Analiză - Curs 1

- 60% final exam
- 30% midterm test (week 8)
- 10% homework + seminar activity
- 5% bonus for extra homework

$\mathbb{N} := \{1, 2, \dots\}$ - natural numbers

$\mathbb{Z} := \{m - n \mid m, n \in \mathbb{N}\} = \{\dots -1, 0, 1, 2 \dots\}$ - integers (whole numbers)

$\mathbb{Q} := \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^*\}$ - rational numbers

$\mathbb{R}$ - real numbers

$\sqrt{2} \notin \mathbb{Q}$: Proof by contradiction

Assume that $\sqrt{2} \in \mathbb{Q}$. Then $\exists\, m, n \in \mathbb{Z}$ such that (s.t.)

$\sqrt{2} = \frac{m}{n}$, with $\gcd(m, n) = 1$, $m = \sqrt{2} \cdot n \uparrow^2$

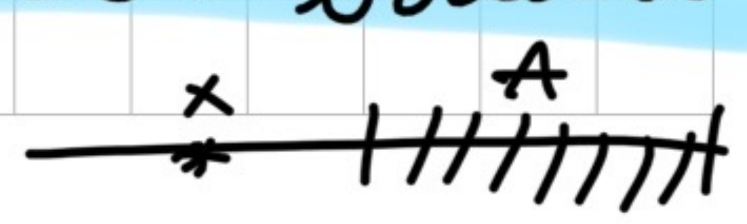
$\Rightarrow m^2 = 2n^2$. Then $m = 2k$, $k \in \mathbb{Z} \Rightarrow 4k^2 = 2n^2 \Rightarrow 2k^2 = n^2$.

Then $n = 2l$, $l \in \mathbb{Z}$ ↯ $\gcd(m, n) \geq 2$

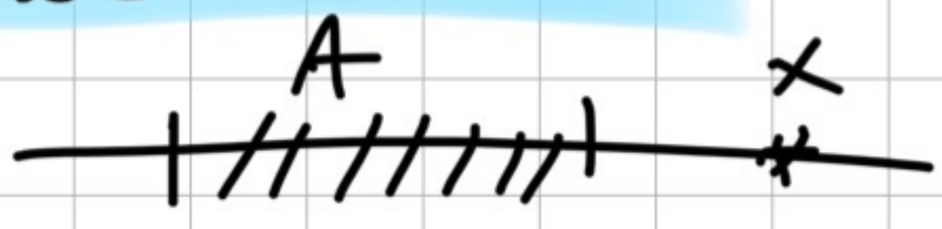
Def: Let $A \subseteq \mathbb{R}$ (subset). We say that:

(mărginit inferior)

- $lb(A) := \{x \in \mathbb{R} \mid x \leq a, (\forall)\, a \in A\} \rightarrow$ lower bounds



- $ub(A) := \{ x \in \mathbb{R} / x \geq a, (\forall) a \in A \}$ → upper bounds



- $min(A) := lb(A) \cap A$
- $max(A) := up(A) \cap A$

- bounded (from) below if $lb(A) \neq \emptyset$
- bounded (from) above if $up(A) \neq \emptyset$
- bounded if it both bounded below and above
- unbounded if it is not bounded

Def: • Every $A \subseteq \mathbb{R}$ (non empty) that is bounded above has a least an upper bound

$sup(A) = min(ub(A))$ → SUPREMUM

• Every $A \subseteq \mathbb{R}$ bounded from below has a greatest lower bound

$inf(A) := max(lb(A))$ → INFIXIMUM

Ex: a) $A = [0,1)$

$lb(A) = \{ x \in \mathbb{R} / x \leq 0 \}$

$ub(A) = \{ x \in \mathbb{R} / x \geq 1 \}$

$min(A) = 0$, $max(A) =$ does not exist

$inf(A) = 0$, $sup(A) = 1$

b) $A = \{ \frac{1}{n} / n \in \mathbb{N} \}$

$lb(A) = \{ x \in \mathbb{R} / x \leq 0 \}$

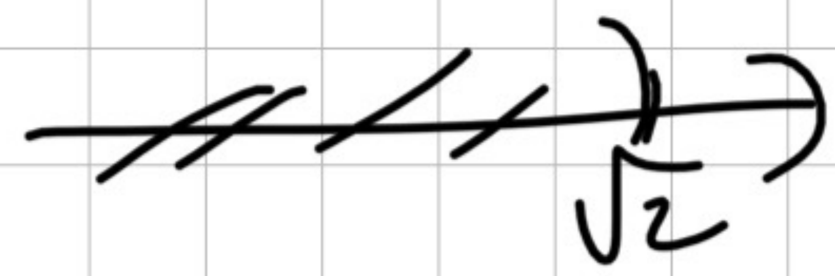
$ub(A) = \{ x \in \mathbb{R} / x \geq 1 \}$

min(A) - does not exist, max(A) = 1

inf(A) = 0, sup(A) = 1

Remark: $\begin{cases} \text{If } \min(A) \text{ exists, then } \inf(A) = \min(A) \\ \text{If } \max(A) \text{ exists, then } \sup(A) = \max(A) \end{cases}$

c) $A = \{ x \in \mathbb{Q} \mid x^2 < 2 \}$



max(A) does not exists, sup(A) - exists $= \sqrt{2}$

- Extended real line $\quad +\infty := \sup(\text{unbounded above set})$

$\qquad -\infty := \inf(\text{unbounded below set})$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

Proposition: Let $A \subseteq B \subseteq \mathbb{R}$

$\left.\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)\right\}$ Hw !

Seminar 1

Ex: 1. b) $A = (-3, 2) \cup \{3\}$

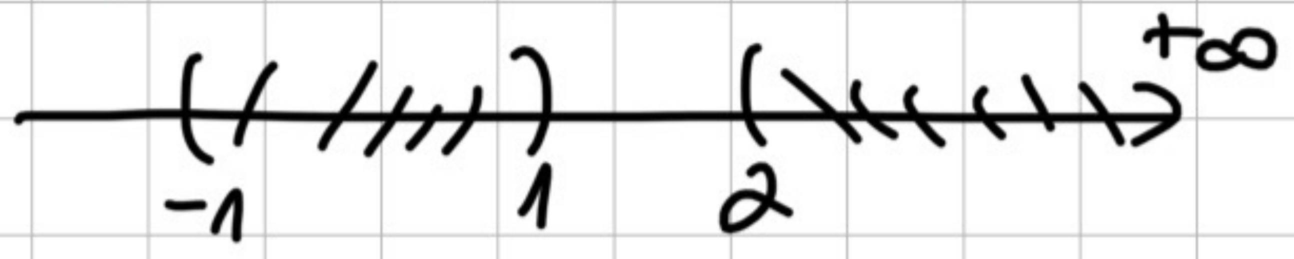
$lb(A) = (-\infty, -3]$

$ub(A) = [3, +\infty)$

min(A) doesn't exist, max(A) = {3}

$\inf(A) = -3$, $\sup(A) = 3$

a) $A = (-1,1) \cup (2,+\infty)$



$lb(A) = (-\infty, -1]$

$ub(A)$ doesn't exist, A is unbounded above

$\nexists \min(A)$, $\nexists \max(A)$

$\inf(A) = -1$, $\sup(A) = +\infty$

c) $A = (-5,5) \cap \mathbb{Z} = \{-4,-3,-2,-1,0,1,2,3,4\}$

$lb(A) = [-\infty, -4]$

$ub(A) = [4, +\infty)$

$\min(A) = -4$, $\max(A) = 4$,

$\inf(A) = -4$, $\sup(A) = 4$

Reminder: $A = \{a_1, a_2, \dots, a_n\}$, $n \in \mathbb{N}$

$\inf(A) = \min(A)$, $\sup(A) = \max(A)$

d) $A = \emptyset$

$lb(A) = \mathbb{R}$, $ub(A) = \mathbb{R}$

$\nexists \min(A)$, $\nexists \max(A)$

$\inf(A) = +\infty$, $\sup(A) = -\infty$

2) a) $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$

$(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

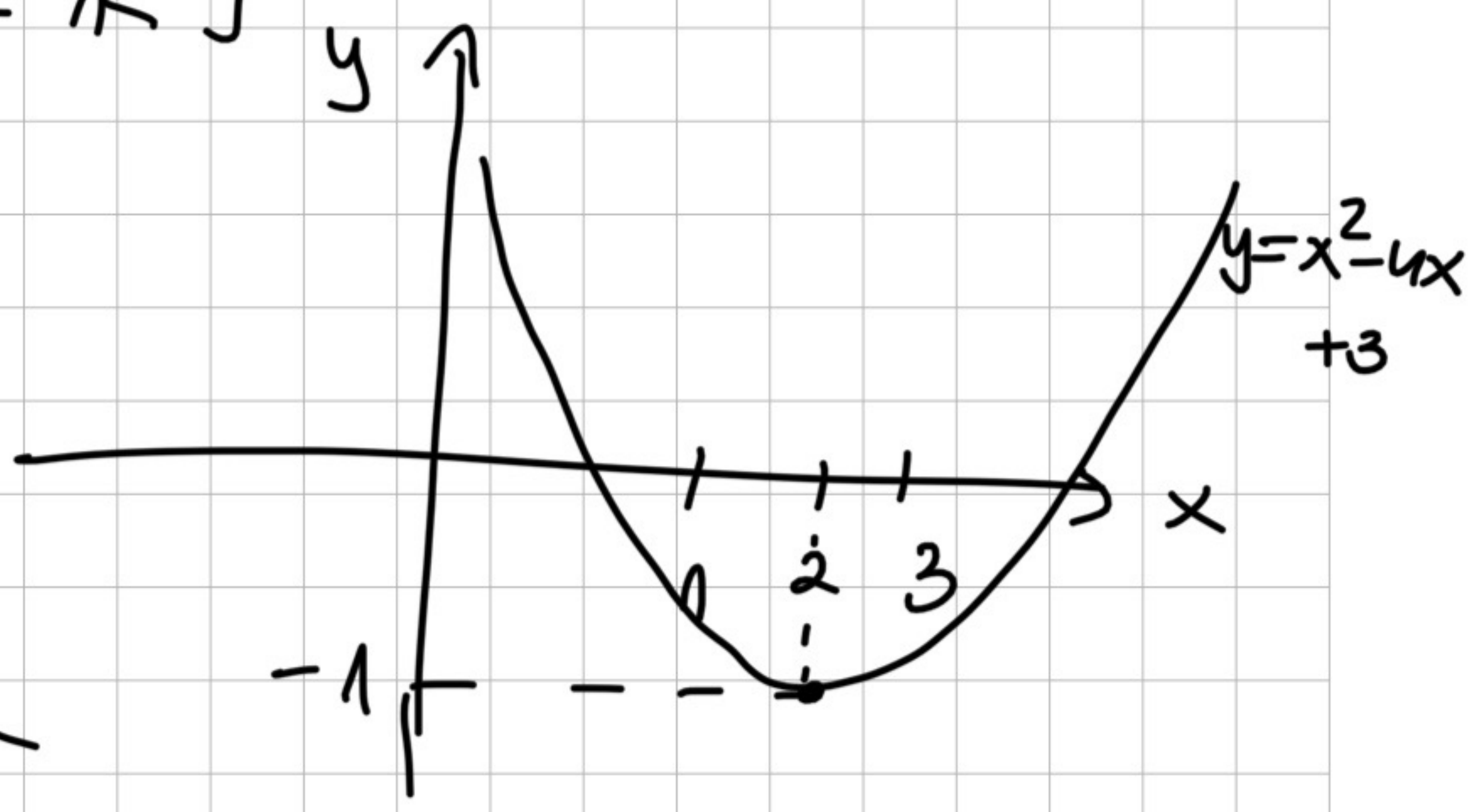
$lb(A) = (-\infty, -\sqrt{2}]$, $ub(A) = [\sqrt{2}, +\infty)$

$\nexists \min(A)$, $\nexists \max(A)$

$\inf(A) = -\sqrt{2}$, $\sup(A) = \sqrt{2}$

b) $A = \{ x^2 - 4x + 3 \mid x \in \mathbb{R} \}$

1, 3



$lb(A) = (-\infty, -1]$

A is unbounded

Completing the square

$$x^2 - 4x + 4 - 1 = \underbrace{(x-2)^2}_{\geq 0} - 1 \geq -1$$

$\min(A) = -1$, $\nexists \max(A)$

$\inf(A) = -1$, $\sup(A) = +\infty$

c) $A = \{ \frac{n}{n+1} \mid n \in \mathbb{N} \}$ $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \to 1 \right)$

$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} < 1$$

$lb(A) = (-\infty, \frac{1}{2}]$

$ub(A) = [1, +\infty)$, $\min(A) = \frac{1}{2}$, $\nexists \max(A)$

$\inf(A) = \frac{1}{2}$, $\sup(A) = 1$

d) $A = \{ 2^{-k} + 3^{-n} \mid k, n \in \mathbb{N} \}$

$\frac{5}{\ } \geq \frac{1}{\ ^{k}} + \frac{1}{\ } > 0$

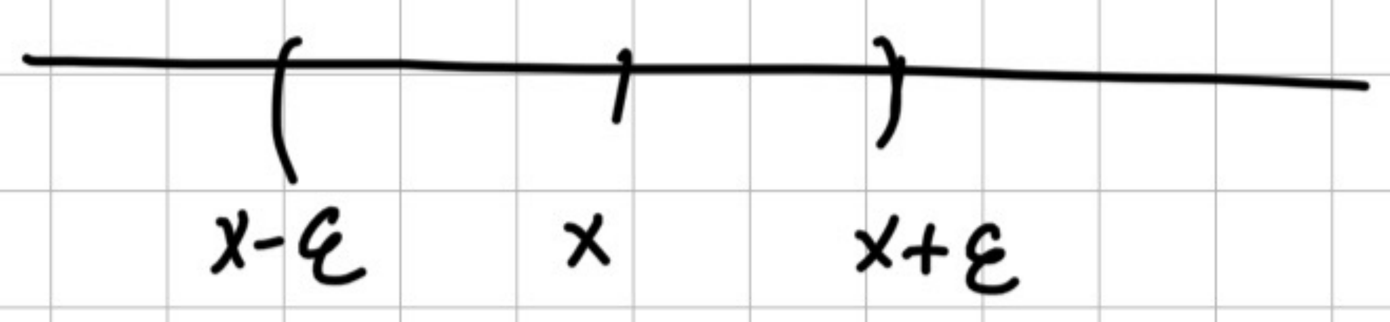
6 2 3^{m}

$lb(A) = (-\infty, 0]$, $ub(A) = [\frac{5}{6}, +\infty)$

$\nexists$ $\min(A)$, $\max(A) = \frac{5}{6} = \sup(A)$

$\inf(A) = 0$

Let $x \in \mathbb{R}$. Let $V \subseteq \mathbb{R}$ is a neighborhood (vecinity) of x. If $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq V$



$V \in \underbrace{\mathcal{V}(x)}$ → contains all the neighborhood of x

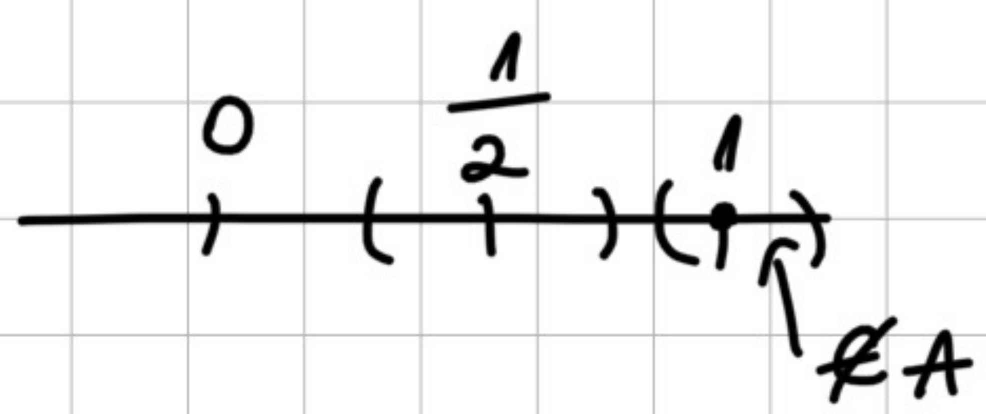
Ex: $x = 0$, $(-1,1) \in \mathcal{V}(0)$

$\{0\} \notin \mathcal{V}(0)$

$(-1,5) \in \mathcal{V}(0)$ (it contains $(-1,1)$ which is a neighborhood of x)

$\operatorname{int}(A) := \{x \in \mathbb{R} \mid \exists\, V \in \mathcal{V}(x), V \subseteq A\}$
↳ interior of A

Ex: $A = [0,1]$



$\operatorname{int}(A) = (0,1)$

$\boxed{\text{If } A = \operatorname{int}(A) \text{, then } A \text{ is open}}$

$\operatorname{cl}(A) := \{x \in \mathbb{R} \mid (\forall)\, V \in \mathcal{V}(x), V \cap A \neq \emptyset\}$
↳ closure of A

Ex: $A = (0,1)$

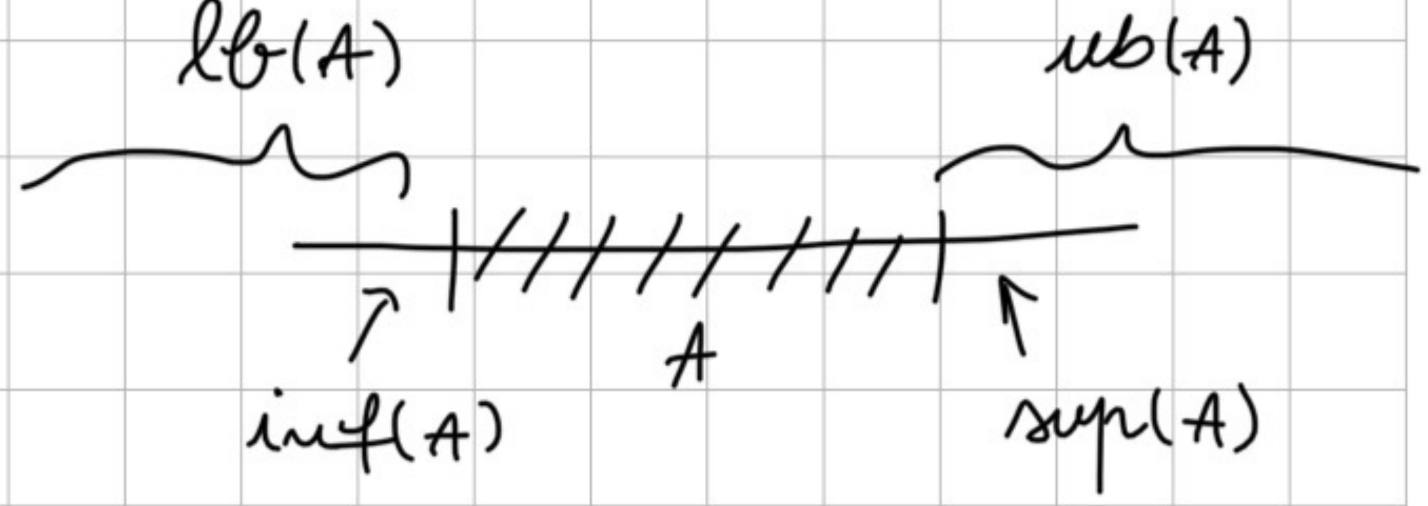
$cl(A) = (0,1) \cup \{0,1\} = [0,1]$

$\boxed{\text{If } cl(A) = A, \text{ then } A \text{ is closed}}$

7.) a) $A = [-1,1] \cup \{2\} \in \mathcal{V}(0)$

$\hookrightarrow (-\frac{1}{2}, \frac{1}{2}) \subseteq A$

Curs 2



- Let $A \subseteq \mathbb{R}$ - bounded

$(\forall)\ \varepsilon > 0, \exists\ x \in A : \sup(A) - \varepsilon < x$

$(\forall)\ \varepsilon > 0, \exists\ x \in A : \inf(A) + \varepsilon > x$

Neighborhood:

Let $x \in \mathbb{R}$, let $V \subseteq \mathbb{R}$. V is a neighborhood of x if $\exists\ \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq V$

$\hookrightarrow V \in \mathcal{V}(x)$ *

$\rightarrow$ ex: $(-1,2) \in \mathcal{V}(1)$

* $\mathcal{V}(x)$ - all the neighborhood of x

V is a neighborhood of x : $V \in \mathcal{V}(x)$

Def: Let $A \subseteq \mathbb{R}$. $int(A) := \{x \in \mathbb{R} \mid \exists\ V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

ex: $int(0,1] = (0,1)$

!! $int(A) \subseteq A$!!

Def: $cl(A) := \{x \in \mathbb{R} \mid (\forall)\ V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

ex: $cl[0,1] = [0,1]$

!! $A \subseteq cl(A)$!!

Ex: $A = \{a_1, a_2, \dots a_n\}$

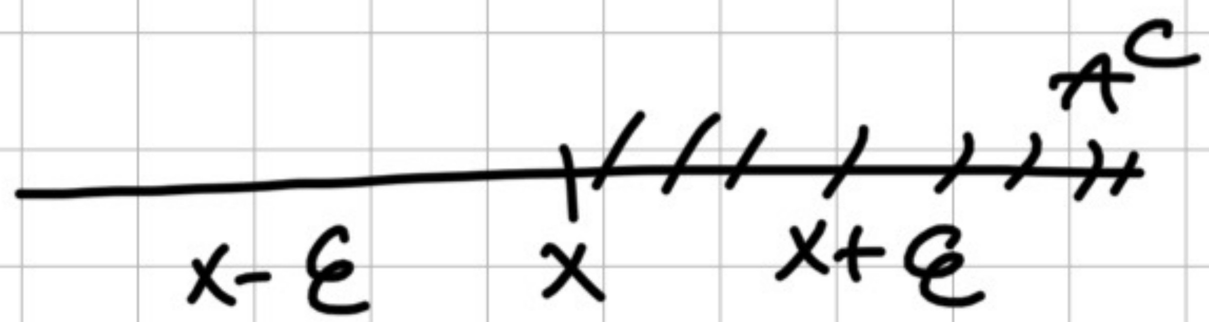
$cl(A) = A$, $int(A) = \emptyset$

Prop: Let $A \subseteq \mathbb{R}$, let $A^c = \mathbb{R} \setminus A$ (the complement)

- if A is open, then A^c is closed.
- if A is closed, A^c is open.

1) Let $x \in A^c$. I want $cl(A^c) \subseteq A^c \Leftarrow A^c \subseteq cl(A^c)$

$$\left.\begin{array}{l} x \leq y \\ y \leq x \end{array}\right\} \Rightarrow x = y$$



Let $x \in cl(A^c)$

$(\forall)\ V \in \mathcal{V}(x): V \cap A^c \neq \emptyset$ (*)

Could $x \in A$? NO $\Rightarrow x \in A^c$

(1) $\Rightarrow$ Assume that $x \in A$ $\left.\begin{array}{l} \\ A \text{ is open} \end{array}\right\} \Rightarrow \exists V \in \mathcal{V}(x), V \subseteq A,$

$V \cap A^c = \emptyset$ contradiction
$\downarrow$
(*)

Sequences:

$(x_n)_{n\in\mathbb{N}}$ or (x_n) is a sequence $S = \{x_n / n \in \mathbb{N}\}$

A sequence is a function $f: \mathbb{N} \to \mathbb{R}$

- (x_n) is bounded if S is bounded
  below: $\exists a \in \mathbb{R}$ s.t. $a \leq x_n, (\forall) n \in \mathbb{N}$
- (x_n) increasing↑ if $x_n \leq x_{n+1}$, $(\forall) n \in \mathbb{N}$

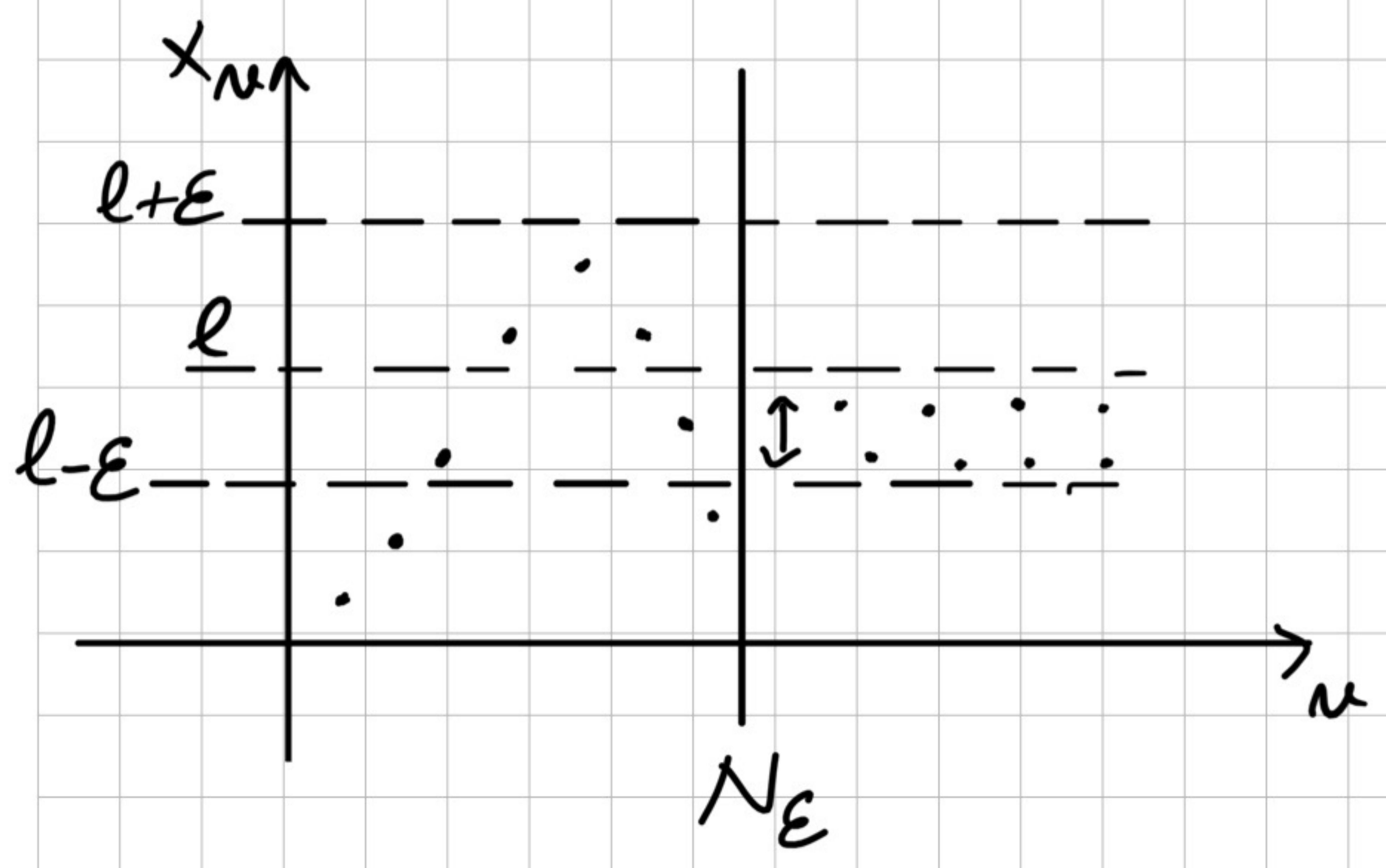
· (x_n) is decreasing if $x_{n-1} \le x_n$, $(\forall)\, n \in \mathbb{N}$

Def: (x_n) has limit $l \in \overline{\mathbb{R}}$, $\lim\limits_{n\to\infty} x_n = l$ if $(\forall)\, V \in \mathcal{V}(l)$, $(\exists)\, N_V \in \mathbb{N}$ such that $x_n \in V$, $(\forall)\, n \ge N_V$

$\boxed{\cdot\ l \in \mathbb{R},\ (x_n) \text{ is convergent}}$

$(\forall)\, \varepsilon > 0, (\exists) N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon$, $(\forall)\, n \geq N_\varepsilon$

$(x_n - l)$ converges to 0



· $l = +\infty$: $\lim\limits_{n\to\infty} x_n = +\infty$ if $(\forall)\, a \in \mathbb{R}$ $\exists\, N_a \in \mathbb{N}$ such that $x_n \ge a_1$, $(\forall)\, n \in N_a$

· $(a_1, +\infty) \in \mathcal{V}(+\infty)$

Prop: (x_n) convergent $\Rightarrow$ (x_n) bounded

ex: $x_n = (-1)^n$ bounded, but NOT convergent

$-1, 1, -1, 1 \ldots$

Prop: (x_n) bounded, monotone $\Rightarrow$ (x_n) convergent

→ **Squeeze Theorem** → T. Cleștelui

$$x_n \le y_n \le z_n,\ (\forall)\, n \in \mathbb{N}$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$l$$

If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l \in \overline{\mathbb{R}} \Rightarrow y_n \to l$

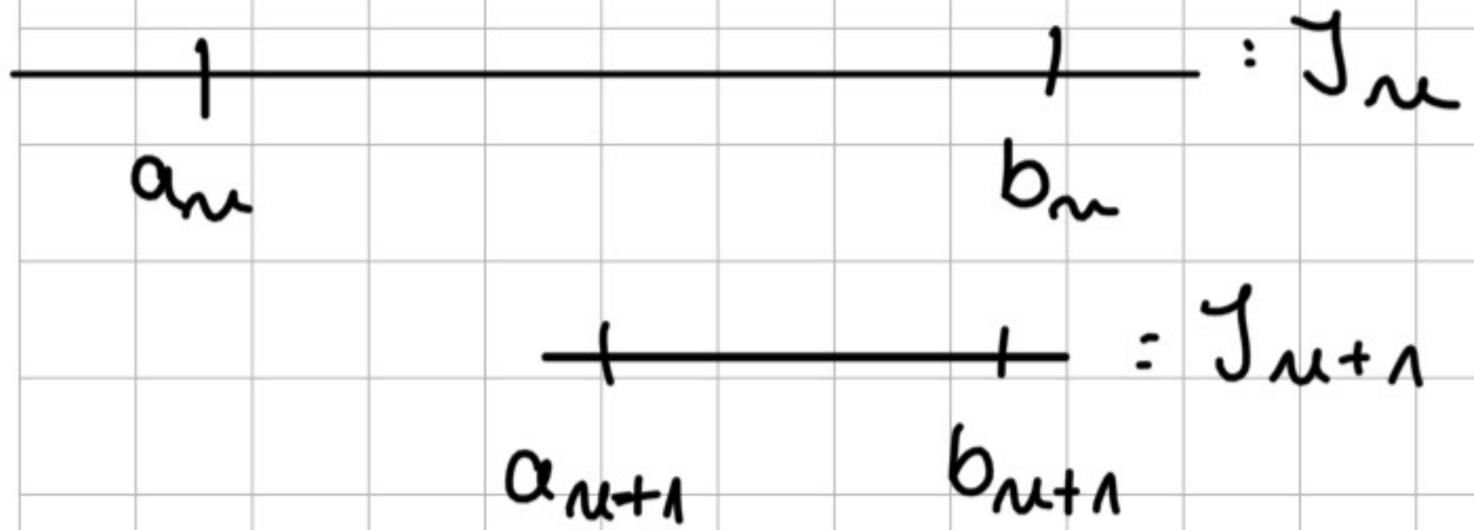
Hw: proof

→ Cantor Nested Interval Theorem

Let (a_n) increasing

(b_n) decreasing $\qquad J_n = [a_n, b_n]$

s.t. $J_{n+1} \subseteq J_n \qquad a_n \le a_{n+1} \le b_{n+1} \le b_n$



If $\lim_{n\to\infty} (b_n - a_n) = 0$ then

$\exists x \in \mathbb{R} \quad \bigcap_{n=1}^{\infty} J_n = \{x\}$

Proof: Let $k \in \mathbb{N}$, $a_k \le a_n \le b_k$, $(\forall)\, n \ge k$

$a_k \le \underbrace{\sup\{a_n / n \ge k\}}_{x} \le b_k \Rightarrow \overbrace{\sup\{a_n\}}^{x} \in J_k, (\forall) k \in \mathbb{N}$

$\Downarrow$

$x = \inf\{b_n\}$

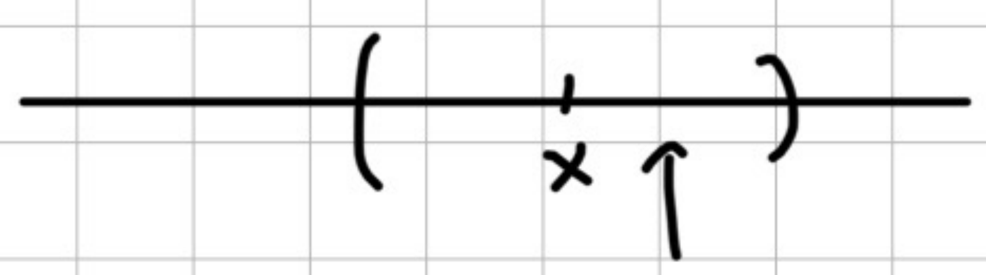
Def: $LIM(x_n) := \{x \in \mathbb{R} / (\forall)\, V \in \mathcal{V}(x)\}$ contains infinitely many terms from (x_n)

$\downarrow$

$\boxed{\text{limit points}}$

Def: $\liminf_{n\to\infty} x_n := \inf(LIM(x_n))$

$\limsup_{n\to\infty} x_n := \sup(LIM(x_n))$



∞ many x_n $(\forall)\, V \in \mathcal{V}$

$\exists x_n \in V, n \in \mathbb{N}$

ex: $x_n = (-1)^n$, $LIM(x_n) = \{-1, 1\}$

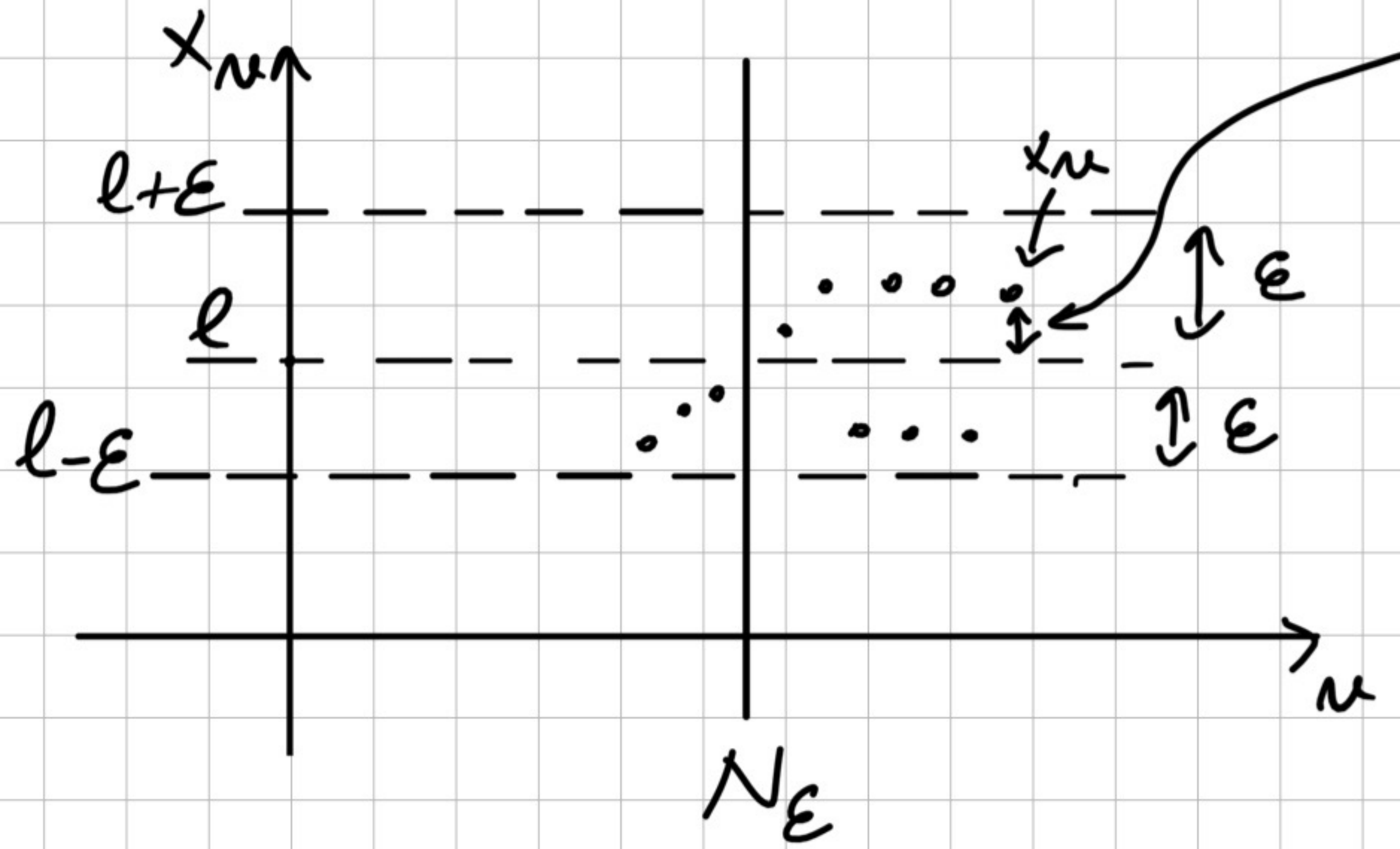
$\liminf = -1$

lim sup = 1

Seminar 2

ε definition:

$\lim_{n\to\infty} x_n = l$: $(\forall)\, \varepsilon > 0$ $\exists$ $N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon$, $(\forall)\, n \geq N_\varepsilon$



1. Proove $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Let $\varepsilon > 0$. I want to find $N_\varepsilon \in \mathbb{N}$

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon, \ (\forall)\, n \geq N_\varepsilon$$

$$\Updownarrow$$

$$\frac{1}{n} < \varepsilon^2 \Leftrightarrow n > \frac{1}{\varepsilon^2}$$

Take $N_\varepsilon > \frac{1}{\varepsilon^2}$, $N_\varepsilon = \left[\frac{1}{\varepsilon^2}\right] + 1$ - rule

For $n \geq N_\varepsilon$, we have that $n > \frac{1}{\varepsilon^2}$

Verification

$$\left|\frac{1}{\sqrt{n}}\right| \underset{n \geq N_\varepsilon}{\leq} \frac{1}{\sqrt{N_\varepsilon}} = \frac{1}{\sqrt{\left[\frac{1}{\varepsilon^2}\right]+1}} < \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon$$

$$\left[\frac{1}{\varepsilon^2}\right] + 1 > \frac{1}{\varepsilon^2} \text{ (logic)}$$

$\Rightarrow \left|\frac{1}{\sqrt{n}}\right| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

* Proove: $\lim_{n\to\infty} \frac{n^2}{n+1} = +\infty \Rightarrow x_n > \varepsilon$

Let $x_n = \frac{n^2}{n+1}$ and $\varepsilon > 0$. I want to find $N_\varepsilon \in \mathbb{N}$

s.t. $\frac{n^2}{n+1} > \varepsilon, (\forall)\, n \geq N_\varepsilon$

$\frac{n^2}{n+1} > \varepsilon \,/\cdot(n+1) \Leftrightarrow n^2 > \varepsilon(n+1) \Leftrightarrow n^2 > \varepsilon n + \varepsilon \Leftrightarrow$

$\Leftrightarrow n^2 - \varepsilon n - \varepsilon > 0, \; \Delta = \varepsilon^2 + 4\varepsilon \Rightarrow n_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2+4\varepsilon}}{2}$

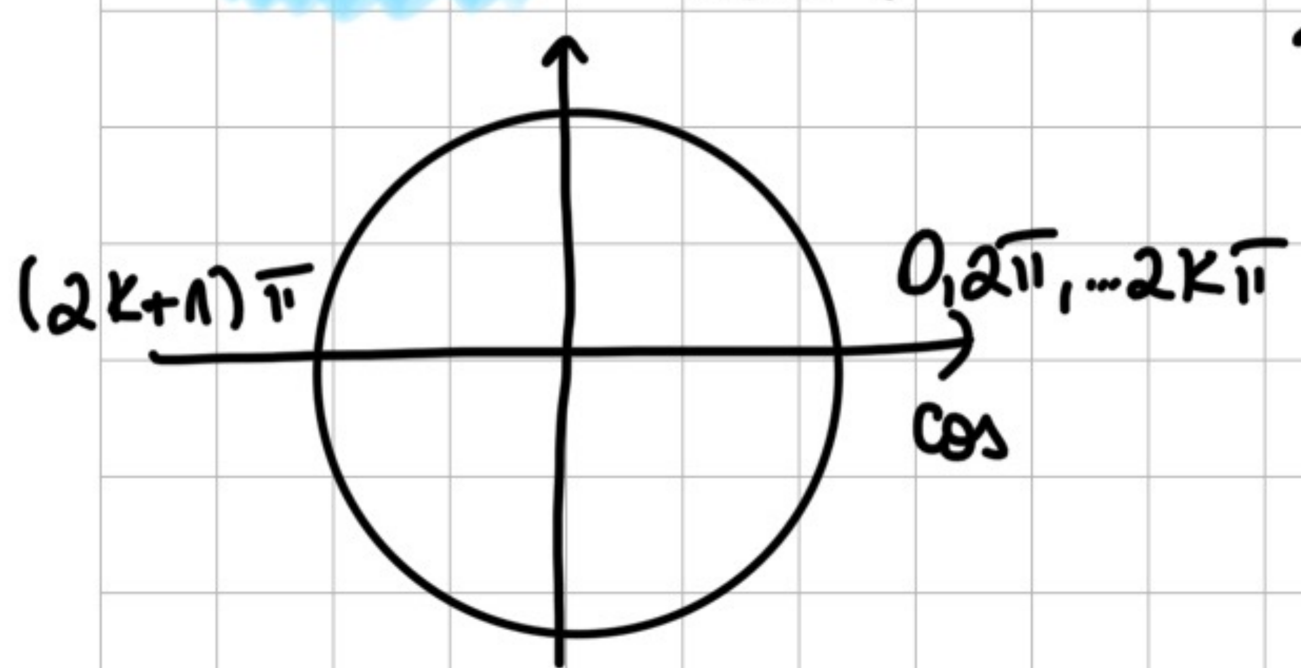
n		n_1		n_2	
$f(n)$	+	0	−	0	+

$N_\varepsilon = \left[\frac{\varepsilon + \sqrt{\varepsilon^2+4\varepsilon}}{2}\right] + 1$

Verification

$\frac{n^2}{n+1} > \frac{N_\varepsilon^2}{N_\varepsilon+1} > \varepsilon \Rightarrow \lim_{n\to\infty} \frac{n^2}{n+1} = +\infty$

2. a) $\cos(n\pi) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$



(x_{2k}, x_{2k+1})

$\text{LIM}(x_n)$ = limits of subsequences that have a limit

$\text{LIM}(x_n) = \{-1, 1\}$

* $y_n = \{ n^2$, even

$\frac{1}{\sqrt{n}}$, n odd

$$\operatorname{LIM}(y_n) = \{0, +\infty\}$$

$$\boxed{\begin{aligned}\liminf_{n\to\infty}(x_n) &:= \inf(\operatorname{LIM}(x_n)) = -1 \\ \limsup_{n\to\infty}(x_n) &:= \sup(\operatorname{LIM}(x_n)) = 1\end{aligned}}$$

!

3. a) $x_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \in (0,1)$ - decreasing $\not\Rightarrow$

$\Downarrow$ $x_n > 0$

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} < 1$, increasing

$$\frac{1}{K(K+1)} = \frac{K+1-K}{K(K+1)} = \frac{1}{K} - \frac{1}{K+1}$$

c) $x_n = \frac{2^n}{n!} > 0$

$$\frac{x_{n+1}}{x_n} = \frac{2^{\cancel{n+1}}}{(\cancel{n+1})!} \cdot \frac{\cancel{n!}}{\cancel{2^n}} = \frac{2}{n+1} < 1,\ n \geq 2$$

- (x_n) - desc $\Rightarrow x_{n+1} < x_n$
- $ub\,(x_n) = x_1 = \frac{2}{1} = 2$
- $lb\,(x_n) = 0$ (termeni pozitivi)

$\Big\}\Rightarrow (x_n)$ - convergent

$$\frac{x_{n+1}}{x_n} = \frac{2}{n+1} \,/\cdot x_n \Rightarrow \underset{\substack{\downarrow \\ l}}{x_{n+1}} = \underset{\substack{\downarrow \\ l}}{x_n} \cdot \underset{\substack{\downarrow \\ 0}}{\frac{2}{n+1}} \Rightarrow l = l\cdot 0 \Rightarrow l = 0$$

4. a) $\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{} =$

$\sqrt{n}+\sqrt{n+1}$

$$= \frac{\sqrt{n}(\cancel{n}+1-\cancel{n})}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(1+\sqrt{\frac{n+1}{n}}\right)} \to \frac{1}{2}$$

b) $(a_1^n + a_2^n + \ldots + a_k^n)^{\frac{1}{n}}, \quad a_i > 0$

$$(2^n + 3^n)^{\frac{1}{n}} = \left\{ 3^n \left[\underbrace{\left(\frac{2}{3}\right)^n}_{\downarrow 0} + 1 \right] \right\}^{\frac{1}{n}} \to l = 3$$

$M = \max a_i, \ i = \overline{1,k}$

5. $e_n = \left(1 + \frac{1}{n}\right)^n$

Binomial formula (Newton):

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \ldots + \binom{n}{k}x^k + \ldots + x^n$$

$$\binom{n}{k} = C_n^k \to \text{„n choose k"} = \frac{n!}{k!(n-k)!} =$$

$$= \frac{\overset{n-1+1}{n}\cdot(\overset{n-2+1}{n-1})\cdot \ldots \cdot (n-k+1)}{k!} \quad k^{th} \text{ number}$$

• for $x = \frac{1}{n}$: $= 1 + 1 + \binom{n}{2}\frac{1}{n^2} + \ldots + \underbrace{\binom{n}{k}\frac{1}{n^k}}_{T_k} + \ldots + \frac{1}{n^n}$

$$T_k = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!\cdot n^k} = \frac{\cancel{n^k}\left[1\cdot\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k+1}{n}\right)\right]}{k!\cdot\cancel{n^k}} < \frac{1}{k!} \Rightarrow \frac{C_n^k}{n^k} < \frac{1}{k!}$$

$$e_n < \underbrace{\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}}$$

prin ind. matem se arată că $k! > 2^{k-1} \Rightarrow \frac{1}{k!} < \frac{1}{2^{k-1}}$ $\Rightarrow$

$$e_n < 1 + \underbrace{1 + \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^{k-1}} + \ldots + \frac{1}{2^{n-1}}}_{S_n}$$

$$1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}, \text{ for } q = \frac{1}{2} \Rightarrow$$

$$\Rightarrow S_n = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\cdot\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$$

$$e_n < 1 + S_n < 3 - \frac{1}{2^{n-1}} < 3 \Rightarrow e_n \in [2,3)$$

$$e_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = 1+1+\binom{n+1}{2}\frac{1}{(n+1)^2} + \ldots + \overbrace{\binom{n+1}{k}\frac{1}{(n+1)^k}} + \ldots + (n+1)\frac{1}{(n+1)^n} + \frac{1}{(n+1)^{n+1}}$$

$$R_k = \frac{1\cdot\left(1-\frac{1}{n+1}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n+1}\right)}{k!} > T_k \Rightarrow e_{n+1} > e_n$$

$$\Rightarrow \lim_{n\to\infty} x_n = e \in (2,3)$$