Note: Probability and measure

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Contents

References: STAT5005 and *Probability: Theory and Examples*, 4th edition, by Richard Durrett, published by Cambridge University Press.

1 Measure Theory

1.1 Expectation

Lemma 1.1. *Let $X \geq 0$, $p > 0$, we have $\mathbb{E}X^p = \int_0^\infty px^{p-1}\mathbb{P}(X > x)\mathrm{d}x$.*

2 Law of Large Numbers

2.1 Almost Surely Convergence

This lemma gives an equivalent relation between expectation and sum of tail probability.

Lemma 2.1. *Let X_i iid and $\varepsilon > 0$, then $\sum_{n=1}^\infty \mathbb{P}(|X_n| > n\varepsilon) \leq \varepsilon^{-1}\mathbb{E}|X_i| \leq \sum_{n=0}^\infty \mathbb{P}(|X_n| > n\varepsilon)$.*

3 Central Limit Theorem

4 Random Walks

Random walk (RW): Let $\boldsymbol{X}_i$ be iid rvs in $\mathbb{R}^d$. Let $\boldsymbol{S}_n = \sum_{i=1}^n \boldsymbol{X}_i$. Then $\{\boldsymbol{S}_n : n \geq 1\}$ is called a RW. Take $\boldsymbol{S}_0 = \boldsymbol{0}$.
Simple random walk (SRW): If $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$, then $\{S_n\}$ is called a SRW in $\mathbb{R}^1$. If $\mathbb{P}(\boldsymbol{X}_i = (1,1)) = \mathbb{P}(\boldsymbol{X}_i = (1,-1)) = \mathbb{P}(\boldsymbol{X}_i = (-1,1)) = \mathbb{P}(\boldsymbol{X}_i = (-1,-1)) = 1/4$, then called a SRW in $\mathbb{R}^2$.

Long-term behavior of RW

Permutable (or exchangeable): An event that does not change under finite permutation of $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots\}$.

- All events in the tail σ-field $\mathcal{T}$ are permutable.
- $\{\omega : \boldsymbol{S}_n(\omega) \in B \text{ i.o.}\}$ is permutable but not tail event.
- $\{\omega : \limsup_{n\to\infty} \boldsymbol{S}_n(\omega)/c_n \geq 1\}$.

Theorem 4.1 (Hewitt-Savage 0-1 law)**.** *If $\boldsymbol{X}_i$ iid and event A is permutable, then $\mathbb{P}(A) = 0$ or 1.*

Theorem 4.2 (Long-term behavior of RW A.1)**.** *For a RW in $\mathbb{R}$, one of the following has probability 1:*
(i) $S_n = 0$ *for all* n ;
(ii) $S_n \to \infty$ *as* $n \to \infty$;
(iii) $S_n \to -\infty$ *as* $n \to \infty$;
(iv) $-\infty = \liminf_n S_n < \limsup_n S_n = \infty$.

For two levels $a < b$, find the probability that RW reaches b before a

Filtration: Let X_i be a sequence of rvs, $\{\mathcal{F}_n := \sigma(X_1, \ldots, X_n)\}_{n=1}^{\infty}$ as an increasing sequence of σ-fields, is called a filtration. We usually take $\mathcal{F}_0 = \{\phi, \Omega\}$.
Stopping time/optional random variable/optimal time/Markov time: $\tau \in \mathbb{N}^+ \cup \{\infty\}$ is a stopping time w.r.t. $\{\mathcal{F}_n\}$ if $\{\tau = n\} \in \mathcal{F}_n$, $\forall n \in \mathbb{N}^+$. (Equivalent def: $\{\tau \leq n\} \in \mathcal{F}_n$ or $\{\tau \geq n+1\} \in \mathcal{F}_n$ for $n \in \mathbb{N}^+$)

- If τ_1, τ_2 are stopping time, then $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$, $\tau_1 + \tau_2$ are stopping times.

A Proofs

A.1 Proofs - section 4

Proof of Theorem 4.2. By the 0-1 law 4.1, $\{\limsup_n S_n \geq c\}$ has probability 0 or 1, meaning that $\limsup_n S_n = c \in [-\infty, \infty]$ w.p.1. Since $S_n \overset{\mathrm{d}}{=} S_{n+1} - X_1$, we have $c = c - X_1$.

(i) If $c \in \mathbb{R}$, then $X_1 \equiv 0$ a.s., so $S_n = 0$ for all n a.s.

If $X_1 \neq 0$ a.s., then $c = -\infty$ or ∞,

(ii) If $c = \infty$, and $\liminf_n S_n = \infty$, then case (ii);

(iii) If $c = -\infty$, and $\liminf_n S_n = -\infty$, then case (iii);

(iv) If $c = \infty$, and $\liminf_n S_n = -\infty$, then case (iv).

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References