Curs 5

Def: Let $A \subseteq \mathbb{R}$, let $x_0 \in \overline{\mathbb{R}}$. We say that:

- x_0 is an accumulation point if $(\forall)\, V \in \mathcal{V}(x_0)$, $V \cap (A - \{x_0\}) \neq \emptyset$
- A' - set of accumulation points of A
- x_0 is an isolated point if $\in A - A'$

Theorem: $x_0 \in A' \iff \exists\, (x_n)$-sequence in $A - \{x_0\}$ so that $x_n \to x_0$

(proof in lecture notes)

ex: a) $A = \{0, 1\}$, $A' = \emptyset$
↳ which are isolated points

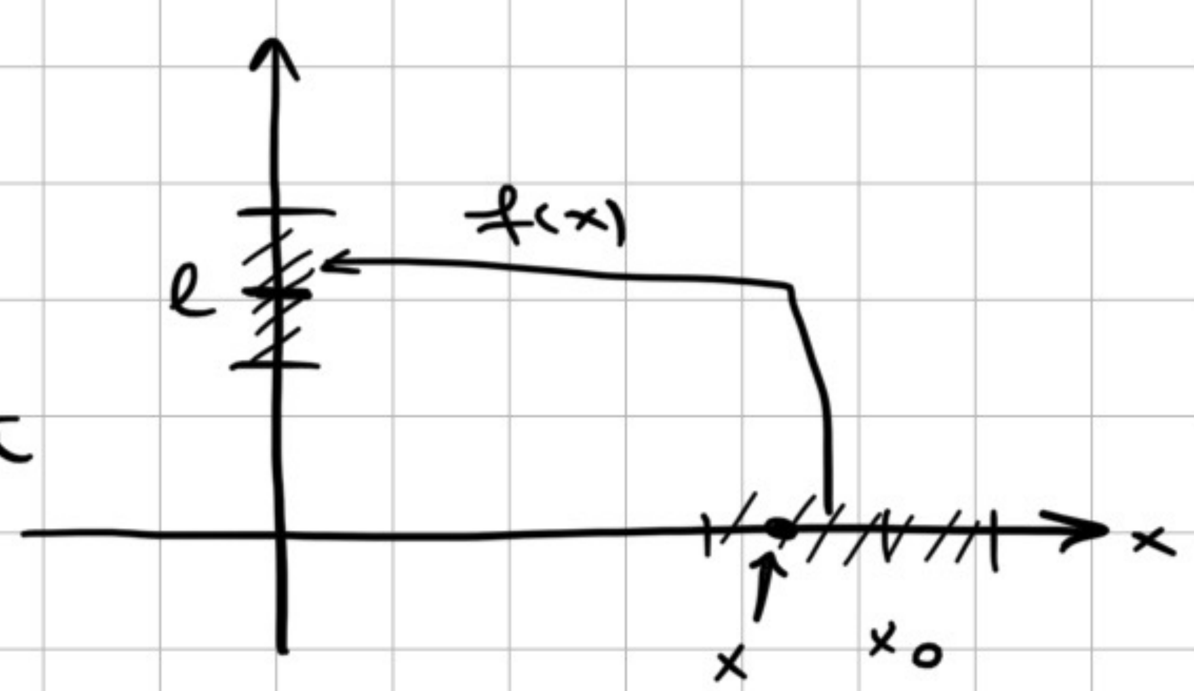
b) $A = \{a_1, a_2, \dots a_n\}$ - finite points, $A' = \emptyset$
↳ all are isolated points

c) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $A' = 0$
↳ isolated points

d) $A = (0,1) \cup (1,2)$, $A' = [0,2]$
no isolated points

Def: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, let $x_0 \in A'$

$\lim\limits_{x \to x_0} f(x) = l \in \overline{\mathbb{R}}$ if $\forall V \in \mathcal{V}(l)$, $\exists\, \mathcal{V}(x_0)$ s.t. $f(x) \in V$, $(\forall)\, x \in U \cap (A - \{x_0\})$



Theorem: $f: A \to \mathbb{R}$, $x_0 \in A'$

$\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R} \iff \forall \varepsilon > 0, \exists\, \delta > 0$ such that

$|f(x) - l| < \varepsilon$, $(\forall)\, x \in A$ s.t. $|x - x_0| < \delta$

Theorem: $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R} \iff \forall$ sequence (x_n) in $A - \{x_0\}$, $x_n \to x_0$, we have $f(x_n) \to l$

ex: a) $\operatorname{sgn}: \mathbb{R} \to \mathbb{R}$, $\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$\nexists \lim\limits_{x \to 0} \operatorname{sgn}(x)$

b) $f: \mathbb{R}^* \to \mathbb{R}$, $f(x) = \sin\left(\frac{1}{x}\right)$

$\cdot\ x_n = \frac{1}{2\pi n}$, $f(x_n) = 0$

$\cdot\ y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, $f(y_n) = 1$

$\begin{matrix} x_n \to 0 \\ y_n \to 0 \end{matrix}$, $f(x_n) \to 0 \neq f(y_n) \to 1 \Rightarrow \nexists \lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$

c) $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$\nexists \lim\limits_{x \to x_0} f(x)$, $\forall\, x_0 \in \mathbb{R}$

$x_n \to x_0$, with $x_n \in \mathbb{Q}$

$y_n \to x_0$, with $y_n \in \mathbb{R} - \mathbb{Q}$

$\begin{matrix} f(x_n) = 1 \\ f(y_n) = 0 \end{matrix} \Rightarrow f(x_n) \to 1 \neq 0 \leftarrow f(y_n) \Rightarrow \nexists \lim\limits_{x \to x_0} f(x)$

Theorem: $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R} \iff \lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = \lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = l$

$\to$ lateral limits

Def: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, let $x_0 \in A' \cap A$

f is cont. at x_n if $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Theorem: $f: A \to \mathbb{R}$, $x_0 \in A' \cap A$, f is cont at $x_0 \iff \forall (x_n)$ sequence $x_n \to x_0$, $f(x_n) \to f(x_0)$

"x close to $x_0 \Rightarrow f(x)$ close to $f(x_0)$"

$\cdot$ **Remark**: if x_0 is an isolated point, any function f is cont at x_0

$f: A \to \mathbb{R}$, $f(A) = \{ y \in \mathbb{R} / \exists\, x \in A, y = f(x) \}$

$\hookrightarrow$ image of A

$\boxed{\begin{array}{l} \cdot\ f \text{ is bounded if } f(A) \text{ is bounded } (\inf f(A), \sup f(A) \in \mathbb{R} \text{ (not infinite)}) \\ \text{Then } f \text{ is bounded and } \exists \min f(A), \max f(A) \end{array}}$

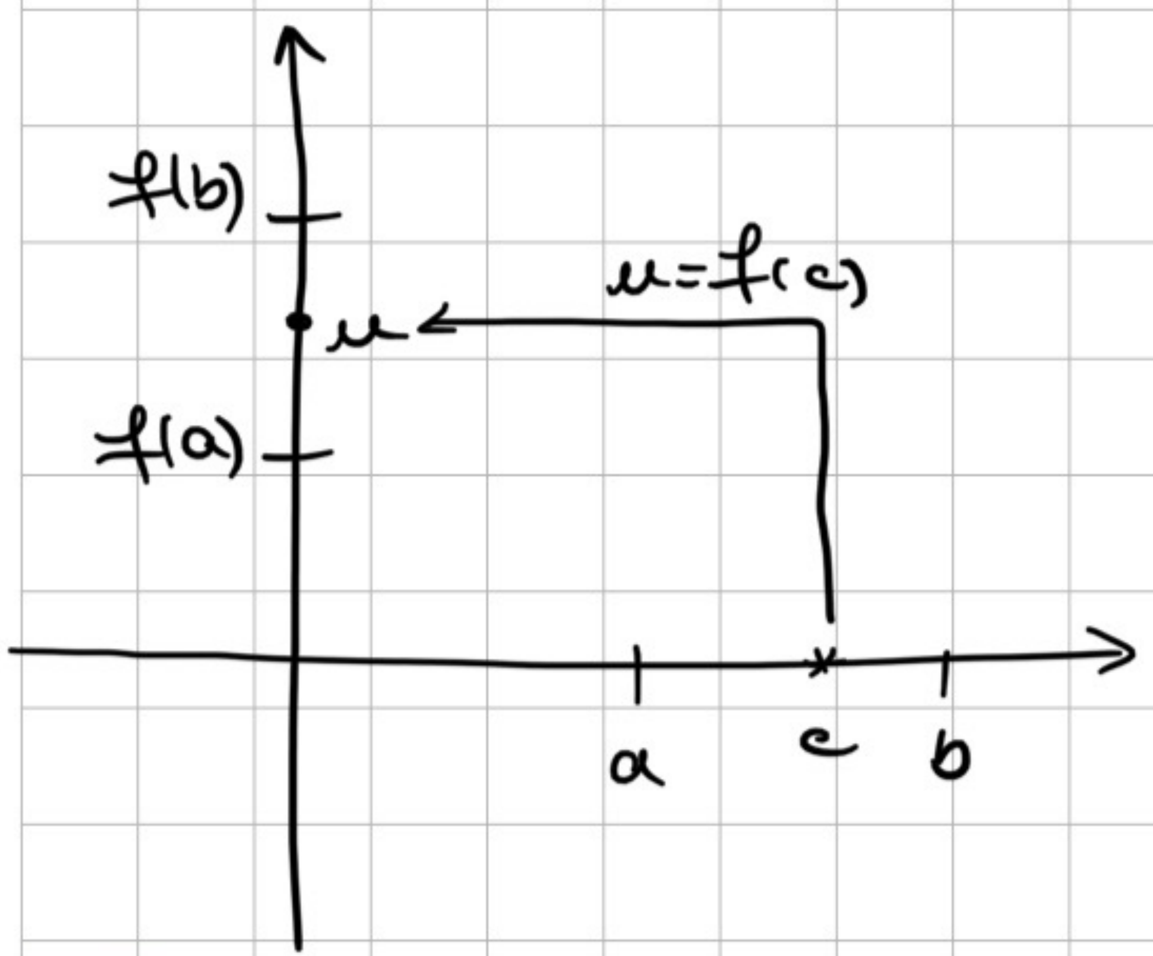
Ex: $f: \underbrace{(0,1)}_{A}$, $f(x) = x$, $f(A) = (0,1) \to$ bounded

$\nexists \min f(A)$, $\nexists \max f(A)$

$\inf A = 0, \sup A = 1$

Proof in the lecture

Theorem: $f: [a,b] \to \mathbb{R}$, f is cont. Say $f(a) < f(b)$, then $(\forall) u \in (f(a), f(b))$,
(Intermediate value property) $\exists\, c \in (a,b)$ such that $u = f(c)$

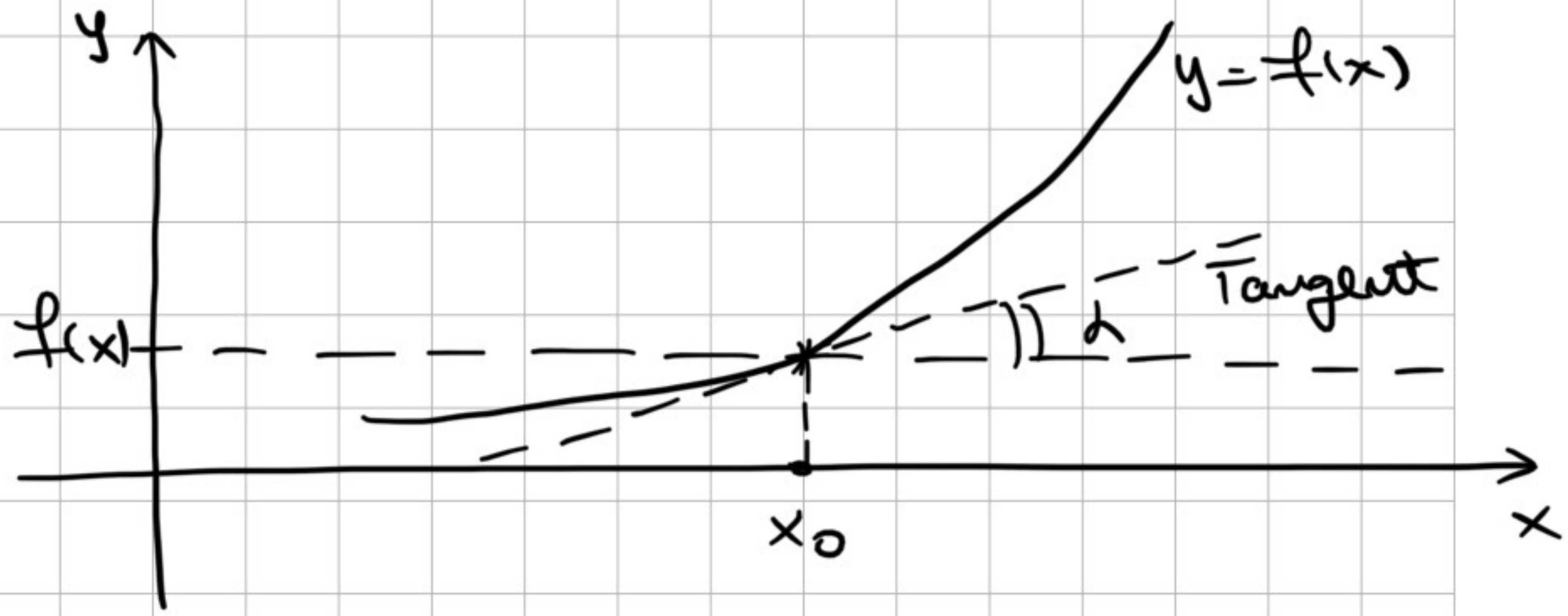


Def: $f: A \to \mathbb{R}$, $x_0 \in A' \cap A$, it has a derivate at x_0 if $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$

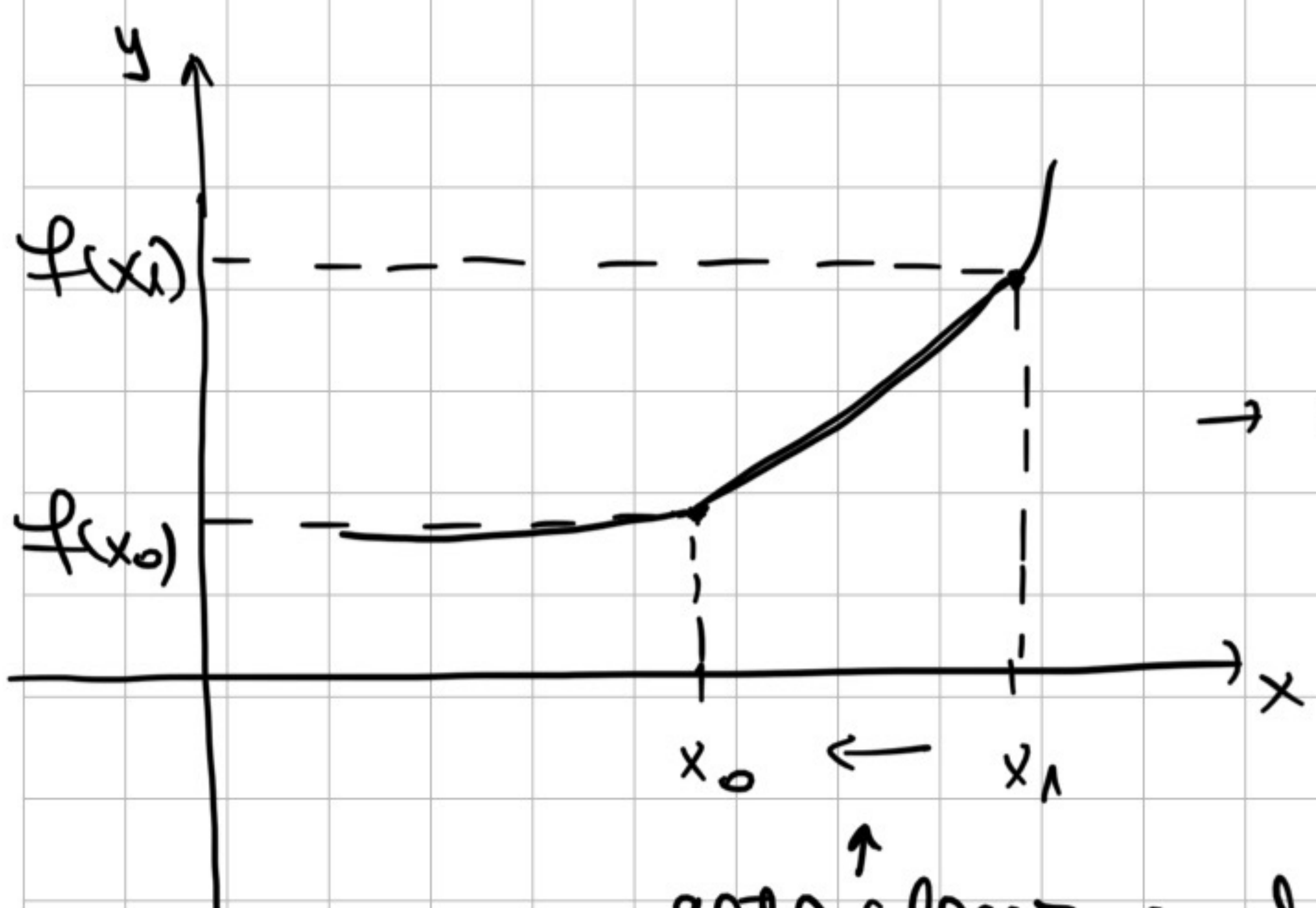
Def: $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$ (finite), then f is differentiable

Gradient of tangent $= f'(x_0)$
(pantă)

Tangent: $y - f(x_0) = f'(x_0) \cdot (x - x_0)$



$f'(x_0) = \tan(\alpha)$ (gradient of tg)



$\to$ gradient $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

goes closer and closer to x_0

Calculus rules:

$(c \cdot f') = c \cdot f'$
$(c \cdot f')(x) = c \cdot f'(x)$
} same thing

- sum rule: $(f+g)' = f' + g'$

- product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$
- quotient rule: $\left(\frac{f}{g}\right) = \frac{f' g - f \cdot g'}{g^2}$
- chain rule: $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dz}$ (differentiate z in respect to x...)
- $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$!

Notation: $\frac{dy}{dx}$ → derivative !

Chain rule proof: $\frac{f(g(x)) - f(g(x_0))}{x - x_0} = (f \circ g)'(x_0) = \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)}$

$\cdot \frac{g(x) - g(x_0)}{x - x_0} = f'(g(x_0)) \cdot g'(x_0)$

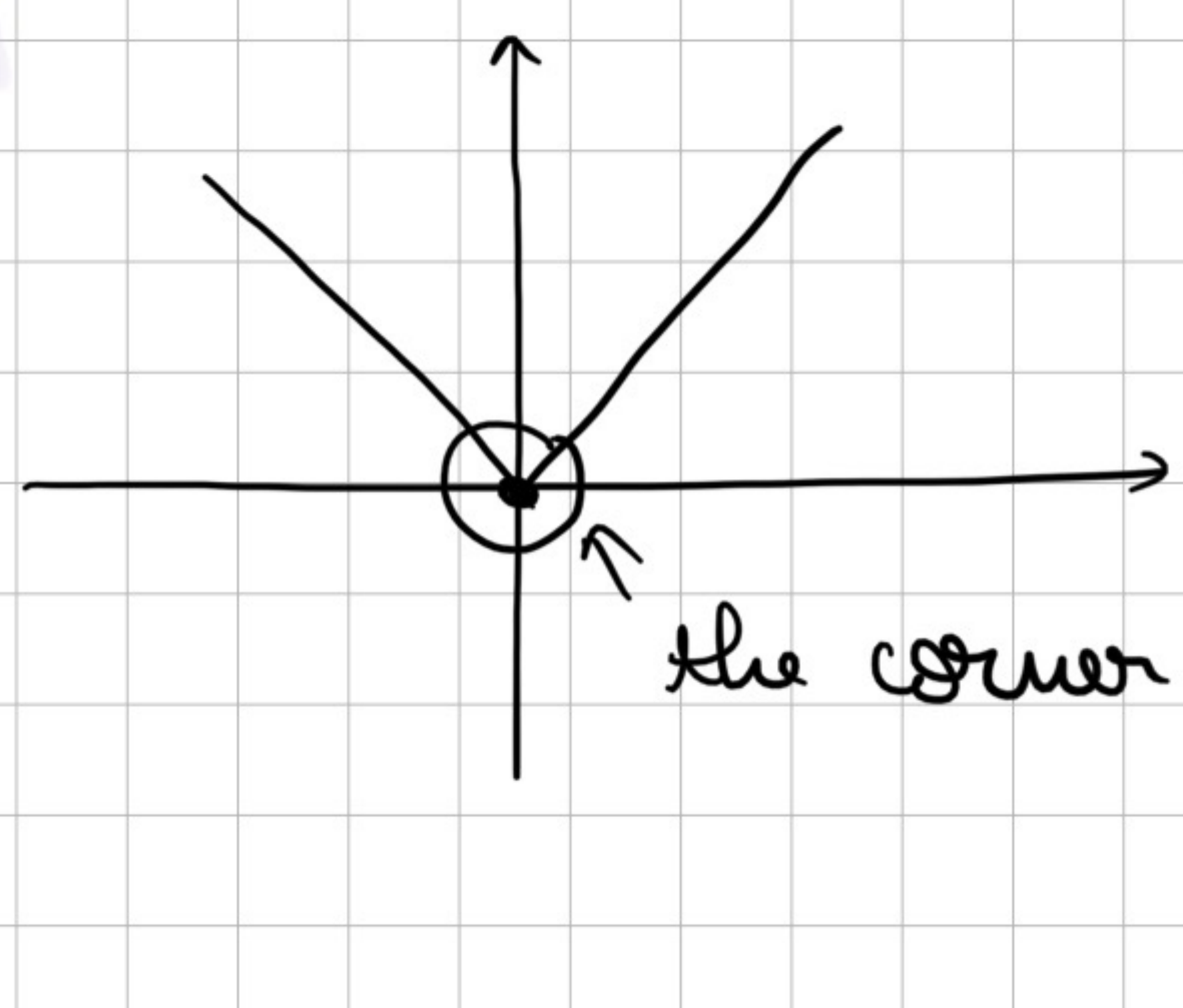
Theorem: f is differentiable ⇒ f is continuous

↳ Proof: $f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$

get $x \to x_0$: $\lim_{x \to x_0} f(x) - f(x_0) = f'(x_0) \cdot 0$

$\lim_{x \to x_0} f(x) = f(x_0)$

Ex:



$f(x) = |x|$, cont but not differentiable

the corner is the problem

Seminar 5

1) $[0,1)\cup\{2\}$, $\mathbb{Z}$, $\mathbb{Q}$

accumulation points

· $x_0 \in A'$ (acc. points, $(\forall)\, V \in \mathcal{V}(x_0)$, $V \cap A - \{x_0\} = \emptyset$)

· $\exists\, (x_n)$ sequence, $x_n \in A - \{x_0\}$ such that $x_n \to x_0$

a) $A = [0,1)$, $A' = [0,1)$

isolated point $\in A - A'$ ($\{2\}$ in this case)

b) $A = \mathbb{Z} = \{\dots -2, -1, 0, 1, 2 \dots\}$

$\exists\, x \in \mathbb{Z}$ is an isolated point

$A' = \{\pm\infty\}$ since we can take $x_n = n \to x_0 (\infty)$

c) $A = \mathbb{Q}$, $A' = \overline{\mathbb{R}}$ → we can have sequences $\frac{n}{1}$ that go to infinity

no isolated points

2. $f: \mathbb{R} \to \mathbb{R}$, discont. everywhere, but $|f|$ - cont. everywhere

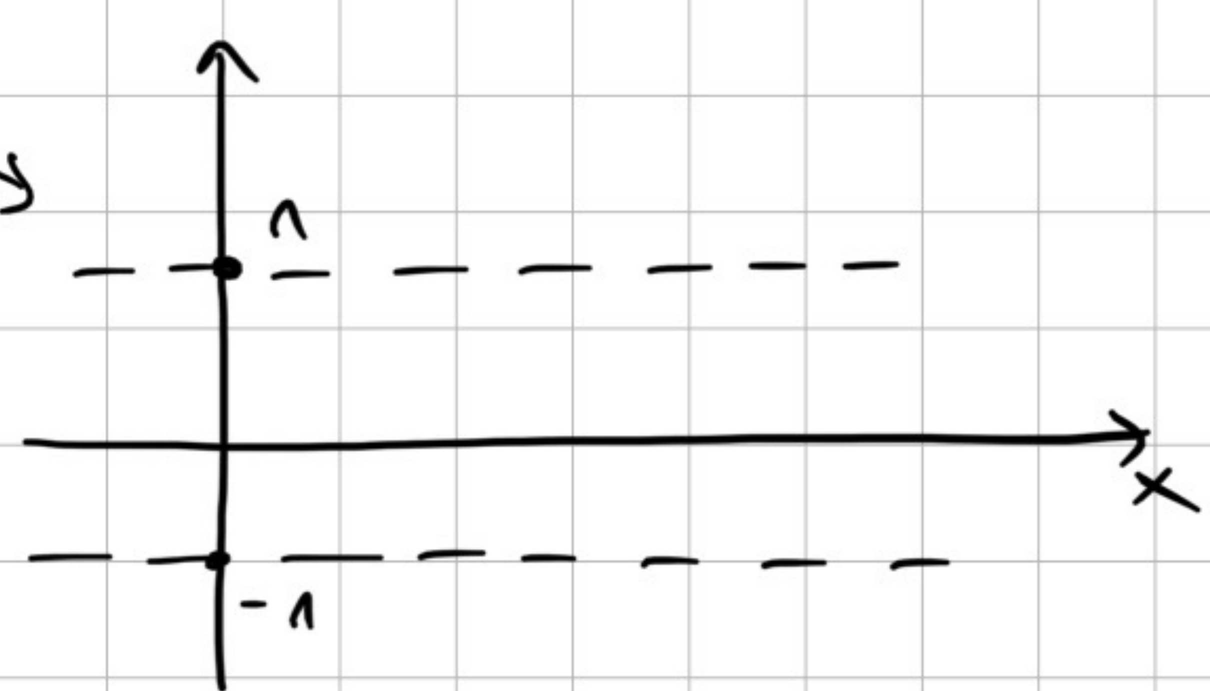
· f cont at $x_0 \in A' - A$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

· f discont at x_0

$$\lim_{x \to x_0} f(x) \neq f(x_0), \quad \nexists \lim_{x \to x_0} f(x)$$

$$f(x) = \begin{cases} +1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} - \mathbb{Q} \end{cases} \to$$



$$\boxed{\lim_{x \to x_0} f(x) = l \in \overline{\mathbb{R}} \text{ if } (\forall)\, x_n,\ x_n \to x_0,\ x \to x_0,\ f(x_n) \to l}$$

get $x_0 \in \mathbb{R}$, $x_n \in \mathbb{Q}$, $x_n \to x_0$

$y_n \in \mathbb{R} - \mathbb{Q}$, $y_n \to x_0$

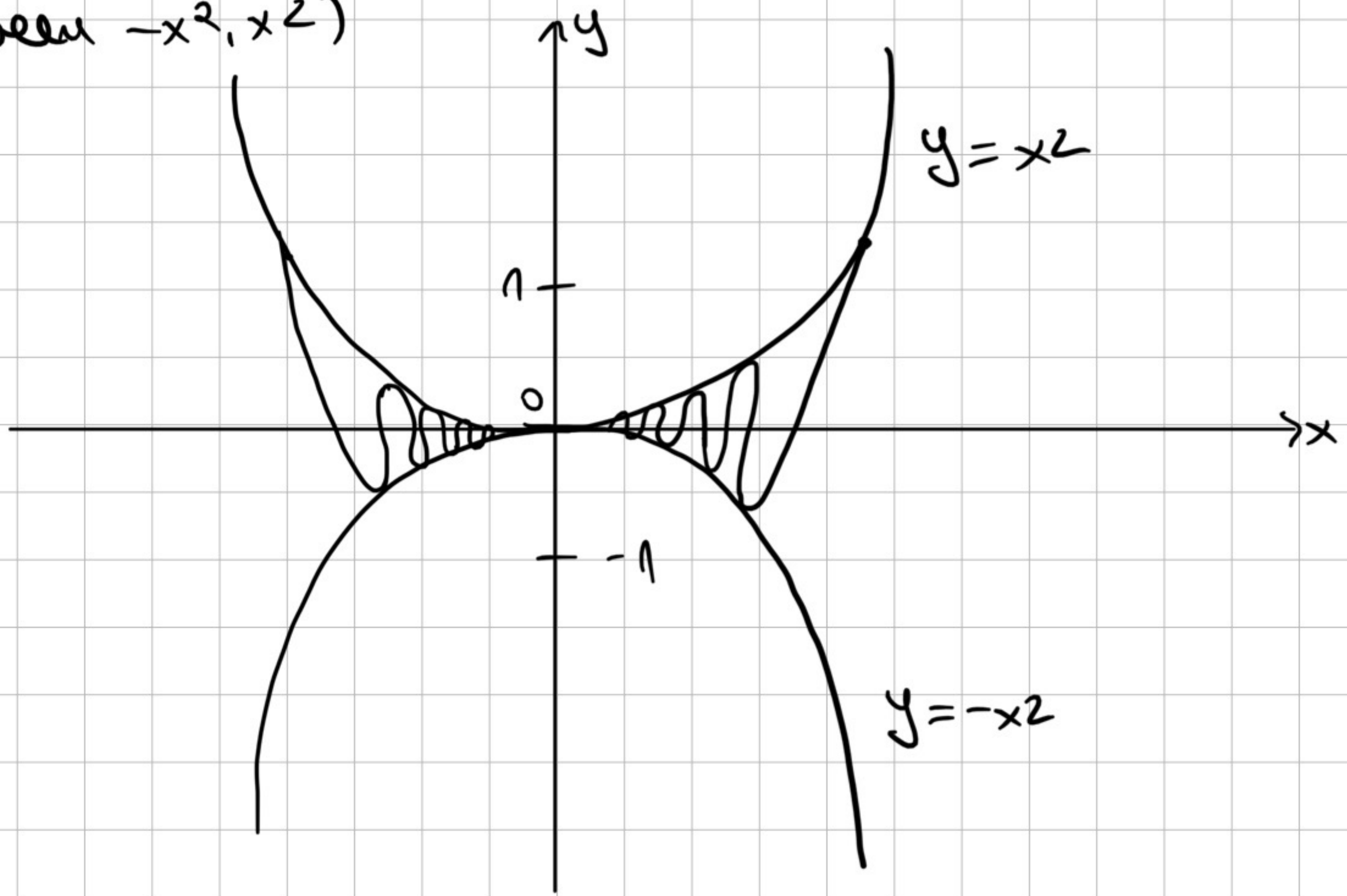
$$\left.\begin{array}{l} f(x_n) = 1 \\ f(y_n) = -1 \end{array}\right\} \Rightarrow f(x_n) \to 1 \neq -1 \leftarrow f(y_n) \Rightarrow \nexists \lim_{x \to x_0} f(x)$$

$$|f(x)| = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} - \mathbb{Q} \end{cases} \Rightarrow |f(x)| - \text{constant} \Rightarrow |f(x)| - \text{cont.}$$

1) $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity and differentiability for f and f', $f: \mathbb{R} \to \mathbb{R}$

$-1 \leq \sin \frac{1}{x} \leq 1 \mid \cdot x^2 \Rightarrow -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ (It squeezes the function between $-x^2, x^2$)



- f is cont at $x \neq 0$ (elementary functions)

$\lim\limits_{x \to 0} f(x) = 0$, so f is cont

$\lim\limits_{x \to 0} f(x) = 0 = f(0) \Rightarrow f$ is cont in 0 (and it is cont everywhere)

If $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \frac{(-1)}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$

- $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{f(x)}{x} = \lim\limits_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$

squeeze theorem

- $f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

- $f'(x)$ is cont when $x \neq 0$

- $\nexists \lim_{x \to 0} f(x)$ (because $\cos \frac{1}{0}$ doesn't exist)

$\Rightarrow f'$ is discont.

$\boxed{\text{Differentiability implies continuity} \\ \text{Not continuous implies not differentiable}}$

6. a) $[x] \leq x < [x] + 1$

b) $x \ln \frac{x+2}{x+1}$, $\lim_{x \to \infty} x \cdot \ln \frac{x+2}{x+1} \overset{\boxed{\infty \cdot 0}}{=} \lim_{x \to \infty} \ln \left(\frac{x+2}{x+1}\right)^x =$

$= \ln\left(\lim_{x \to \infty} \frac{(x+2)^x}{(x+1)^x}\right) = \ln\left(\lim_{x \to \infty} \left(1 + \frac{1}{x+1}\right)^x\right) = \ln\left(\lim_{x \to \infty} \left(1 + \frac{1}{x+1}\right)^{x \cdot \frac{1}{x+1} \cdot \frac{x+1}{1}}\right)$

$= \ln\left(e^{\lim_{x \to \infty} \frac{x}{x+1}}\right) = \ln(e^1) = 1$

d) $\lim_{\substack{x \to 0 \\ x > 0}} x^x \overset{\boxed{0^0}}{=} \lim_{x \downarrow 0} e^{\ln x^x} = \lim_{x \downarrow 0} e^{x \ln x} = \lim_{x \downarrow 0} e^0 = 1$

$\boxed{a = e^{\ln a}} \Rightarrow x^x = e^{\ln x^x} = e^{x \ln x} \overset{*}{=} e^0 = 1$

$^* \lim_{x \downarrow 0} x \ln x \overset{0 \cdot (-\infty)}{=} \lim_{x \downarrow 0} \frac{\ln x}{\frac{1}{x}} \overset{l'H}{=} \lim_{x \downarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \downarrow 0} (-x) = 0$

(we did e) in the same way)

f) $\lim_{x \to \infty} x\left(\left(1 + \frac{1}{x}\right)^x - e\right) \overset{\infty \cdot 0}{=} \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)^x - e}{\frac{1}{x}} \overset{l'H}{=} \lim_{x \to \infty} \frac{x\left(1 + \frac{1}{x}\right)^{x-1} \left(1 + \frac{1}{x}\right)'}{-\frac{1}{x^2}} =$

$= \lim_{x \to \infty} x\left(1 + \frac{1}{x}\right)^{x-1}$.

$f(x) = \left(1 + \frac{1}{x}\right)^x$

$f'(x) = ?$

$$f(x) = e^{\ln|f(x)|} = e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$f'(x) = e^{x\ln\left(1+\frac{1}{x}\right)} \cdot \left(\ln\left(1+\frac{1}{x}\right) + \cancel{x} \cdot \frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^{\cancel{2}}}\right) = e^{\overbrace{x\ln\left(1+\frac{1}{x}\right)}^{\to 1}} \left(\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right) =$$

$$= e \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}}{\frac{1}{x^2}} \overset{\frac{0}{0}}{\underset{l'H}{=}} e \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2} + \frac{1}{(x+1)^2}}{\frac{-2}{x^3}} = e \lim_{x\to\infty} \frac{\frac{-1}{x^2+x} + \frac{1}{(x+1)^2}}{\frac{-2}{x^3}}$$

$$= e \lim_{x\to\infty} \frac{\frac{-(x+1)}{x(x+1)^2} + \frac{x}{x(x+1)^2}}{\frac{-2}{x^3}} = e \lim_{x\to\infty} \frac{\frac{1}{x(x+1)^2}}{\frac{-2}{x^3}} = e \lim_{x\to\infty} \frac{1}{x(x+1)^2} \cdot \frac{x^3}{-2} = \frac{e}{2}$$