Seminar 8

7.9 $S = \{ (x,y,z) \in \mathbb{R}^3 \mid x=0 \}$

$T = \langle (0,1,1), (1,1,0) \rangle$

$S \cap T = \{ (x,y,z) \in \mathbb{R}^3 \mid \begin{array}{l} x=0 \\ (x,y,z) = \alpha(0,1,1) + \beta(1,1,0) \end{array} \} = \{ (x,y,z) \in \mathbb{R}^3 \mid \begin{array}{l} 0=\beta \\ y=\alpha+\beta \\ z=\alpha \end{array} \}$

$= \{ (0,y,z) \in \mathbb{R}^3 \mid y=\alpha, z=\alpha \} = \{ (0,\alpha,\alpha) \mid \alpha \in \mathbb{R} \} = \langle (0,1,1) \rangle$

$S \oplus T \iff \begin{cases} V = S+T \\ S \cap T = 0 \end{cases}$

$\Updownarrow$

$\dim(S+T) = \dim(V) = \dim(S) + \dim(T)$

8.5 Solve the fllw linear systems using the Gauss and the Gauss-Jordan methods:

i) $\begin{cases} 2x+2y+3z=3 \\ x-y=1 \\ -x+2y+z=2 \end{cases}$ $A = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array} \right)$

$A = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array} \right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 2 & 2 & 3 & 1 \\ -1 & 2 & 1 & 2 \end{array} \right)$ ↗ pivot $\underset{L_3 = L_3 + L_1}{\overset{L_2 = L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array} \right)$ → pivot row (doesn't get changed) $\overset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array} \right)$

$\overset{L_3 = L_3 - 4L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array} \right)$

Gauss method - revert to the system and solve it manually

Gauss-Jordan method - do the elimination the opposite way

Gauss:

$\begin{cases} x-y=1 \\ y+z=3 \\ -z=-11 \end{cases} \iff \begin{cases} z=11 \\ y+11=3 \\ x=y+1 \end{cases} \iff \begin{cases} z=11 \\ y=-8 \\ x=-7 \end{cases}$

Gauss - Jordan :

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \overset{L_2 \leftrightarrow L_2+L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & -1 & -11 \end{array}\right) \overset{L_1 \leftrightarrow L_1+L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & -1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \Rightarrow \begin{cases} x=-7 \\ y=-8 \\ z=11 \end{cases}$$

ii) $\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \\ x+y-4z=2 \end{cases}$

$$A=\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{L_3 \leftrightarrow L_3-L_1}{\overset{L_2 \leftrightarrow L_2-2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \; L_3 \leftrightarrow L_3+L_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftrightarrow L_1-2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

↳ an ecuation is unnecessary

$$\Rightarrow \begin{cases} x-7z=1 \\ y+3z=1 \end{cases} \overset{z=\alpha}{\Longrightarrow} \begin{cases} x=1+7\alpha \\ y=1-3\alpha \\ z=\alpha \end{cases}$$

iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

$$A=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{\substack{L_3 \leftrightarrow L_3-2L_1 \\ L_4 \leftrightarrow L_4-L_1}}{\overset{L_2 \leftrightarrow L_2-L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & 1 \end{array}\right) \overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & -2 & 0 & 1 \end{array}\right) \underset{L_4 \leftrightarrow L_4-2L_2}{\overset{L_3 \leftarrow L_3-3L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

From the last line we get $0=-4$ which is absurd, so the sistem is incompatible

8.6 $\begin{cases} 2x_1+x_2+x_3+x_4=1 \\ x_1+2x_2-x_3+4x_4=2 \\ x_1+5x_2-4x_3+11x_4=\lambda \end{cases}$ $(\lambda \in \mathbb{R})$

$$A=\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_2 \leftrightarrow L_1}{\sim} \left(\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \underset{L_3 \leftrightarrow L_3-L_1}{\overset{L_2 \leftrightarrow L_2-2L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda-2 \end{array}\right) \overset{L_3 \leftrightarrow L_3+L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{array}\right)$$

if $\lambda \neq 5$, the system is incompatible

if $\lambda=5$, $\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftrightarrow L_1+\frac{2}{3}L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x_1 + x_3 - \frac{2}{3} x_4 = 0 \\ -3x_2 + 3x_3 - 7x_4 = -3 \\ x_3 = \alpha \\ x_4 = \beta \end{cases} \Leftrightarrow \begin{cases} x_1 = -\alpha + \frac{2}{3} \beta \\ x_2 = 1 + \alpha - \frac{7}{3} \beta \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$