# Note: Math Tools

Nov 2024

*Lecturer:* *Typed by: Zhuohua Shen*

## Contents



References:

- CUHKSZ: MAT3007 - Optimization I
- CUHKSZ: MAT3220 - Optimization II. Textbook:
- 1. *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*, Amir Beck.
  2. *Convex Optimization*, S. Boyd and L. Vandenberghe.
  3. *Nonlinear Programming*, D. Bertsekas.
  4. *First-Order Methods in Optimization*, Amir Beck

# 1 Nonlinear Optimization

## 1.1 KKT Conditions

**Theorem 1.1** (The Fritz-John necessary conditions)**.** *Let $\boldsymbol{x}^*$ be a local minimum of the problem*

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

*where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. Then $\exists$ multipliers $\lambda_0, \dots, \lambda_m \geq 0$, which are not all zeros, such that*

$$\lambda_0 \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\boldsymbol{x}^*) = \mathbf{0}$$
$$\lambda_i g_i(\boldsymbol{x}^*) = 0, \quad i = 1, 2, \dots, m.$$

A major drawback of the Fritz-John conditions is, they allow $\lambda_0 = 0$. Under an additional regularity condition, we can assume $\lambda_0 = 1$. Let $I(\boldsymbol{x}^*)$ be the set of active constraints at $\boldsymbol{x}^*$:

$$I(\boldsymbol{x}^*) = \{i : g_i(\boldsymbol{x}^*) = 0\}.$$

**Theorem 1.2** (The KKT conditions for inequality constrained problems)**.** *Let $\boldsymbol{x}^*$ be a local minimum of*

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

*where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\boldsymbol{x}^*)\}_{i \in I(\boldsymbol{x}^*)}$ are linearly independent. Then $\exists \lambda_1, \dots, \lambda_m \geq 0$ such that*

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\boldsymbol{x}^*) = \mathbf{0}$$
$$\lambda_i g_i(\boldsymbol{x}^*) = 0, \quad i = 1, 2, \dots, m.$$

**Theorem 1.3** (The KKT conditions for inequality/equality constrained problems)**.** *Let $\boldsymbol{x}^*$ be a local minimum of*

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_j(\boldsymbol{x}) = 0, \quad j = 1, \dots, p \end{aligned} \tag{1}$$

*where $f, g_1, \dots, g_m, h_1, \dots, h_p \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\boldsymbol{x}^*), \nabla h_j(\boldsymbol{x}^*), i \in I(\boldsymbol{x}^*), j = 1, \dots, p\}$ are linearly independent.*

*Then* $\exists \lambda_1, \dots, \lambda_m \geq 0$, $\mu_1, \dots, \mu_p \in \mathbb{R}$, *such that*

$$\begin{aligned} \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\boldsymbol{x}^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(\boldsymbol{x}^*) = \mathbf{0}, \\ \lambda_i g_i(\boldsymbol{x}^*) = 0, \quad i = 1, 2, \dots, m. \end{aligned} \tag{2}$$

Consider problem (1), a feasible point $\boldsymbol{x}^*$ is called a KKT point if $\exists \lambda_1, \dots, \lambda_m \geq 0$, $\mu_1, \dots, \mu_p \in \mathbb{R}$, such that (2) holds. $\boldsymbol{x}^*$ is called regular if $\{\nabla g_i(\boldsymbol{x}^*), \nabla h_j(\boldsymbol{x}^*), i \in I(\boldsymbol{x}^*), j = 1, \dots, p\}$ are linearly independent.

- The additional requirement of regularity is not required in linearly constrained problems in which no such assumption is needed.