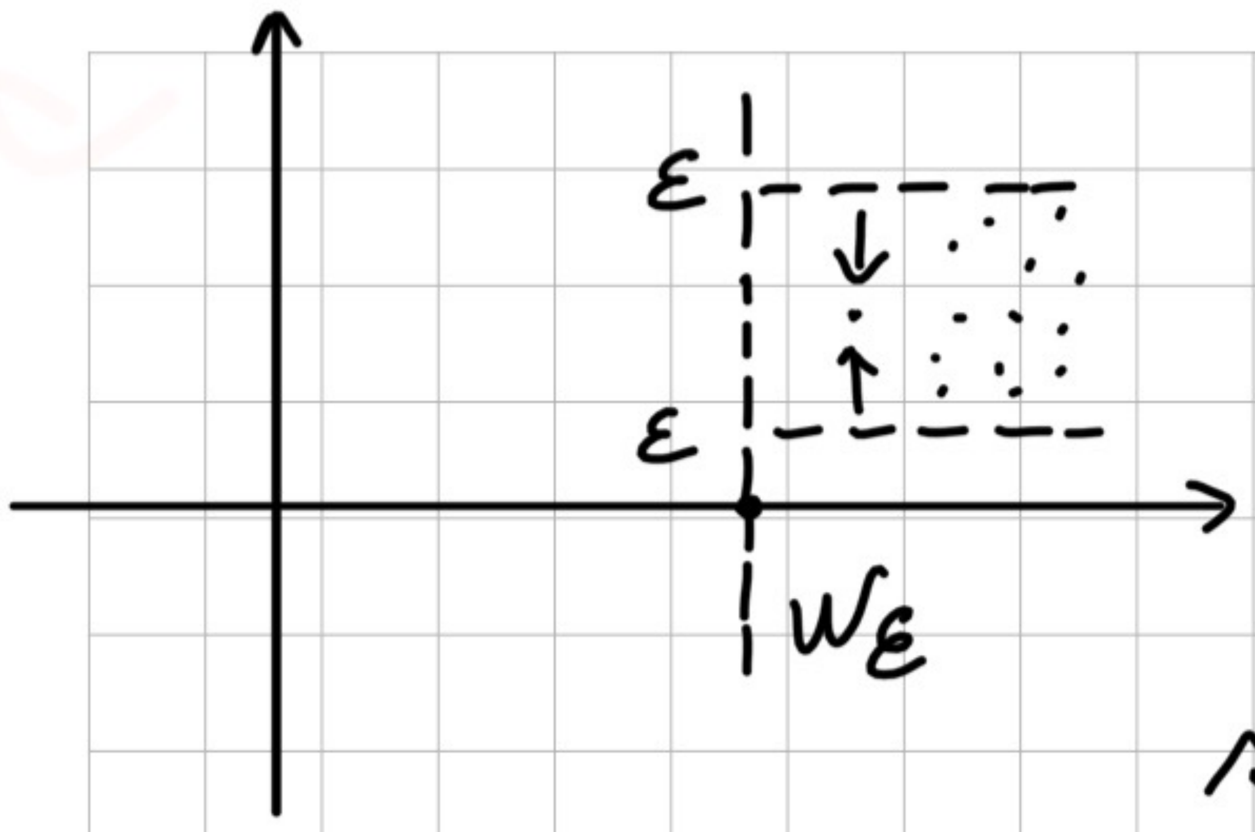
Cantor's Nested Intervals thm:

$[a_n, b_n]$ with $b_n - a_n \to 0$, then $[a_n, b_n] = \{x\}$

- From $[a_k, b_k]$ pick $x_{n_k} \in [a_k, b_k]$. Then $x_{n_k} \xrightarrow{\text{converges}} x$

Def: (x_n) is a Cauchy sequence (fundamental) if

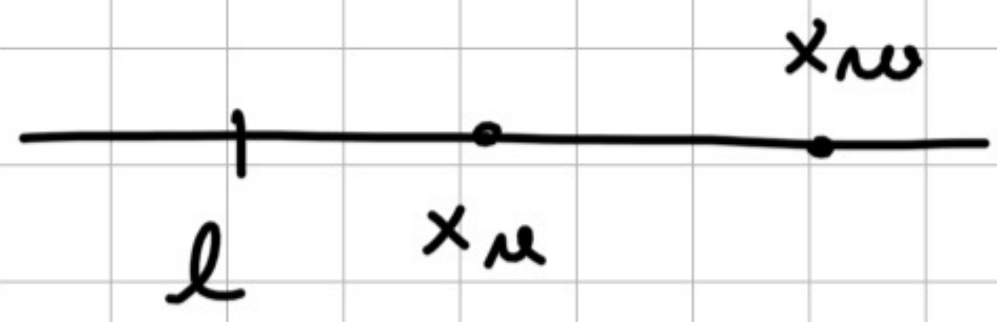
$\varepsilon, \exists\, n_\varepsilon$ s.t. $|x_m - x_n| < \varepsilon, \forall\, m, n \geq n_\varepsilon$



Thm: (x_n) - conv. $\Rightarrow (x_n)$ is Cauchy

Proof: $\lim\limits_{n\to\infty} x_n = l \Rightarrow (\forall)\, \varepsilon > 0, \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $|x_n - l| < \frac{\varepsilon}{2}, (\forall)\, n \geq n_\varepsilon$

Let $m, n \geq n_\varepsilon$: $|x_m - x_n| = |x_m - l + l - x_n|$



$|a \pm b| \leq |a| + |b| \rightarrow$ Triangle ineq

$|x_m - x_n| \leq \underbrace{|x_m - l|}_{< \frac{\varepsilon}{2}} + \underbrace{|x_n - l|}_{< \frac{\varepsilon}{2}} < \varepsilon$. Hence $(x_m - x_n) < \varepsilon$, $(\forall)\, m, n \geq n_\varepsilon$

$\boxed{\text{Thm: } (x_n) \text{ is Cauchy} \Rightarrow (x_n) \text{ is convergent}}$

Proof: (x_n) is Cauchy $\Rightarrow (x_n)$ is bounded

$\Rightarrow (x_n)$ has a convergent subsequence, call it (x_{n_k}) : $x_{n_1}, x_{n_2} \dots$

$\lim\limits_{k\to+\infty} x_{n_k} = l \in \mathbb{R}$.

Let $\varepsilon > 0$. There is $k_\varepsilon \in \mathbb{N}$ s.t. $|x_{n_k} - l| < \frac{\varepsilon}{2}, (\forall)\, k \geq k_\varepsilon$

- (x_n) is Cauchy $\Rightarrow \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $|x_m - x_n| < \frac{\varepsilon}{2}, (\forall)\, m, n \geq n_\varepsilon$

- $|x_n - l| = |x_n - x_{n_k} + x_{n_k} - l| \leq \underbrace{|x_n - x_{n_k}|}_{< \frac{\varepsilon}{2}} + \underbrace{|x_{n_k} - l|}_{< \frac{\varepsilon}{2}}, (\forall)\, n, k \geq \max\{k_\varepsilon, n_\varepsilon\}$

Ex: $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

$x_{n+1} > x_n$

(x_n) is increasing

$x_{2n} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > n \cdot \frac{1}{2n} \Rightarrow x_{2n} - x_n > \frac{1}{2}$

(x_n) is not Cauchy $\Rightarrow (x_n)$ is not convergent

Series

- Let (x_n) be a sequence, $\sum_{n=1}^{\infty} x_n \rightarrow$ sequence

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots + x_n + \dots$$

- $S_n = x_1 + \dots + x_n \rightarrow$ partial sum of the series

$$S_n = \sum_{k=1}^{n} x_k$$

- $\sum_{n=1}^{\infty} x_n$ converges if (S_n) converges

Examples:

- **Geometric series**

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \dots q^n = \frac{1-q^{n+1}}{1-q}$$

$$S_n = 1 + \cancel{q} + \cancel{q^2} + \dots + \cancel{q^n}$$

$$q S_n = \cancel{q} + \cancel{q^2} + q^3 + \dots + q^{n+1}$$

$$(q-1) S_n = q^{n+1} - 1 \; ; \; S_n = \frac{q^{n+1}-1}{q-1} = \frac{1-q^{n+1}}{1-q}$$

$$\sum_{n=0}^{\infty} q^n = \lim_{n\to\infty} S_n = \begin{cases} \frac{1}{1-q}, & |q|<1 \\ +\infty, & q>1 \\ \nexists, & q \leq -1 \end{cases}$$

- **Harmonic series:**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots = +\infty \text{ (divergent)}$$

Proposition: if $\sum_{n=1}^{\infty} x_n$ convergent, then $\lim_{n\to\infty} x_n = 0$

• Series with nonnegative terms

$x_n \geq 0$, $\sum_{n\geq 1} x_n \nearrow$ (S_n) increasing

Thm (Comparison test):

Let $\sum_{n\geq 1} x_n$, $\sum_{n\geq 1} y_n$ with nonnegative terms

If $x_n \leq y_n$, $(\forall)\ n \geq n_0$, $n_0 \in \mathbb{N}$:

1. If $\sum_{n\geq 1} y_n$ converges, then $\sum_{n\geq 1} x_n$ converges
2. If $\sum_{n\geq 1} x_n$ diverges then $\sum_{n\geq 1} y_n$ diverges

Thm (Comparison test)

If $\lim_{n\to\infty} \frac{x_n}{y_n} = l \in \mathbb{R}$, then $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ have the same nature

ex: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = +\infty$ has the same nature as $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

$\left(\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1\right)$

Thm (Ratio test):

Let $\sum_{n\geq 1} x_n$ with $x_n \geq 0$. If $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \in \mathbb{R}$:

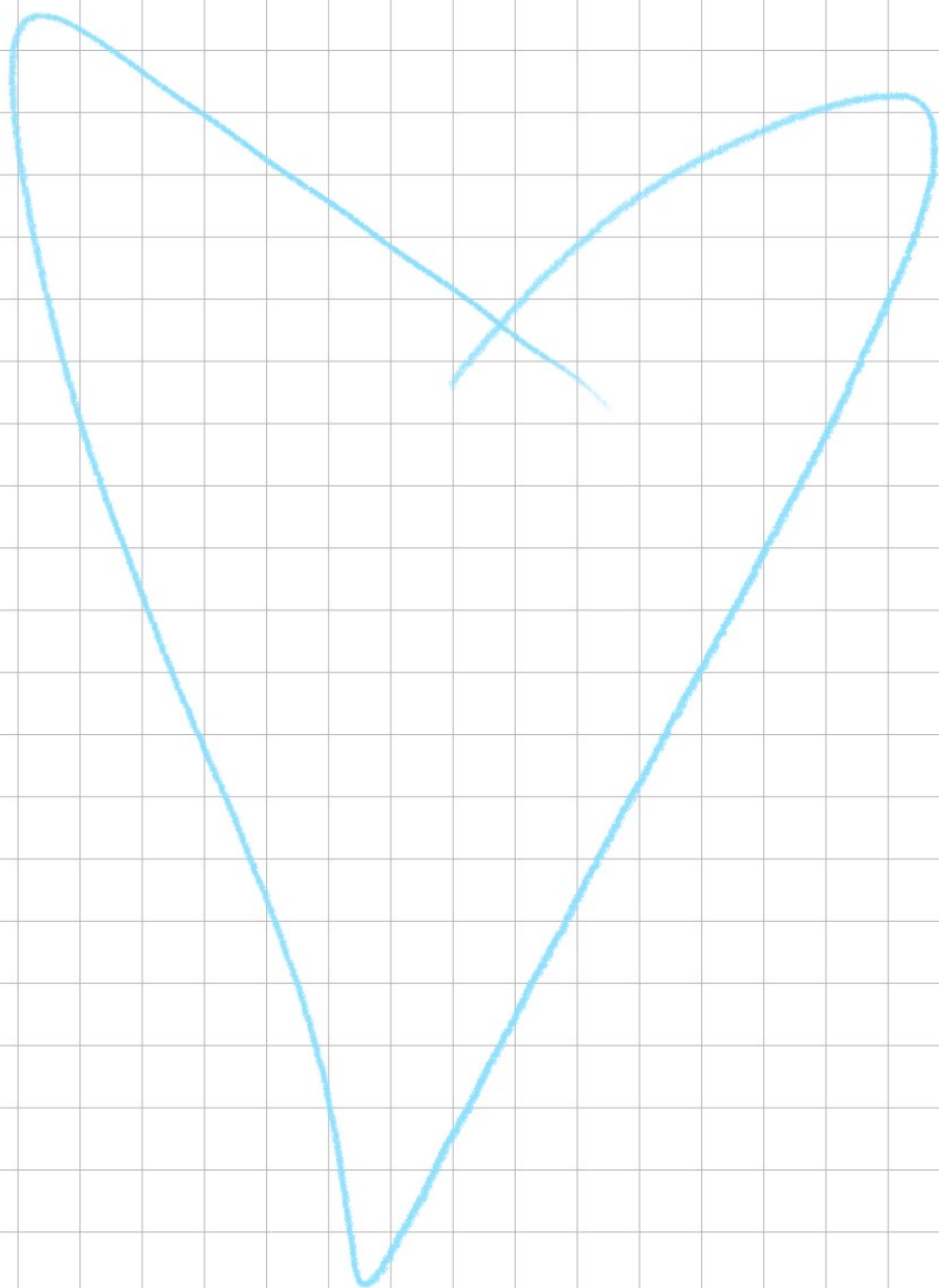
1. If $l < 1$, then $\sum_{n\geq 1} x_n$ converges
2. If $l > 1$, then $\sum_{n\geq 1} x_n$ diverges

Remark: If $l = 1$, inconclusive

etch: idea is that x_n behaves like a geo. progression with ratio l (Complete proof is in the lecture notes)

ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, $(\forall)\, x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{x^{\cancel{n+1}}}{(\cancel{n+1})!} \cdot \frac{\cancel{n!}}{\cancel{x^n}} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1 \Rightarrow$$



Seminar 3

Geometric Series

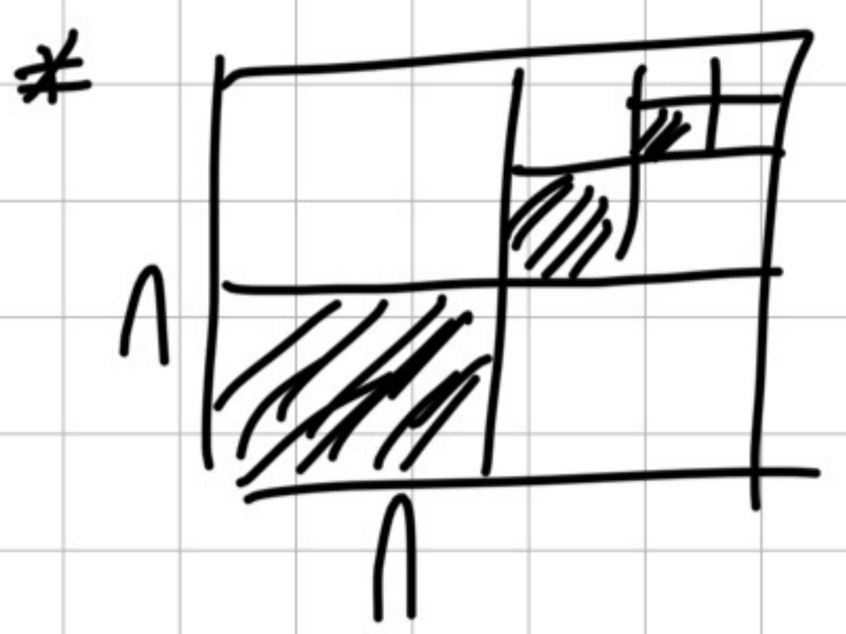
- $\sum_{n=0}^{\infty} q^n = 1+q+q^2+\dots = \frac{1}{1-q}$ if $|q|<1$

$$1+q+q^2+\dots+q^n = \frac{1-q^{n+1}}{1-q}$$

- $\sum_{n=1}^{\infty} q^n = q+q^2+\dots = \frac{1}{1-q}-1$

1) a) $$\sum_{n\geq 1} \frac{2}{3^n} = 2\sum_{n\geq 1}\frac{1}{3^n} = 2\left(\frac{1}{3}+\frac{1}{3^2}+\dots\right) = 2\left(\frac{1}{1-\frac{1}{3}}-1\right)$$

$$= 2\left(\frac{3}{2}-1\right) = 3-2 = 1$$

* 

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\dots = \frac{1}{1-\frac{1}{4}}-1 = \frac{4}{3}-1 = \frac{1}{3}$$

(the area which is ////)

b) $$\sum_{n\geq 1}\frac{2n+1}{n!} = \sum_{n\geq 1}\frac{2n}{n!}+\sum_{n\geq 1}\frac{1}{n!} = 2e+e-1 = 3e-1$$

* $$\sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\sum_{n\geq 1}\frac{2n}{n!} = 2\sum_{n\geq 1}\frac{1}{(n-1)!} = 2\sum_{k\geq 0}\frac{1}{k!} = 2e$$

$k=(n-1)!$

c) $$\sum_{n\geq 1}\frac{1}{4n^2-1} = \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) = \frac{1}{2}\cdot 1$$

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Telescoping series

$$1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1} = 1 - \underbrace{\frac{1}{2n+1}}_{\downarrow \atop 0} = 1$$

Comparison test : $\frac{x_n}{y_n} \to l \in (0, +\infty)$, then $\sum x_n$ and $\sum y_n$ have the same nature

$$\frac{x_n}{y_n} = \frac{1}{4n^2-1} \cdot n^2 = \frac{n^2}{4n^2-1} \to \frac{1}{4} \Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6} \text{ is convergent}$$

Ratio test : $\frac{x_{n+1}}{x_n} \to l$, if $l < 1$: $\sum x_n$ conv.

if $l > 1$: $\sum x_n$ div.

e) $\sum\limits_{n \geq 1} \frac{n}{2^n}$ - conv / div ?

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1 \Rightarrow \sum_{n \geq 1} x_n \text{ is}$$

convergent

$\hookrightarrow$
$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n-1}{2^{n-1}} + \frac{n}{2^n} + \dots$$

$$= \frac{1}{2}\left(1 + \frac{2}{2} + \frac{3}{2^2} + \dots + \frac{n}{2^{n-1}} + \dots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1+1}{2} + \frac{2+1}{2^2} + \frac{3+1}{2^3} + \dots + \frac{n-1+1}{2^{n-1}} + \dots\right)$$

$$= \frac{1}{2}\Big(S + \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} + \dots}_{2}\Big)$$

$$S = \frac{1}{2}(S+2) = \frac{1}{2}S + 1 \Rightarrow \frac{1}{2}S = 1 \Rightarrow S = 2$$

$$* \; f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}, \text{ if } |x| < 1$$

$$f'(x) = 1 + 2x + \ldots + n x^{n-1} + \ldots = \sum_{n \geq 1} n x^{n-1}$$

$$f'(x) = \left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{n \geq 1} n x^n = \frac{x}{(1-x)^2} = S(x)$$

$$\Rightarrow \sum_{n \geq 1} \frac{n}{2^n} = S\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

$$x = \frac{1}{2}$$

$$\sum_{n \geq 1} n x^n = \sum_{n \geq 1} n \left(\frac{1}{2}\right)^n = \sum \frac{n}{2^n}$$

f) $\sum_{n \geq 1} \frac{1}{\sqrt[3]{n}}$ - diverges

$\sqrt[3]{n} < n \Rightarrow \frac{1}{\sqrt[3]{n}} > \frac{1}{n} \rightarrow \sum_{n \geq 1} \frac{1}{n} = +\infty$ (Harmonic Series)

$$\Rightarrow \sum_{n \geq 1} \frac{1}{\sqrt[3]{n}} > \sum_{n \geq 1} \frac{1}{n} = +\infty \Rightarrow \sum_{n \geq 1} \frac{1}{\sqrt[3]{n}} = +\infty \Rightarrow$$

$$\boxed{\sum_{n \geq 1} \frac{1}{n^p} = +\infty, \text{ if } p \leq 1}$$

Root test: $\sqrt[n]{x_n} \rightarrow l$, if $l < 1$, $\sum x_n$ converges

$\sqrt[n]{x_n} = x_n^{\frac{1}{n}}$ if $l > 1$, $\sum x_n$ diverges

3) a) $\sum_{n \geq 1} \left(\frac{n}{n+1}\right)^{n^2}$ - conv / div

Root test: $x_n^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n \rightarrow \frac{1}{e} < 1$

$\left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n \rightarrow e$ e invers $\left(\frac{n}{n+1} \,/\, \frac{n+1}{n}\right)$

$\Rightarrow \sum_{n\geq 1} x_n$ - is convergent

b) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{R}$

Ratio test: $\frac{x_{n+1}}{x_n} = \frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n} = x\cdot\underbrace{\frac{n^p}{(n+1)^p}}_{1} \to x$

If $x<1 \Rightarrow \sum_{n\geq 1} x_n$ - is convergent

If $x>1 \Rightarrow \sum_{n\geq 1} x_n$ - is divergent

If $x=1$, $\sum_{n\geq 1}\frac{1}{n^p}$ $\to$ $p\leq 1$: $\sum_{n\geq 1}\frac{1}{n^p}$ - div

$p>1$: $\sum_{n\geq 1}\frac{1}{n^p}$ - conv