Seminar 6

Def: $V : K-vs$

$v_1, \dots v_n \in V$ - are $\boxed{\text{linear independent}}$ if $(\forall)\, \alpha_1, \dots \alpha_n \in K$,
$\alpha_1 v_1 + .. + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

ex 6.2 Prove that the following vectors are linear indep.:

ii) $v_1 = (1,2,3,4)$
$v_2 = (2,3,4,1)$
$v_3 = (3,4,1,2)$
$v_4 = (4,1,2,3)$

Prove that:

$$\alpha_1(1,2,3,4) + \alpha_2(2,3,4,1) + \alpha_3(3,4,1,2) + \alpha_4(4,1,2,3) = 0$$

$$= (\alpha_1, 2\alpha_1, 3\alpha_1, 4\alpha_1) + (2\alpha_2, 3\alpha_2, 4\alpha_2, \alpha_2) + (3\alpha_3, 4\alpha_3, \alpha_3, 2\alpha_3) + (4\alpha_4, \alpha_4, 2\alpha_4, 3\alpha_4)$$

$$= (\underbrace{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4}_{=0}, \underbrace{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + \alpha_4}_{=0}, \underbrace{3\alpha_1 + 4\alpha_2 + \alpha_3 + 2\alpha_4}_{=0}, \underbrace{4\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4}_{=0})$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0 \\ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + \alpha_4 = 0 \\ 3\alpha_1 + 4\alpha_2 + \alpha_3 + 2\alpha_4 = 0 \\ 4\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 = 0 \end{cases}, \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{C_3 \leftarrow C_3 - 3C_1 \\ C_4 \leftarrow C_4 - 4C_1}]{C_2 \leftarrow C_2 - 2C_1} \begin{vmatrix} 1 & \cancel{0} & \cancel{0} & \cancel{0} \\ 2 & -1 & -2 & -7 \\ 3 & -2 & -8 & -10 \\ 4 & -7 & -10 & -13 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 2 & 7 \\ 2 & 8 & 10 \\ 7 & 10 & 13 \end{vmatrix} = 160 \neq 0 \Rightarrow$$

$\Rightarrow$ the sistem is compatible det. $\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is the only sol

ex 6.4. $v_1 = (1,-2,0,-1)$
$v_2 = (2,1,1,0)$

$v_3 = (0, a, 1, 2)$

$a \in \mathbb{R} = ?$ s.t. v_1, v_2, v_3 are linear dependent

we have to prove that $\exists\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (not all 0)

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \alpha_1(1, -2, 0, -1) + \alpha_2(2, 1, 1, 0) + \alpha_3(0, a, 1, 2) =$

$(\alpha_1, -2\alpha_1, 0, -\alpha_1) + (2\alpha_2, \alpha_2, \alpha_2, 0) + (0, a\alpha_3, \alpha_3, 2\alpha_3) =$

$= (\alpha_1 + 2\alpha_2 + 0, -2\alpha_1 + \alpha_2 + a\alpha_3, 0 + \alpha_2 + \alpha_3, -\alpha_1 + 0 + 2\alpha_3) = (0, 0, 0, 0)$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 0 = 0 \\ -2\alpha_1 + \alpha_2 + a\alpha_3 = 0 \\ 0 + \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + 0 + 2\alpha_3 = 0 \end{cases}, \quad A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & a \\ 0 & \boxed{1} & \boxed{1} \\ -1 & \boxed{0} & \boxed{2} \end{pmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2 \neq 0$$

$$\Delta_{31} = \begin{vmatrix} -2 & 1 & a \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = a - 5, \quad \Delta_{32} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 0$$

v_1, v_2, v_3 - lin. dep. $\Leftrightarrow$ rank$(A) < 3 \Leftrightarrow \Delta_{31} = 0 \Rightarrow a - 5 = 0$, $a \in \mathbb{R}$ $\Rightarrow a = 5$

Def: V: K-vs, $B = \{v_1, v_2, \dots v_n\}$

B - basis for V $\overset{def}{\Leftrightarrow}$ $\begin{cases} \textcircled{!} \cdot V = \langle v_1, \dots, v_n \rangle \\ \textcircled{!} \cdot v_1, v_2, \dots v_n \text{ linearly independent} \end{cases}$

$\overset{OR}{\Leftrightarrow}$ $(\forall)\, v \in V$, $\exists!\, \alpha_1, \alpha_2 \dots \alpha_n$ s.t. $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$

If this is the case, then we denote $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$!

ex 6.7 $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Prove that (E_1, E_2, E_3, E_4), (A_1, A_2, A_3, A_4) are basis for $\mathcal{M}_2(\mathbb{R})$ and determ. the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the 2 bases

- for list (E_1, E_2, E_3, E_4) :

Let $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ s.t. $v = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4$

$$v = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

$\Rightarrow a = \alpha_1, b = \alpha_2, c = \alpha_3, d = \alpha_4$

unique solutions, therefore $\underbrace{(E_1, E_2, E_3, E_4)}_{\mathcal{E}}$ is a basis of $[B]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

- for list (A_1, A_2, A_3, A_4)

Let $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ s.t. $v = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$

$$v = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_3 & \alpha_3 \\ \alpha_3 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_4 & \alpha_4 \\ \alpha_4 & \alpha_4 \end{pmatrix}$$

$$\Rightarrow v = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_1 + \alpha_4 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a \\ \alpha_2 + \alpha_3 + \alpha_4 = b \end{cases}$$

$\alpha_3 + \alpha_4 = c$

$\alpha_1 \quad + \quad \alpha_4 = d \Rightarrow \alpha_4 = d - \alpha_1 \Rightarrow \alpha_3 + d - \alpha_1 = c \Leftrightarrow$

$\Leftrightarrow \alpha_3 = c - d + \alpha_1$