

Note: Math Tools

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Lecturer:

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Chapter 1

Complex Analysis

References:

- CUHKSZ: MAT3253 - Complex Variables notes by Kenneth Shum (2022-2023 Spring)

1.1 Complex Numbers

Polar form of complex numbers $z = x + iy = r(\cos \theta + i \sin \theta)$ for $r, \theta \geq 0$.

- If $z_k = r_k(\cos \theta_k + i \sin \theta_k)$ for $k = 1, 2$, then $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.
- If $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.
- If $\Re(z_1), \Re(z_2) > 0$, then $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$, where principal arguments in $(-\pi, \pi]$ are used.

Properties of complex numbers (i) $(z^*)^* = z$; (ii) $z^* = z$ iff $z \in \mathbb{R}$; (iii) $z z^* = |z|^2 = x^2 + y^2$; (iv) $z_1, z_2 \in \mathbb{C}$, $(z_1 + z_2)^* = z_1^* + z_2^*$, $(z_1 z_2)^* = z_1^* z_2^*$; (v) $\Re(z) = (z + z^*)/2$, $\Im(z) = (z - z^*)/(2i)$; (vi) $|z_1 + z_2| \leq |z_1| + |z_2|$; (vii) $z_1 \neq z_2$, then $|z_2 - z_1|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$; (viii) $|z_1 + z_2|^2 \leq |z_1|^2 + 2|\Re(z_1 z_2^*)| + |z_2|^2$, and $|\Re(z_1 z_2^*)| \leq |z_1| |z_2|$.

- (DeMoivre formula) $\forall n \in \mathbb{Z}, \theta \in \mathbb{R}$, $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$
- (Binomial formula) $(z_1 + z_2)^m = \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k}$ for $m \in \mathbb{N}^+$, $z_1, z_2 \in \mathbb{C}$.
- (Geometric series) $\sum_{k=0}^n z^k = (1 - z^{n+1})/(1 - z)$.

n -th root of a complex number w is the n -th root of z_0 if $w^n = z_0$.

- (n -th root of unity) $\forall n \in \mathbb{N}^+$, the solution of $z^n = 1$ is $z = \cos(2\pi k/n) + i \sin(2\pi k/n)$, $k = 0, \dots, n-1$. If we write $w = \cos(2\pi/n) + i \sin(2\pi/n)$, then the n -th root is w^k , $k = 0, \dots, n-1$.

Example 1.1.1.

- (Summation of $\cos k\theta$)

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}, \quad 0 < \theta < 2\pi.$$

- (Chebyshev polynomials) Let $m = n/2$ if n is even and $(n-1)/2$ if n is odd, then

$$\cos n\theta = \sum_{k=0}^m \binom{n}{2k} (-1)^k \cos^{n-2k}(\theta) \sin^{2k}(\theta), \quad n \in \mathbb{N}.$$

Write $x = \cos \theta$, the above becomes a polynomial $T_n(x)$ of degree n in the variable x .

1.1.1 Transformation

Linear fractional/Möbius/bilinear transformation

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

- $b = 0, c = 0, d = 1$, rotation $f(z) = az = re^{i\theta}z$;
- $a = 1, c = 0, d = 1$, translation $f(z) = z + b$;
- $a = 0, b = 1, c = 1, d = 0$, inversion function $f(z) = 1/z$, that maps circles and straight lines to circles and straight lines;
- $f(z) = rz$, $0 < r \in \mathbb{R}$, scaling.

All four types of transformation maps circle/line to circle/line. If $ad - bc = 0$, then $f(z)$ is a constant.

When $z = -d/c$, $f(z) = \infty$, we extend the domain. The **Riemann sphere** is three-dimensional sphere with the south pole touching the origin of the complex plane. The **stereographic projection** is a function that maps a complex number $z = x + iy$ in the complex plane to the point $P(x, y)$ on the Riemann sphere such that $(x, y), P(x, y)$ and the north pole of the sphere are colinear. The north pole of the sphere does not correspond to any point on the complex

plane and is called the **point at infinity**, and is denoted by the symbol ∞ . The Riemann sphere is often called the **one-point compactification** of the complex plane.

Extended complex number system/extended complex plane $\bar{\mathbb{C}}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

- Given complex numbers $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, define a linear fractional transformation on $\bar{\mathbb{C}}$ by

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c, z \neq \infty \\ \infty & \text{if } z = -d/c \\ a/c & \text{if } z = \infty \end{cases}$$

which is a bijection on the Riemann sphere.

1.2 Complex functions

1.2.1 Complex sequences and series

- Distance $d(z_1, z_2) = |z_1 - z_2|$.
- Open disc** of radius r centered at z_0 : $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. Neighborhood of ∞ is $\{z \in \mathbb{C} : |z| > R\}$ for some large R .
- Convergence** of complex sequence $(z_n)_{n=1}^{\infty}$: converges to $L \in \mathbb{C}$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|z_n - L| < \epsilon, \forall n > N$. We write $\lim_{n \rightarrow \infty} z_n = L$. $(z_n)_{n=1}^{\infty}$ converges to ∞ iff $1/|z_n| \rightarrow 0$ as $n \rightarrow \infty$ ($|z_n| \rightarrow \infty$). It **diverges** if z_n does not converge to any $L \in \mathbb{C}$ ($\rightarrow \infty$ is also divergent for \mathbb{C}).
- Cauchy sequence** $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|z_m - z_n| \leq \epsilon, \forall m, n \geq N$.
 - If $z_n = x_n + iy_n$, then z_n is Cauchy iff x_n, y_n Cauchy.
 - z_n converges iff z_n is Cauchy.
- Complex series** $\sum_{k=1}^{\infty} z_k := \lim_n \sum_{k=1}^n z_k$ if the limit exists. We call it **converges absolutely** if $\sum_{k=1}^{\infty} |z_k|$ converges. We call it **converges conditionally** if $\sum_{k=1}^{\infty} z_k$ converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.
 - (n-th term test)** If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_n |z_n| = 0$. If $|z_n| \not\rightarrow 0$, then $\sum_k z_k$ diverges.
 - (Absolute convergence test)** $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ converges.
 - (Limit ratio test)** Assume $\lim_n |a_{n+1}/a_n|$ exists and is equal to L . (a) $L > 1 \Rightarrow \sum_{k=1}^{\infty} a_n$ diverges, (b) $L < 1 \Rightarrow \sum_{k=1}^{\infty} a_n$ converges absolutely, (c) $L = 1$, no conclusion.
 - If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converges absolutely, then $(\sum_{k=0}^{\infty} a_k)(\sum_{k=0}^{\infty} b_k) = (\sum_{k=0}^{\infty} c_k)$, where $c_k = \sum_{j=0}^k a_j b_{k-j}$.
 - If a series converges absolutely, then a series obtained by rearranging the terms converges to the same limit.

1.2.2 Basic complex functions

Power series A complex power series centered at the origin is a series in the form $\sum_{k=0}^{\infty} a_k z_k^k$, $a_k \in \mathbb{C}$.

Definition 1.2.1. For $z \in \mathbb{C}$, define

- (complex exponential function)**

$$e^z := \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we have $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $e^{-z} = (e^z)^{-1}$, $e^z \neq 0, \forall z \in \mathbb{C}$, and $e^{a+ib} = e^a e^{ib}$, $a, b \in \mathbb{R}$.

- (complex trigonometric, hyperbolic trigonometric)**

$$\begin{aligned} \sin(z) &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, & \cos(z) &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, & \tan(z) &:= \frac{\sin(z)}{\cos(z)}, \\ \sinh(z) &:= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, & \cosh(z) &:= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, & \tanh(z) &:= \frac{\sinh(z)}{\cosh(z)}. \end{aligned}$$

They are all converges absolutely.

Theorem 1.2.2. $\forall z \in \mathbb{C}$,

- (Euler's formula)** $e^{iz} = \cos z + i \sin z$

•

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, & \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \\ \cosh(z) &= \frac{e^z + e^{-z}}{2}, & \sinh(z) &= \frac{e^z - e^{-z}}{2}, \end{aligned}$$

thus we have $\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

Since $e^z = e^{z+2\pi ki}$, $k \in \mathbb{Z}$, the inverse function of e^z is multi-valued. For $0 \neq w \in \mathbb{C}$, define the **complex log function** as

$$\log(w) := \log|w| + i(\arg(w) + 2\pi k), \quad k \in \mathbb{Z}.$$

Define the **principal complex log function** as

$$\text{Log}(w) := \log|w| + i\arg(w), \quad \arg(w) \in (-\pi, \pi] \text{ or } [0, 2\pi).$$

Given $0 \neq z \in \mathbb{C}$, define the **complex power** by

$$z^w := \exp(w \log(z)).$$

The angle function, parametric curve and winding number

Suppose $\theta(z)$ is continuous, $\theta(z_0) = 0$, then as $z \rightarrow z_0$ from the right, $\theta(z) \rightarrow 2\pi \neq 0$. To prevent closed cycle around the origin, let the domain of **angle function** be the half plane.

- If $H = \{x + iy : y > 0\}$, the range is $(0, \pi)$, then for $z \in H$, define $F(x, y) := \cos^{-1}(x/\sqrt{x^2 + y^2})$.
- If $H_\alpha = \{x + iy : y > \tan(\alpha)x\}$ for $\alpha > 0$, define $F_\alpha := F(e^{-i\alpha}z) + \alpha$.

The **parametric curve** is a function $\gamma : [a, b] \rightarrow \mathbb{C}$ continuous, $\gamma(t)$ is the location at time t . If $\gamma(a) = \gamma(b)$, then we call it **closed curve**.

Given $\gamma : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$, divide $[a, b]$ into $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ s.t. $\gamma(t)$, $t_k \leq t \leq t_{k+1}$, is inside $H_{\alpha(k)}$ for $k = 0, \dots, n-1$. Define

- **change in angle** in the k th part: $F_{\alpha(k)}(\gamma(t_{k+1})) - F_{\alpha(k)}(\gamma(t_k))$,
- **overall change of angle**: $\sum_{k=0}^{n-1} [F_{\alpha(k)}(\gamma(t_{k+1})) - F_{\alpha(k)}(\gamma(t_k))]$,
- **branch**: A continuous angle function as a function of t .

Note that it doesn't depend on the sub-division of the curve and how we parameterize the curve. The **winding number/index** of a closed parametric curve not passing through the origin is $(2\pi)^{-1}(\text{change in angle})$.

1.2.3 Complex differentiability

Limit and continuity

Chapter 2

Linear Algebra

References:

- CUHKSZ: MAT2040 - Linear Algebra, by Dr. Dongxu Ji (2020-2021 Summer).
- CUHK: STAT5030 - Linear Models, by Prof. Yuanyuan Lin (2024-2025 Spring)
- Peng Ding - *Linear Model and Extensions*: appendix A.
- Robb J. Muirhead - *Aspects of multivariate statistical theory* [2]: appendix A.
- Ronald Christensen - *Plane Answers to Complex Questions: The Theory of Linear Models* [1], 2nd: Appendix B.

Notations:

- **Column space/range/image** of $n \times m$ matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_m)$ is $\text{Col}(A) = \{\alpha_1 \mathbf{a}_1 + \dots + \alpha_m \mathbf{a}_m : \alpha_1, \dots, \alpha_m \in \mathbb{R}\}$.
- **Row space** of A is $\text{Col}(A^\top)$.
- **Kernel/null space/nullspace** of A is $\ker(A) = \{\mathbf{v} \in \mathbb{R}^m : A\mathbf{v} = \mathbf{0}_n\}$.
- **Rank** of A is $r(A)$.
- **Trace** of A is $\text{tr}(A)$.
- **Eigenvalues** of A are $\lambda(A)$.

2.1 Spaces

2.1.1 Spaces, rank, and related factorization

Theorem 2.1.1 (Column spaces and rank). *Let $A \in \mathbb{R}^{n \times m}$.*

1. $\text{Col}(A) = \text{Col}(A^\top A)$, so $r(A) = r(A^\top A)$.
2. If $B \in \mathbb{R}^{m \times m}$ is nonsingular, then $\text{Col}(AB) = \text{Col}(A)$, so $r(AB) = r(A)$.

Theorem 2.1.2 (Properties of rank). *Let $A \in \mathbb{R}^{n \times m}$.*

1. $r(A) = r(A^\top)$.
2. $r(A) \leq \min(n, m)$.
3. $r(AB) \leq \min\{r(A), r(B)\}$.
4. If A, B have same size, then $r(A + B) \leq r(A) + r(B)$.
5. If $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times m}$, and A and C are nonsingular, then $r(ABC) = r(B)$.
6. If $B \in \mathbb{R}^{m \times p}$ such that $AB = \mathbf{0}$, then $r(B) \leq m - r(A)$.
7. A is full column/row rank iff $A^\top A/AA^\top$ is nonsingular.
8. The system of equations $A\mathbf{x} = \mathbf{c}$ is consistent iff $r(A) = r([A, \mathbf{c}])$.

Theorem 2.1.3 (Rank-related factorization).

1. (**Full-rank factorization**) If $A \in \mathbb{R}^{n \times m}$ with $r(A) = k$, then $A = BC$ for some full column rank $B \in \mathbb{R}^{n \times k}$ and full row rank $C \in \mathbb{R}^{k \times m}$.
2. (**Non-negative definite**) If $n \times n$ $A \succeq \mathbf{0}$, $r(A) = r$, then
 - $\exists B \in \mathbb{R}^{n \times r}$ of rank r such that $A = BB^\top$;
 - $\exists C \in \mathbb{R}^{n \times n}$ nonsingular such that $A = C \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} C^\top$.

2.2 Inversion

Theorem 2.2.1 (Push-through identity). Let $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{k \times n}$, then

$$(I_n + UV)^{-1}U = U(I_n + VU)^{-1}. \quad (2.1)$$

Theorem 2.2.2 (Sherman–Morrison–Woodbury formula/matrix inversion lemma). Let $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{k \times k}$, $U \in \mathbb{R}^{n \times k}$, and $V \in \mathbb{R}^{k \times n}$. A is invertible. Then

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}. \quad (2.2)$$

Specially,

1. if $A = I_n$ and $C = I_k$, then

$$(I_n + UV)^{-1} = I_n - U(I_k + VU)^{-1}V.$$

2. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $A + \mathbf{u}\mathbf{v}^\top$ is invertible iff $1 + \mathbf{v}^\top A^{-1}\mathbf{u} \neq 0$. In this case, we have the [Sherman–Morrison formula](#)

$$(A + \mathbf{u}\mathbf{v}^\top)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}}.$$

2.2.2 Types of inversion

Left and right inverse

Definition 2.2.3 (Left/right inverse). Let $A \in \mathbb{R}^{n \times m}$. The **left inverse** $A_{left}^{-1} \in \mathbb{R}^{m \times n}$ satisfies $A_{left}^{-1}A = I_m$, and the **right inverse** $A_{right}^{-1} \in \mathbb{R}^{m \times n}$ satisfies $AA_{right}^{-1} = I_n$.

We can find these inverses by Theorem 2.1.2: if

- A full column rank $r(A) = m$, then $A^\top A$ nonsingular, and $A_{left}^{-1} = (A^\top A)^{-1}A^\top$; A^- is also a left inverse.
- A full row rank $r(A) = n$, then AA^\top nonsingular, and $A_{right}^{-1} = A^\top(AA^\top)^{-1}$; A^- is also a right inverse.

Moore–Penrose inverse

It is used when left/right inverse cannot be obtained.

Definition 2.2.4 (Moore–Penrose inverse/pseudoinverse). Let $A \in \mathbb{R}^{n \times m}$. $A^+ \in \mathbb{R}^{m \times n}$ is defined as a **Moore–Penrose (M-P) inverse** of A if

1. AA^+ and A^+A are symmetric;
2. $AA^+A = A$;
3. $A^+AA^+ = A^+$.

Example 2.2.5 (M-P inverse of a diagonal matrix). Let $D \in \mathbb{R}^{n \times m}$, WLOG, assume $n \geq m$. $D_{ii} = d_i$ for $i = 1, \dots, m$, while others are zero. Then $D^+ \in \mathbb{R}^{m \times n}$ satisfies $D_{ii}^+ = 1/d_i$ if $d_i \neq 0$ and zero otherwise, $i = 1, \dots, m$.

Theorem 2.2.6. Each matrix A has an A^+ .

If $A = \mathbf{0}$, then $A^+ = \mathbf{0}$. If $A \neq \mathbf{0}$, two ways to construct A^+ :

1. If A is symmetric with eigendecomposition $A = P^\top DP$, then $A^+ = P^\top D^+ P$, where D^+ is given in Ex. 2.2.5.
2. Full-rank factorization $A = B_{n \times r} C_{r \times m}$ of rank r . We have

$$A^+ = C^\top(CC^\top)^{-1}(B^\top B)^{-1}B^\top.$$

3. SVD $A = UDV^\top$, then $A^+ = VD^+U^\top$.

Theorem 2.2.7. Let $A \in \mathbb{R}^{n \times m}$.

1. The M-P inverse is unique.
2. $(A^\top)^+ = (A^+)^\top$.
3. $r(A^+) = r(A)$.
4. If A is symmetric, then $A^+ = (A^+)^\top$.
5. If A is nonsingular, then $A^{-1} = A^+$.
6. If A is symmetric idempotent, then $A^+ = A$.
7. If $r(A) = m$, then $A^+ = A_{left}^{-1} = (A^\top A)^{-1}A^\top$.
8. If $r(A) = n$, then $A^+ = A_{right}^{-1} = A^\top(AA^\top)^{-1}$.

9. The matrices AA^+ , A^+A , $I_n - AA^+$, and $I_m - A^+A$ are all symmetric idempotent.
10. AA^+ (A^+A) is a p.p.m. onto $\text{Col}(A)$ ($\text{Col}(A^\top)$).

Generalized Inverse

Definition 2.2.8 (Generalized inverse). Let $A \in \mathbb{R}^{n \times m}$. The **generalized inverse (G-inverse)** $A^- \in \mathbb{R}^{m \times n}$ satisfies

$$AA^-A = A.$$

It is obvious that the M-P inverse is also a G-inverse. $\forall A$, G-inverse exists.

1. If A nonsingular, then only $A^- = A^{-1}$.
2. If $A = \mathbf{0}_{n \times m}$, then $A^- = \mathbf{0}$.
3. If D is diagonal like Example 2.2.5, then let $D^- = D^+$.
4. If A is symmetric with eigendecomposition $A = P^\top DP$, then let $A^- = A^+$, symmetric.
5. If $r(A) = r$, then by SVD, $A = U_{n \times n} \begin{bmatrix} \Sigma_{r \times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V_{m \times m}^\top$. Let $A^- = V \begin{bmatrix} \Sigma^{-1} & B \\ C & D \end{bmatrix} U^\top$, $\forall B, C, D$.
6. If G_1, G_2 are G-inverse of A , then so is G_1AG_2 .

Note that

- G-inverse may not be unique by the above construction.
- A is symmetric, A^- may not be symmetric.

Theorem 2.2.9 (Properties of G-inverse). Let $A \in \mathbb{R}^{n \times m}$ with $\text{rank } k > 0$.

1. $r(A^-) \geq k$.
2. A^-A and AA^- are idempotent.
3. $\ker(A^-A) = \ker(A)$ and $\text{Col}(AA^-) = \text{Col}(A)$.
4. $r(A^-A) = r(AA^-) = k$.
5. $A^-A = I_m$ (i.e., A^- is a left inverse of A) iff $r(A) = m$.
6. $AA^- = I_n$ (i.e., A^- is a right inverse of A) iff $r(A) = n$.
7. $\text{tr}(A^-A) = \text{tr}(AA^-) = k = r(A)$.
8. If A^- is any G-inverse of A , then $(A^-)^\top$ is a G-inverse of A^\top .
9. The system of equations $A\mathbf{x} = \mathbf{c}$ is consistent iff $\forall A^-$ of A , $AA^-\mathbf{c} = \mathbf{c}$ (i.e. $A^-\mathbf{c}$ is a solution).
10. If G, H are G-inverses of $(A^\top A)$, then
 - (a) $AGA^\top A = AHA^\top A = A$, i.e., $A(A^\top A)^-A^\top A = A$ for any G-inverse $(A^\top A)^-$.
 - $(A^\top A)^-A^\top$ is a G-inverse of A for any G-inverse of $A^\top A$.
 - $A(A^\top A)^-$ is a G-inverse of A^\top for any G-inverse of $A^\top A$.
 - (b) $AGA^\top = AHA^\top$, i.e., $A(A^\top A)^-A^\top$ the same for any G-inverse $(A^\top A)^-$.
 - (c) Since $A^\top A$ is symmetric, $\exists (A^\top A)^-$ symmetric such that $A(A^\top A)^-A^\top$ symmetric. So by ii, $A(A^\top A)^-A^\top$ symmetric for all $(A^\top A)^-$.
 - (d) $A(A^\top A)^-A^\top$ is the p.p.m. onto $\text{Col}(A)$, see Example 2.3.5.

2.3 Special matrices

2.3.1 Idempotent matrices

Definition 2.3.1 (Idempotent matrices). $A \in \mathbb{R}^{n \times n}$ if $A^2 = A$.

Theorem 2.3.2. A is idempotent.

1. All idempotent matrices(except I) are singular.
2. $r(A) = \text{tr}(A)$.
3. $\lambda(A)$ is either 0 or 1.
4. If A symmetric, all $\lambda(A)$'s are 0 or 1, then A is idempotent.

2.3.2 Perpendicular projection matrices

See appendix B of [1].

Definition 2.3.3 (Perpendicular projection matrices). $M \in \mathbb{R}^{n \times n}$ is a **perpendicular projection matrix (p.p.m.)** onto $\text{Col}(X)$ ($X \in \mathbb{R}^{n \times m}$) iff

1. $\forall \mathbf{v} \in \text{Col}(X), M\mathbf{v} = \mathbf{v}$. (Projection - Idempotent)
2. $\forall \mathbf{w} \perp \text{Col}(X), M\mathbf{w} = \mathbf{0}$. (Perpendicularity - Symmetric)

If M satisfies $M\mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in \text{Col}(M)$, then M is called the **projection operator (matrix)** onto $\text{Col}(M)$ along $\text{ker}(M)$. Any projection operator that is not p.p.m. is called an **oblique projection operator**.

Any idempotent matrix M is a projection operator onto $\text{Col}(M)$.

Theorem 2.3.4 (Properties of p.p.m.). Let M be a p.p.m. onto $\text{Col}(X)$ ($X \in \mathbb{R}^{n \times m}$).

1. p.p.m.'s are unique.
2. $\text{Col}(M) = \text{Col}(X)$.
3. M is idempotent and symmetric.
4. $MX = X$.
5. M is a p.p.m. onto $\text{Col}(M)$ iff M is idempotent and symmetric.
6. Let $\mathbf{o}_1, \dots, \mathbf{o}_r \in \mathbb{R}^n$ be orthonormal basis of $\text{Col}(X)$. Let $O = [\mathbf{o}_1 \ \dots \ \mathbf{o}_r]$. Then $OO^\top = \sum_{i=1}^r \mathbf{o}_i \mathbf{o}_i^\top$ is the p.p.m. onto $\text{Col}(X)$.

There are two methods to find the p.p.m. of $\text{Col}(X)$. The first one is to find the orthonormal basis of $\text{Col}(X)$, the second one uses the G-inverse shown in Example 2.3.5.

Example 2.3.5. Let $X \in \mathbb{R}^{n \times m}$. Then by the property of $(A^\top A)^-$ in Theorem 2.2.9, $K = X(X^\top X)^- X^\top \in \mathbb{R}^{n \times n}$ is the p.p.m. (also unique) onto $\text{Col}(X)$. We have

1. K is idempotent and symmetric.
2. $\text{r}(K) = \text{r}(X)$.
3. $KX = X$ and $X^\top K = X^\top$.
4. $K = XX^+$.

Theorem 2.3.6 (Relationships between two p.p.m. (Thm B.45–49 [1])). Let M_\bullet be $n \times n$.

1. Let M_1, M_2 be two p.p.m. $M_1 + M_2$ is the p.p.m. onto $\text{Col}(M_1, M_2)$ iff $\text{Col}(M_1) \perp \text{Col}(M_2)$.
2. If M_1, M_2 symmetric, $\text{Col}(M_1) \perp \text{Col}(M_2)$, and $(M_1 + M_2)$ p.p.m., then M_1 and M_2 are p.p.m.
3. Let M and M_0 be p.p.m. with $\text{Col}(M_0) \subset \text{Col}(M)$. Then
 - $M - M_0$ is a p.p.m.; and
 - $\text{Col}(M - M_0) = \text{Col}(M_0)^\perp$ w.r.t. $\text{Col}(M)$.
 - $\text{r}(M) = \text{r}(M_0) + \text{r}(M - M_0)$.

One particular application: I_n is the p.p.m. onto \mathbb{R}^n . \forall other p.p.m. M_0 , $\text{Col}(M_0) \subset \text{Col}(I_n) = \mathbb{R}^n$, $I_n - M_0$ is a p.p.m. onto $\text{Col}(M_0)^\perp$ in \mathbb{R}^n .

Chapter 3

Optimization

References:

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 1. *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*, Amir Beck.
 2. *Convex Optimization*, S. Boyd and L. Vandenberghe.
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3.1 Nonlinear Optimization

3.1.1 KKT Conditions

Theorem 3.1.1 (The Fritz-John necessary conditions). *Let \mathbf{x}^* be a local minimum of the problem*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. Then \exists multipliers $\lambda_0, \dots, \lambda_m \geq 0$, which are not all zeros, such that

$$\begin{aligned} \lambda_0 \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) &= \mathbf{0} \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

A major drawback of the Fritz-John conditions is, they allow $\lambda_0 = 0$. Under an additional [regularity condition](#), we can assume $\lambda_0 = 1$. Let $I(\mathbf{x}^*)$ be the set of active constraints at \mathbf{x}^* :

$$I(\mathbf{x}^*) = \{i : g_i(\mathbf{x}^*) = 0\}.$$

Theorem 3.1.2 (The KKT conditions for inequality constrained problems). *Let \mathbf{x}^* be a local minimum of*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\mathbf{x}^)\}_{i \in I(\mathbf{x}^*)}$ are linearly independent. Then $\exists \lambda_1, \dots, \lambda_m \geq 0$ such that*

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) &= \mathbf{0} \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Theorem 3.1.3 (The KKT conditions for inequality/equality constrained problems). *Let \mathbf{x}^* be a local minimum of*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, p \end{aligned} \tag{3.1}$$

where $f, g_1, \dots, g_m, h_1, \dots, h_p \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\mathbf{x}^), \nabla h_j(\mathbf{x}^*), i \in I(\mathbf{x}^*), j = 1, \dots, p\}$ are linearly independent.*

Then $\exists \lambda_1, \dots, \lambda_m \geq 0, \mu_1, \dots, \mu_p \in \mathbb{R}$, such that

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^p \mu_j \nabla h_j(\mathbf{x}^*) &= \mathbf{0}, \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned} \tag{3.2}$$

Consider problem (1), a feasible point \mathbf{x}^* is called a **KKT point** if $\exists \lambda_1, \dots, \lambda_m \geq 0, \mu_1, \dots, \mu_p \in \mathbb{R}$, such that (3.2) holds. \mathbf{x}^* is called **regular** if $\{\nabla g_i(\mathbf{x}^*), \nabla h_j(\mathbf{x}^*), i \in I(\mathbf{x}^*), j = 1, \dots, p\}$ are linearly independent.

- The additional requirement of regularity is not required in linearly constrained problems in which no such assumption is needed.

Bibliography

- [1] R. Christensen et al. *Plane answers to complex questions*, volume 35. Springer, 2002. [2](#), [2.3.2](#), [2.3.6](#)
- [2] R. J. Muirhead. *Aspects of multivariate statistical theory*. John Wiley & Sons, 1982. [2](#)