Final Exam #1

1. Study the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$, with $k > 1$. **1p**

2. (a) Draw the interior and the boundary of the set $\{(x, y) \in \mathbb{R}^2 \mid |x| < |y| < 1\}$. **1p**

   (b) Let $x, y \in \mathbb{R}^n$ be orthogonal vectors. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. **1p**

3. Find the second order Taylor polynomial for $f(x, y) = \sqrt{x^2 + y^2}$ around $(1, 1)$. **1p**

4. Find and classify all the critical points of $f(x, y) = x^3 - 3x + y^2$. **1p**

5. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let $b \in \mathbb{R}^n$. Consider the function

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \frac{1}{2}x^T Ax - b^T x.$$

   (a) Prove that f has a unique minimum, which satisfies the equation $Ax = b$. **1p**

   (b) Write a gradient descent method for finding the minimum of f. **1p**

6. Let the probabilities $p_1, p_2, p_3 \in (0, 1)$ with $p_1 + p_2 + p_3 = 1$. Consider the function

$$f : \mathbb{R}^3 \to \mathbb{R},\ f(p_1, p_2, p_3) = -\sum_{i=1}^{3} p_i \log_2(p_i),$$

   known as information entropy (a measure of uncertainty for the probability distribution).

   (a) Using Lagrange multipliers, find p_1, p_2, p_3 that maximize the entropy function f. **0.75p**

   (b) Generalize to n probabilities $p_1, \ldots, p_n$. **0.25p**

7. Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a, b > 0$.

   (a) Find the equation of the tangent line to the ellipse at a point (x_0, y_0). **1p**

   (b) Find the area enclosed by the ellipse, for example by using a double integral. **1p**

Time: 2h. The marks in the final exam add up to **10p**.

Midterm Test Retake

1. Find inf, sup, min, max, the interior and the closure of the set $\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots\}$.
   1.5p (6×0.25p)

2. Study the convergence of the following series:

   (a) $\displaystyle\sum_{n\geq 1} \frac{a^n n!}{n^n}$, with $a > e$. **1p**

   (b) $\displaystyle\sum_{n\geq 1} \frac{1}{n\sqrt[n]{n}}$. **1p**

   (c) $\displaystyle\sum_{n\geq 0} \frac{a(a+1)\ldots(a+n)}{n!}$, with $a > 0$. **1p**

   (d) $\displaystyle\sum_{n\geq 1} \frac{(\ln n)^k}{n^2}$, with $k > 1$. **1p**

3. Find the sum and the radius of convergence for the following power series:

   (a) $\displaystyle\sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$. **1p**

   (b) $\displaystyle\sum_{n\geq 1} \frac{n}{x^n}$. **1p**

4. Find the Taylor series around zero and its radius of convergence for the following functions:

   (a) $\sinh(x) := \frac{1}{2}(e^x - e^{-x})$. **1.25p**

   (b) $(1+x)^\alpha$, with $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. **1.25p**

Time: 1h. The marks in the midterm test add up to **10p**.