

Note: Probability and measure

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Lecturer:

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References: STAT5005 and *Probability: Theory and Examples*, 4th edition, by Richard Durrett, published by Cambridge University Press.

1 Measure Theory

1.1 Expectation

Lemma 1.1. Let $X \geq 0$, $p > 0$, we have $\mathbb{E}X^p = \int_0^\infty px^{p-1}\mathbb{P}(X > x)dx$.

2 Law of Large Numbers

2.1 Almost Surely Convergence

This lemma gives an equivalent relation between expectation and sum of tail probability.

Lemma 2.1. Let X_i iid and $\varepsilon > 0$, then $\sum_{n=1}^\infty \mathbb{P}(|X_n| > n\varepsilon) \leq \varepsilon^{-1}\mathbb{E}|X_1| \leq \sum_{n=0}^\infty \mathbb{P}(|X_n| > n\varepsilon)$.

3 Central Limit Theorem

4 Random Walks

Random walk (RW): Let \mathbf{X}_i be iid rvs in \mathbb{R}^d . Let $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_i$. Then $\{\mathbf{S}_n : n \geq 1\}$ is called a RW. Take $\mathbf{S}_0 = \mathbf{0}$.

Simple random walk (SRW): If $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$, then $\{S_n\}$ is called a SRW in \mathbb{R}^1 . If $\mathbb{P}(\mathbf{X}_i = (1, 1)) = \mathbb{P}(\mathbf{X}_i = (1, -1)) = \mathbb{P}(\mathbf{X}_i = (-1, 1)) = \mathbb{P}(\mathbf{X}_i = (-1, -1)) = 1/4$, then called a SRW in \mathbb{R}^2 .

4.1 Stopping Times (A.1.1)

Long-term behavior of RW

Permutable (or exchangeable): An event that does not change under finite permutation of $\{X_1, X_2, \dots\}$.

- All events in the tail σ -field \mathcal{T} are permutable.
- $\{\omega : S_n(\omega) \in B \text{ i.o.}\}$ is permutable but not tail event.
- $\{\omega : \limsup_{n \rightarrow \infty} S_n(\omega)/c_n \geq 1\}$.

Theorem 4.1 (Hewitt-Savage 0-1 law). *If X_i iid and event A is permutable, then $\mathbb{P}(A) = 0$ or 1 .*

Theorem 4.2 (Long-term behavior of RW). *For a RW in \mathbb{R} , one of the following has probability 1:*

- (i) $S_n = 0$ for all n ;
- (ii) $S_n \rightarrow \infty$ as $n \rightarrow \infty$;
- (iii) $S_n \rightarrow -\infty$ as $n \rightarrow \infty$;
- (iv) $-\infty = \liminf_n S_n < \limsup_n S_n = \infty$.

For two levels $a < b$, find the probability that RW reaches b before a

Filtration: Let X_i be a sequence of rvs, $\{\mathcal{F}_n := \sigma(X_1, \dots, X_n)\}_{n=1}^\infty$ as an increasing sequence of σ -fields, is called a filtration. We usually take $\mathcal{F}_0 = \{\phi, \Omega\}$.

Stopping time/optional random variable/optimal time/Markov time: $\tau \in \mathbb{N}^+ \cup \{\infty\}$ is a stopping time w.r.t. $\{\mathcal{F}_n\}$ if $\{\tau = n\} \in \mathcal{F}_n, \forall n \in \mathbb{N}^+$. (Equivalent def: $\{\tau \leq n\} \in \mathcal{F}_n$ or $\{\tau \geq n+1\} \in \mathcal{F}_n$ for $n \in \mathbb{N}^+$)

- Constant $\tau = n$ is a stopping time.
- If τ_1, τ_2 are stopping time, then $\tau_1 \wedge \tau_2, \tau_1 \vee \tau_2, \tau_1 + \tau_2$ are stopping times.
- **Hitting time of A :** let A measurable, then $\tau = \inf\{n \geq 1 : S_n \in A\}$ is a stopping time.
- σ -field \mathcal{F}_N = the information known at time N . Def: \mathcal{F}_N is the collection of sets A that have $A \cup \{N = n\} \in \mathcal{F}_n, \forall n < \infty$. Example: $\{N \leq n\} \in \mathcal{F}_N$, i.e., N is \mathcal{F}_N -measurable.

Theorem 4.3 (Wald's equation). *Let X_i iid and τ be a stopping time.*

1. (Wald's first equation) *If $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}\tau < \infty$, then $\mathbb{E}S_\tau = \mathbb{E}X_1\mathbb{E}\tau$.*
2. (Wald's second equation) *If $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = \sigma^2 < \infty, \mathbb{E}\tau < \infty$, then $\mathbb{E}S_\tau^2 = \sigma^2\mathbb{E}\tau$.*

Example 4.4 (Results for 1-d SRW). *For 1-d SRW, let $a, b \in \mathbb{Z}, a < 0 < b$. Let $N = \inf\{n : S_n \notin (a, b)\} = \inf\{n : S_n = a \text{ or } b\}$. Then*

1. $\mathbb{E}N < \infty$,
2. $S_N = a \text{ or } b$,
3. $\mathbb{P}(S_N = a) = b/(b-a), \mathbb{P}(S_N = b) = -a/(b-a)$,
4. $\mathbb{E}N = \mathbb{E}S_N^2 = (-a)b$.

4.2 Recurrence vs. Transience (A.1.2)

When RW return to 0? We consider SRW on \mathbb{R}^d and define its first, second, ..., n th returning time to the origin to be

$$\begin{aligned}\tau_1 &= \inf\{m \geq 1 : S_m = \mathbf{0}\}, \\ \tau_n &= \inf\{m > \tau_{n-1} : S_m = \mathbf{0}\}.\end{aligned}$$

Theorem 4.5. *For any RW, the following are equivalent:*

- (i) $\mathbb{P}(\tau_1 < \infty) = 1$
- (ii) $\mathbb{P}(\tau_n < \infty) = 1, \forall n = 1, 2, 3, \dots$
- (iii) $\mathbb{P}(S_m = \mathbf{0} \text{ i.o.}) = 1$
- (iv) $\sum_{m=1}^\infty \mathbb{P}(S_m = \mathbf{0}) = \infty$.

- $\mathbb{P}(\tau_n < \infty) = (\mathbb{P}(\tau_1 < \infty))^n$.

Recurrent: If $\mathbb{P}(\tau_1 < \infty) = 1$, then the RW is called recurrent.

Transient: If $\mathbb{P}(\tau_1 < \infty) < 1$, then the RW is called Transient.

Theorem 4.6 (Recurrence of SRW). *SRW is recurrent in $d \leq 2$ and transient in $d \geq 3$.*

- We define the first time a random walk starting from a reaches b : $\tau_{a \rightarrow b} := \inf\{m \geq 1 : a + S_m = b\}$. It can be proved that $\mathbb{P}(\tau_1 < \infty) = 1$ iff $\mathbb{P}(\tau_{a \rightarrow b} < \infty) = 1, \forall a, b$.

4.3 Reflection Principle and Arcsine Distribution (A.1.3)

What is the distribution of the time spent above 0? We consider the SRW, $d = 1$, and think of the sequence S_1, \dots, S_n as being represented by a polygonal line with segments $(k-1, S_{k-1}) \rightarrow (k, S_k)$.

Theorem 4.7 (Reflection Principle).

- (Reflection principle for numbers) If $x, y > 0$, then the number of paths from $(0, x)$ to (n, y) that are 0 at some time is equal to the number of paths from $(0, -x)$ to (n, y) .
- (Reflection principle for SRW) Let X_i be SRW with $d = 1$. Then $\forall b \in \mathbb{N}^+$,

$$\mathbb{P}(\max_{1 \leq k \leq n} S_k \geq b) = 2\mathbb{P}(S_n > b) + \mathbb{P}(S_n = b).$$

Theorem 4.8 (Hit 0 time). $\mathbb{P}(S_1 > 0, \dots, S_{2n} > 0) = \frac{1}{2}\mathbb{P}(S_{2n} = 0)$, and $\mathbb{P}(S_1 \neq 0, \dots, S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0)$.

Arcsine distribution: a continuous distribution with density $\frac{1}{\pi\sqrt{x(1-x)}}$, $x \in (0, 1)$. Define

$$\begin{aligned} L_{2n} &:= \sup\{m \leq 2n : S_m = 0\}, & (\text{last time at } 0) \\ F_n &:= \inf\{0 \leq m \leq n : S_m = \max_{0 \leq k \leq n} S_k\}, & (\text{first time at maximum}) \\ \pi_{2n} &:= \text{number of } k : 1 \leq k \leq 2n \text{ such that the line } (k-1, S_{k-1}) \rightarrow (k, S_k) \text{ is above the } x\text{-axis.} \end{aligned}$$

Theorem 4.9 (Arcsine law). $\frac{L_{2n}}{2n}, \frac{F_n}{n}, \frac{\pi_{2n}}{2n}$ all converge in distribution to the arcsine distribution.

5 Martingales

5.1 Conditional expectation A.2.1

Definition 5.1 (Conditional expectation). X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}|X| < \infty$. \mathcal{A} is a σ -field and $\mathcal{A} \subset \mathcal{F}$. We define the **conditional expectation** of X given \mathcal{A} , $\mathbb{E}(X|\mathcal{A})$, to be any random variable Y satisfying

- (i) Y is \mathcal{A} -measurable, and
- (ii) $\forall A \in \mathcal{A}, \mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]$.
 - For rvs X and Y , $\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$.
 - For set A , $\mathbb{P}(A|\mathcal{A}) := \mathbb{E}(\mathbf{1}_A|\mathcal{A})$.
 - For set A, B , $\mathbb{P}(A|B) := \mathbb{P}(A \cap B)/\mathbb{P}(B)$ given $\mathbb{P}(B) > 0$.

- If Y satisfies (i) and (ii), then $\mathbb{E}|Y| \leq \mathbb{E}|X| < \infty$.
- $\mathbb{E}Y = \mathbb{E}X$, i.e., $\mathbb{E}[\mathbb{E}(X|\mathcal{A})] = \mathbb{E}X$.
- **Uniqueness:** If Y' also satisfies (i) and (ii), then $Y = Y'$ a.s. Any such Y is said to be a **version** of $\mathbb{E}(X|\mathcal{A})$.
- **Existence:** By Radon-Nikodym theorem.

Example 5.2.

- If X is \mathcal{A} -measurable, then $\mathbb{E}(X|\mathcal{A}) = X$. So a constant $c = \mathbb{E}(c|\mathcal{A})$. (Know X , the “best guess” is X)
- If $\sigma(X) \perp \mathcal{A}$, then $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}X$. (Don’t know anything about X , the best guess is its mean)
- Suppose $\Omega_1, \Omega_2, \dots$ is a finite or infinite partition of Ω into disjoint sets with positive probability, and let $\mathcal{A} = \sigma(\Omega_1, \dots)$, then $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}(X\mathbf{1}_{\Omega_i})/\mathbb{P}(\Omega_i)$ on Ω_i .
 - Let $\mathcal{A} = \{\emptyset, \Omega\}$, then $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}X$.
- (Bayes’s formula) Let $G \in \mathcal{G}$, then $\mathbb{P}(G|A) = \int_G \mathbb{P}(A|\mathcal{G})d\mathbb{P} / \int_\Omega \mathbb{P}(A|\mathcal{G})d\mathbb{P}$. When \mathcal{G} is a σ -field generated by a partition, this reduces to the usual Bayes’ formula $\mathbb{P}(G_i|A) = \mathbb{P}(A|G_i)P(G_i) / \sum_j \mathbb{P}(A|G_j)P(G_j)$.

5.2 Almost surely convergence

Theorem 5.3 (Martingale convergence theorem). Suppose S_n is a submartingale and $\liminf \mathbb{E}S_n^+ < \infty$ (or $\sup \mathbb{E}S_n^+ < \infty$), then $S_n \rightarrow S$ a.s. with $\mathbb{E}|S| < \infty$.

Theorem 5.4. Suppose S_n is a supermartingale. If $S_n \geq 0$, then $S_n \rightarrow S$ a.s. and $\mathbb{E}S \leq \mathbb{E}S_1 < \infty$.

6 Techniques

6.1 Convergence

6.1.1 Convergence of random series

Let X_i be a sequence of rvs. $S_n = \sum_{i=1}^n X_i$. By 0-1 law, $\mathbb{P}(\lim_n S_n \text{ exists}) = 0 \text{ or } 1$.

To show the convergence of random series:

1.

To show the divergence of random series:

1. If SLLN holds, $S_n/n \rightarrow \mu$ a.s., if $\mu > 0$, then $S_n \rightarrow \infty$ a.s.
 2. Use S'_n , $S'_n = S_n$ e.v. a.s., that is, $\mathbb{P}(S_n \neq S'_n \text{ i.o.}) = 0$. And show $S'_n \rightarrow \infty$ a.s.
- Next, we consider $S_n/f(n)$.

A Proofs

A.1 Proofs - 4

A.1.1 Proofs - 4.1

Proof of Theorem 4.2. By the 0-1 law 4.1, $\{\limsup_n S_n \geq c\}$ has probability 0 or 1, meaning that $\limsup_n S_n = c \in [-\infty, \infty]$ w.p.1. Since $S_n \stackrel{d}{=} S_{n+1} - X_1$, we have $c = c - X_1$.

(i) If $c \in \mathbb{R}$, then $X_1 \equiv 0$ a.s., so $S_n = 0$ for all n a.s.

If $X_1 \neq 0$ a.s., then $c = -\infty$ or ∞ ,

(ii) If $c = \infty$, and $\liminf_n S_n = \infty$, then case (ii);

(iii) If $c = -\infty$, and $\liminf_n S_n = -\infty$, then case (iii);

(iv) If $c = \infty$, and $\liminf_n S_n = -\infty$, then case (iv). □

Proof of Theorem 4.3. Prove 1: First suppose $X_i \geq 0$. We have

$$\mathbb{E}S_\tau = \mathbb{E} \sum_{i=1}^{\tau} X_i = \mathbb{E} \sum_{i=1}^{\infty} X_i \mathbf{1}_{\{\tau \geq i\}} = \sum_{i=1}^{\infty} \mathbb{E}X_i \mathbf{1}_{\{\tau \geq i\}} = \sum_{i=1}^{\infty} \mathbb{E}X_i \mathbb{E} \mathbf{1}_{\{\tau \geq i\}} = \mathbb{E}X_1 \mathbb{E}\tau,$$

where the 3rd equality uses Fubini by $X_i \geq 0$, and the 4th uses $\{\tau \geq i\} \in \mathcal{F}_{i-1}$. For general case, since $\sum_{i=1}^{\infty} \mathbb{E}|X_i| \mathbf{1}_{\{\tau \geq i\}} = \sum_{i=1}^{\infty} \mathbb{E}|X_i| \mathbb{E} \mathbf{1}_{\{\tau \geq i\}} < \infty$, we can still use the Fubini.

Prove 2: If $\tau < n$, then $\tau \wedge n = \tau \wedge (n-1)$, so $S_{\tau \wedge n}^2 = S_{\tau \wedge (n-1)}^2$; if $\tau \geq n$, we have $\tau \wedge n = n$ and $\tau \wedge (n-1) = n-1$, so $S_{\tau \wedge n}^2 = S_n^2 = (S_{n-1} + X_n)^2 = (S_{\tau \wedge (n-1)} + X_n)^2$. Hence write

$$S_{\tau \wedge n}^2 = S_{\tau \wedge (n-1)}^2 + (2X_n S_{n-1} + X_n^2) \mathbf{1}_{\{\tau \geq n\}}.$$

Note that all the following expectations exist,

$$\begin{aligned} \mathbb{E}S_{\tau \wedge n}^2 &= \mathbb{E}S_{\tau \wedge (n-1)}^2 + \mathbb{E}(2X_n S_{n-1} \mathbf{1}_{\{\tau \geq n\}}) + \mathbb{E}[X_n^2 \mathbf{1}_{\{\tau \geq n\}}] \\ &= \mathbb{E}S_{\tau \wedge (n-1)}^2 + \sigma^2 \mathbb{P}(\tau \geq n) && \text{(stopping time, independence, and } \mathbb{E}X_i = 0) \\ &= \dots && \text{(reduce to } n-2, n-3, \dots) \\ &= \sigma^2 \sum_{i=1}^n \mathbb{P}(\tau \geq i). \end{aligned}$$

In the 1st line, the expectation $\mathbb{E}X_n S_{n-1}$ exists since both rvs are in \mathcal{L}^2 . By the last line, $\|S_{\tau \wedge n} - S_{\tau \wedge m}\|^2 = \sigma^2 \sum_{i=m+1}^n \mathbb{P}(\tau \geq i) \rightarrow 0$ as $n, m \rightarrow \infty$, $\{S_{\tau \wedge n}\}_n$ is a Cauchy sequence in \mathcal{L}^2 , so letting $n \rightarrow \infty$ gives the result. □

Proof of Example 4.4. 1. For any positive integer k , by dividing the interval $(0, k(b-a))$ into k subintervals of equal length and considering an extreme case behavior (keep going upwards) of the random walk within each subinterval, we obtain

$$\begin{aligned} \mathbb{E}N &= \sum_{i=0}^{\infty} \mathbb{P}(N > i) \leq (b-a) \sum_{k=0}^{\infty} \mathbb{P}(N > k(b-a)) \\ &\leq (b-a) \sum_{k=0}^{\infty} \mathbb{P}((X_{(j-1)(b-a)+1}, \dots, X_{j(b-a)}) \neq (1, \dots, 1), j = 1, \dots, k) \\ &\leq (b-a) \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^{b-a}}\right)^k < \infty. \end{aligned}$$

2. It is obvious.

3. By Wald's first equation 4.3, $0 = \mathbb{E}S_N = a\mathbb{P}(S_N = a) + b\mathbb{P}(S_N = b)$, we also have $1 = \mathbb{P}(S_N = a) + \mathbb{P}(S_N = b)$, so solve for the result.

4. By Wald's second equation 4.3 and $\sigma = 1$, we have $\mathbb{E}N = \mathbb{E}S_N^2 = a^2\mathbb{P}(S_N = a) + b^2\mathbb{P}(S_N = b)$, and use 3. \square

A.1.2 Proofs - 4.2

Proof of Theorem 4.5. We have

$$\begin{aligned}
\mathbb{P}(\tau_2 < \infty) &= \mathbb{P}(\tau_1 < \infty, \tau_2 - \tau_1 < \infty) \\
&= \sum_{m,n=1}^{\infty} \mathbb{P}(\tau_1 = m, \tau_2 - \tau_1 = n) \\
&= \sum_{m,n=1}^{\infty} \mathbb{P}(\mathbf{X}_1 + \cdots + \mathbf{X}_m = \mathbf{0}, \mathbf{X}_1 + \cdots + \mathbf{X}_u \neq \mathbf{0}, \forall 1 \leq u < m; \\
&\quad \mathbf{X}_{m+1} + \cdots + \mathbf{X}_{m+n} = \mathbf{0}, \mathbf{X}_{m+1} + \cdots + \mathbf{X}_{m+v} \neq \mathbf{0}, \forall 1 \leq v < n) \\
&= \sum_{m,n=1}^{\infty} \mathbb{P}(\tau_1 = m) \mathbb{P}(\tau_1 = n) \tag{iid} \\
&= (\mathbb{P}(\tau_1 < \infty))^2.
\end{aligned}$$

Similarly, we can prove $\mathbb{P}(\tau_n < \infty) = (\mathbb{P}(\tau_1 < \infty))^n$. So (i) and (ii) are equivalent. They are equivalent to (iii) by examining their meanings. Finally,

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathbb{P}(\mathbf{S}_m = \mathbf{0}) &= \sum_{m=0}^{\infty} \mathbb{E} \mathbf{1}_{\{\mathbf{S}_m = \mathbf{0}\}} = \mathbb{E} \sum_{m=0}^{\infty} \mathbf{1}_{\{\mathbf{S}_m = \mathbf{0}\}} = \mathbb{E} \sum_{n=0}^{\infty} \mathbf{1}_{\{\tau_n < \infty\}} \\
&= \sum_{n=0}^{\infty} \mathbb{P}(\tau_n < \infty) = \sum_{n=0}^{\infty} \mathbb{P}(\tau_1 < \infty)^n = \frac{1}{1 - \mathbb{P}(\tau_1 < \infty)}.
\end{aligned}$$

So (i) and (iv) are equivalent. \square

Proof of Theorem 4.6. In $d = 1$, use (iv) in Theorem 4.5 to show.

$$\begin{aligned}
\sum_{m=1}^{\infty} P(S_m = 0) &= \sum_{n=1}^{\infty} P(S_{2n} = 0) && \text{(can only return to 0 at even steps)} \\
&= \sum_{n=1}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} && \text{(combinatorics)} \\
&\sim \sum_{n=1}^{\infty} \frac{\sqrt{2\pi 2n} \left(\frac{2n}{e}\right)^{2n}}{(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n)^2} \frac{1}{2^{2n}} && \text{(Stirling's formula)} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty.
\end{aligned}$$

In $d = 2$, note that in order for $S_{2n} = 0$, we must for some $0 \leq m \leq n$ have m up steps, m down steps, $n - m$ to the left, and $n - m$ to the right, so

$$\begin{aligned}
\mathbb{P}(\mathbf{S}_{2n} = \mathbf{0}) &= \frac{1}{4^{2n}} \sum_{m=0}^n \binom{2n}{m} \binom{2n-m}{m} \binom{2n-2m}{n-m} = \frac{1}{4^{2n}} \sum_{m=0}^n \frac{(2n)!}{m!m!(n-m)!(n-m)!} \\
&= \frac{1}{4^{2n}} \binom{2n}{n} \sum_{m=0}^n \binom{n}{m} \binom{n}{n-m} = 4^{-2n} \binom{2n}{n}^2 \sim (\pi n)^{-1/2}
\end{aligned}$$

by Stirling's formula. So its sum is ∞ , still recurrent.

For $d = 3$, more complicated combinatorics give $\mathbb{P}(\mathbf{S}_{2n} = \mathbf{0}) \asymp \frac{1}{n^{3/2}}$, summing up to a finite number; hence transient. In even higher dimensions, the probabilities become even smaller; hence all transient. \square

A.1.3 Proofs - 4.3

Proof of Theorem 4.7. To show the first result, suppose $(0, s_0), (1, s_1), \dots, (n, s_n)$ is a path from $(0, x)$ to $(0, y)$. Let $K = \inf\{k : s_k = 0\}$. Let $s'_k = -s_k$ for $k \leq K$ and $s'_k = s_k$ for $K \leq k \leq n$. Then (k, s'_k) , $0 \leq k \leq n$, is a path from $(0, -x)$ to (n, y) . Conversely, given a path from $(0, -x)$ to (n, y) , we can also construct a reflected path. We have a one-to-one correspondence between the two classes of paths, so their numbers must be equal.

Then, we have

$$\begin{aligned}
\mathbb{P}(\max_{1 \leq k \leq n} S_k \geq b) &= P(\max_{1 \leq k \leq n} S_k \geq b, S_n > b) + P(\max_{1 \leq k \leq n} S_k \geq b, S_n < b) + P(\max_{1 \leq k \leq n} S_k \geq b, S_n = b) \\
&= \mathbb{P}(S_n > b) + \mathbb{P}(\max_{1 \leq k \leq n} S_k \geq b, S_n > b) + \mathbb{P}(S_n = b) \\
&= 2\mathbb{P}(S_n > b) + \mathbb{P}(S_n = b),
\end{aligned}$$

which prove the second result. \square

Proof of Theorem 4.8. To count the number of paths from $(0, 0)$ to (n, x) , denote $a, b \in \mathbb{N}$ be the number of positive steps and b negative steps, respectively. $n = a + b$, and $x = a - b$, where $x \in [-n, n]$, and $n - x$ is even. The number of paths from $(0, 0)$ to (n, x) is $N_{n,x} = \binom{n}{a}$.

Since

$$\mathbb{P}(S_1 > 0, \dots, S_{2n} > 0) = \sum_{r=1}^{\infty} \mathbb{P}(S_1 > 0, \dots, S_{2n-1} > 0, S_{2n} = 2r).$$

Now we count the number of paths of $(1, 1) \rightarrow (2n, 2r)$, that are never 0. Since the total number of paths of $(1, 1) \rightarrow (2n, 2r)$ is $N_{2n-1, 2r-1}$, the number of these paths touching 0 is the number of paths of $(1, -1) \rightarrow (2n, 2r)$, i.e., $N_{2n-1, 2r+1}$, by reflection principle, we have the number of paths of $(1, 1) \rightarrow (2n, 2r)$ never touching 0 is $N_{2n-1, 2r-1} - N_{2n-1, 2r+1}$. Hence

$$\sum_{r=1}^{\infty} \mathbb{P}(S_1 > 0, \dots, S_{2n-1} > 0, S_{2n} = 2r) = \sum_{r=1}^{\infty} \frac{1}{2} \frac{1}{2^{2n-1}} (N_{2n-1, 2r-1} - N_{2n-1, 2r+1}) = \frac{1}{2^{2n}} N_{2n-1, 1},$$

where the $1/2$ in the 2nd term guarantees $S_1 > 0$. Since $\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} N_{2n-1, -1} + \frac{1}{2^{2n}} N_{2n-1, 1} = 2 \cdot \frac{1}{2^{2n}} N_{2n-1, 1}$,

$$\mathbb{P}(S_1 > 0, \dots, S_{2n} > 0) = \frac{1}{2^{2n}} N_{2n-1, 1} = \frac{1}{2} \mathbb{P}(S_{2n} = 0).$$

Symmetry implies $\mathbb{P}(S_1 < 0, \dots, S_{2n} < 0) = (1/2) \mathbb{P}(S_{2n} = 0)$. Then the proof is completed. \square

A.2 Proofs - 5

A.2.1 Proofs - 5.1

References