

Note: Math Tools

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Lecturer:

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Chapter 1

Complex Analysis

References:

- CUHKSZ: MAT3253 - Complex Variables notes by Kenneth Shum (Spring 2023)

1.1 Complex Numbers

Polar form of complex numbers $z = x + iy = r(\cos \theta + i \sin \theta)$ for $r, \theta \geq 0$.

- If $z_k = r_k(\cos \theta_k + i \sin \theta_k)$ for $k = 1, 2$, then $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.
- If $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.
- If $\Re(z_1), \Re(z_2) > 0$, then $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$, where principal arguments in $(-\pi, \pi]$ are used.

Properties of complex numbers (i) $(z^*)^* = z$; (ii) $z^* = z$ iff $z \in \mathbb{R}$; (iii) $z z^* = |z|^2 = x^2 + y^2$; (iv) $z_1, z_2 \in \mathbb{C}$, $(z_1 + z_2)^* = z_1^* + z_2^*$, $(z_1 z_2)^* = z_1^* z_2^*$; (v) $\Re(z) = (z + z^*)/2$, $\Im(z) = (z - z^*)/(2i)$; (vi) $|z_1 + z_2| \leq |z_1| + |z_2|$; (vii) $z_1 \neq z_2$, then $|z_2 - z_1|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$; (viii) $|z_1 + z_2|^2 \leq |z_1|^2 + 2|\Re(z_1 z_2^*)| + |z_2|^2$, and $|\Re(z_1 z_2^*)| \leq |z_1| |z_2|$.

- (**DeMoivre formula**) $\forall n \in \mathbb{Z}, \theta \in \mathbb{R}$, $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$
- (**Binomial formula**) $(z_1 + z_2)^m = \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k}$ for $m \in \mathbb{N}^+$, $z_1, z_2 \in \mathbb{C}$.
- (Geometric series) $\sum_{k=0}^n z^k = (1 - z^{n+1})/(1 - z)$.

n -th root of a complex number w is the n -th root of z_0 if $w^n = z_0$.

- (n -th root of unity) $\forall n \in \mathbb{N}^+$, the solution of $z^n = 1$ is $z = \cos(2\pi k/n) + i \sin(2\pi k/n)$, $k = 0, \dots, n-1$. If we write $w = \cos(2\pi/n) + i \sin(2\pi/n)$, then the n -th root is w^k , $k = 0, \dots, n-1$.

Example 1.1.1.

- (*Summation of $\cos k\theta$*)

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}, \quad 0 < \theta < 2\pi.$$

- (*Chebyshev polynomials*) Let $m = n/2$ if n is even and $(n-1)/2$ if n is odd, then

$$\cos n\theta = \sum_{k=0}^m \binom{n}{2k} (-1)^k \cos^{n-2k}(\theta) \sin^{2k}(\theta), \quad n \in \mathbb{N}.$$

Write $x = \cos \theta$, the above becomes a polynomial $T_n(x)$ of degree n in the variable x .

1.1.1 Transformation

Linear fractional/Möbius/bilinear transformation

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

- $b = 0, c = 0, d = 1$, rotation $f(z) = az = re^{i\theta}z$;
- $a = 1, c = 0, d = 1$, translation $f(z) = z + b$;
- $a = 0, b = 1, c = 1, d = 0$, inversion function $f(z) = 1/z$, that maps circles and straight lines to circles and straight lines;
- $f(z) = rz$, $0 < r \in \mathbb{R}$, scaling.

All four types of transformation maps circle/line to circle/line. If $ad - bc = 0$, then $f(z)$ is a constant.

When $z = -d/c$, $f(z) = \infty$, we extend the domain. The **Riemann sphere** is three-dimensional sphere with the south pole touching the origin of the complex plane. The **stereographic projection** is a function that maps a complex number $z = x + iy$ in the complex plane to the point $P(x, y)$ on the Riemann sphere such that $(x, y), P(x, y)$ and the north pole of the sphere are colinear. The north pole of the sphere does not correspond to any point on the complex

plane and is called the **point at infinity**, and is denoted by the symbol ∞ . The Riemann sphere is often called the *one-point compactification* of the complex plane.

Extended complex number system/extended complex plane $\bar{\mathbb{C}}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

- Given complex numbers $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, define a linear fractional transformation on $\bar{\mathbb{C}}$ by

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c, z \neq \infty \\ \infty & \text{if } z = -d/c \\ a/c & \text{if } z = \infty \end{cases}$$

which is a bijection on the Riemann sphere.

1.2 Complex functions

1.2.1 Complex sequences and series

- Distance $d(z_1, z_2) = |z_1 - z_2|$.
- Open disc** of radius r centered at z_0 : $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. Neighborhood of ∞ is $\{z \in \mathbb{C} : |z| > R\}$ for some large R .
- Convergence** of complex sequence $(z_n)_{n=1}^\infty$: converges to $L \in \mathbb{C}$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|z_n - L| < \epsilon, \forall n > N$. We write $\lim_{n \rightarrow \infty} z_n = L$. $(z_n)_{n=1}^\infty$ converges to ∞ iff $1/|z_n| \rightarrow 0$ as $n \rightarrow \infty$ ($|z_n| \rightarrow \infty$). It **diverges** if z_n does not converge to any $L \in \mathbb{C}$ ($\rightarrow \infty$ is also divergent for \mathbb{C}).
- Cauchy sequence** $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|z_m - z_n| \leq \epsilon, \forall m, n \geq N$.
 - If $z_n = x_n + iy_n$, then z_n is Cauchy iff x_n, y_n Cauchy.
 - z_n converges iff z_n is Cauchy.
- Complex series** $\sum_{k=1}^\infty z_k := \lim_n \sum_{k=1}^n z_k$ if the limit exists. We call it **converges absolutely** if $\sum_{k=1}^\infty |z_k|$ converges. We call it **converges conditionally** if $\sum_{k=1}^\infty z_k$ converges but $\sum_{k=1}^\infty |z_k|$ diverges.
 - (n-th term test)** If $\sum_{k=1}^\infty z_k$ converges, then $\lim_n |z_n| = 0$. If $|z_n| \not\rightarrow 0$, then $\sum_k z_k$ diverges.
 - (Absolute convergence test)** $\sum_{k=1}^\infty |z_k|$ converges, then $\sum_{k=1}^\infty z_k$ converges.
 - (Limit ratio test)** Assume $\lim_n |a_{n+1}/a_n|$ exists and is equal to L . (a) $L > 1 \Rightarrow \sum_{k=1}^\infty a_n$ diverges, (b) $L < 1 \Rightarrow \sum_{k=1}^\infty a_n$ converges absolutely, (c) $L = 1$, no conclusion.
 - If $\sum_{k=0}^\infty a_k$ and $\sum_{k=0}^\infty b_k$ converges absolutely, then $(\sum_{k=0}^\infty a_k)(\sum_{k=0}^\infty b_k) = (\sum_{k=0}^\infty c_k)$, where $c_k = \sum_{j=0}^k a_j b_{k-j}$.
 - If a series converges absolutely, then a series obtained by rearranging the terms converges to the same limit.

1.2.2 Basic complex functions

Power series A complex power series centered at the origin is a series in the form $\sum_{k=0}^\infty a_k z_k^k$, $a_k \in \mathbb{C}$.

Definition 1.2.1. For $z \in \mathbb{C}$, define

- (complex exponential function)**

$$e^z := \exp(z) := \sum_{n=0}^\infty \frac{z^n}{n!},$$

we have $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $e^{-z} = (e^z)^{-1}$, $e^z \neq 0, \forall z \in \mathbb{C}$, and $e^{a+ib} = e^a e^{ib}$, $a, b \in \mathbb{R}$.

- (complex trigonometric, hyperbolic trigonometric)**

$$\begin{aligned} \sin(z) &:= \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}, & \cos(z) &:= \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!}, & \tan(z) &:= \frac{\sin(z)}{\cos(z)}, \\ \sinh(z) &:= \sum_{n=0}^\infty \frac{z^{2n+1}}{(2n+1)!}, & \cosh(z) &:= \sum_{n=0}^\infty \frac{z^{2n}}{(2n)!}, & \tanh(z) &:= \frac{\sinh(z)}{\cosh(z)}. \end{aligned}$$

They are all converges absolutely.

Theorem 1.2.2. $\forall z \in \mathbb{C}$,

- (Euler's formula)** $e^{iz} = \cos z + i \sin z$

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$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, & \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \\ \cosh(z) &= \frac{e^z + e^{-z}}{2}, & \sinh(z) &= \frac{e^z - e^{-z}}{2}, \end{aligned}$$

thus we have $\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

Since $e^z = e^{z+2\pi ki}$, $k \in \mathbb{Z}$, the inverse function of e^z is multi-valued. For $0 \neq w \in \mathbb{C}$, define the **complex log function** as

$$\log(w) := \log|w| + i(\arg(w) + 2\pi k), \quad k \in \mathbb{Z}.$$

Define the **principal complex log function** as

$$\text{Log}(w) := \log|w| + i\arg(w), \quad \arg(w) \in (-\pi, \pi] \text{ or } [0, 2\pi).$$

Given $0 \neq z \in \mathbb{C}$, define the **complex power** by

$$z^w := \exp(w \log(z)).$$

The angle function, parametric curve and winding number

Suppose $\theta(z)$ is continuous, $\theta(z_0) = 0$, then as $z \rightarrow z_0$ from the right, $\theta(z) \rightarrow 2\pi \neq 0$. To prevent closed cycle around the origin, let the domain of **angle function** be the half plane.

- If $H = \{x + iy : y > 0\}$, the range is $(0, \pi)$, then for $z \in H$, define $F(x, y) := \cos^{-1}(x/\sqrt{x^2 + y^2})$.
- If $H_\alpha = \{x + iy : y > \tan(\alpha)x\}$ for $\alpha > 0$, define $F_\alpha := F(e^{-i\alpha}z) + \alpha$.

The **parametric curve** is a function $\gamma : [a, b] \rightarrow \mathbb{C}$ continuous, $\gamma(t)$ is the location at time t . If $\gamma(a) = \gamma(b)$, then we call it **closed curve**.

Given $\gamma : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$, divide $[a, b]$ into $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ s.t. $\gamma(t)$, $t_k \leq t \leq t_{k+1}$, is inside $H_{\alpha(k)}$ for $k = 0, \dots, n-1$. Define

- **change in angle** in the k th part: $F_{\alpha(k)}(\gamma(t_{k+1})) - F_{\alpha(k)}(\gamma(t_k))$,
- **overall change of angle**: $\sum_{k=0}^{n-1} [F_{\alpha(k)}(\gamma(t_{k+1})) - F_{\alpha(k)}(\gamma(t_k))]$,
- **branch**: A continuous angle function as a function of t .

Note that it doesn't depend on the sub-division of the curve and how we parameterize the curve. The **winding number/index** of a closed parametric curve not passing through the origin is $(2\pi)^{-1}(\text{change in angle})$.

1.2.3 Complex differentiability

Limit and continuity

Chapter 2

Matrix Algebra

Chapter 3

Optimization

References:

- CUHKSZ: MAT3007 - Optimization I
- CUHKSZ: MAT3220 - Optimization II. Textbook:
- 1. *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*, Amir Beck.
- 2. *Convex Optimization*, S. Boyd and L. Vandenberghe.
- 3. *Nonlinear Programming*, D. Bertsekas.
- 4. *First-Order Methods in Optimization*, Amir Beck

3.1 Nonlinear Optimization

3.1.1 KKT Conditions

Theorem 3.1.1 (The Fritz-John necessary conditions). *Let \mathbf{x}^* be a local minimum of the problem*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. Then \exists multipliers $\lambda_0, \dots, \lambda_m \geq 0$, which are not all zeros, such that

$$\begin{aligned} \lambda_0 \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) &= \mathbf{0} \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

A major drawback of the Fritz-John conditions is, they allow $\lambda_0 = 0$. Under an additional [regularity condition](#), we can assume $\lambda_0 = 1$. Let $I(\mathbf{x}^*)$ be the set of active constraints at \mathbf{x}^* :

$$I(\mathbf{x}^*) = \{i : g_i(\mathbf{x}^*) = 0\}.$$

Theorem 3.1.2 (The KKT conditions for inequality constrained problems). *Let \mathbf{x}^* be a local minimum of*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \end{aligned}$$

where $f, g_1, \dots, g_m \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\mathbf{x}^)\}_{i \in I(\mathbf{x}^*)}$ are linearly independent. Then $\exists \lambda_1, \dots, \lambda_m \geq 0$ such that*

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) &= \mathbf{0} \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Theorem 3.1.3 (The KKT conditions for inequality/equality constrained problems). *Let \mathbf{x}^* be a local minimum of*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, p \end{aligned} \tag{3.1}$$

where $f, g_1, \dots, g_m, h_1, \dots, h_p \in C^1(\mathbb{R}^n)$. If $\{\nabla g_i(\mathbf{x}^), \nabla h_j(\mathbf{x}^*), i \in I(\mathbf{x}^*), j = 1, \dots, p\}$ are linearly independent.*

Then $\exists \lambda_1, \dots, \lambda_m \geq 0, \mu_1, \dots, \mu_p \in \mathbb{R}$, such that

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^p \mu_j \nabla h_j(\mathbf{x}^*) &= \mathbf{0}, \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \dots, m. \end{aligned} \tag{3.2}$$

Consider problem (1), a feasible point \mathbf{x}^* is called a **KKT point** if $\exists \lambda_1, \dots, \lambda_m \geq 0, \mu_1, \dots, \mu_p \in \mathbb{R}$, such that (3.2) holds. \mathbf{x}^* is called **regular** if $\{\nabla g_i(\mathbf{x}^*), \nabla h_j(\mathbf{x}^*), i \in I(\mathbf{x}^*), j = 1, \dots, p\}$ are linearly independent.

- The additional requirement of regularity is not required in linearly constrained problems in which no such assumption is needed.