Operation through Enrichment

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1 Introduction

Formal systems are often defined without intrinsic connection to how they actually *operate* in practice. In everyday computation, the *structure* of the program is separate from the *dynamics* - but this disparity is the primary source of error. If these are unified in one mathematical object, the system can be proven *correct by construction.* **Operational semantics** [13] is an essential tool in language design and verification, which formally specifies program behavior by *labelled transition systems*, or labelled directed graphs:

$$(\lambda x.x + x\ ,\ 2) \xrightarrow{\beta} 2 + 2 \xrightarrow{+} 4$$

The idea is to *reify* operational semantics via **enrichment** [6]. In the categorical representation of an algebraic theory, the objects are *types*, and the morphisms are *terms*; hence to represent the actual *process* of computation, we need the higher-level notion of *rewriting* one term into another - the hom-object or "thing of morphisms" between two terms should be not a set but a *category*-like structure, where these 2-morphisms represent *rewrites*. For instance, the SK-combinator calculus is the "abstraction-free" λ-calculus (see §8):

$$(((Sx)y)z) \overset{3}{\underset{1}{\Longrightarrow}} ((xz)(yz)) \qquad\qquad ((Kx)y) \overset{2}{\underset{1}{\Longrightarrow}} x$$

A **Lawvere theory** [8] defines an algebraic structure abstractly, as a category $\mathcal{T}$ generated by powers of a single object s and morphisms $s^n \to s$ representing n-ary operations, satisfying equations. This represents the *theory* of a kind of algebra, which can be modelled in a category $\mathcal{C}$ by a power-preserving functor $\mu : \mathcal{T} \to \mathcal{C}$. This is a very general notion of "algebra" - computational formalisms are also presented by generators and relations: in particular, a **term calculus** represents a *formal language* by sorts, term constructors, and congruence rules.

Enriched Lawvere theories for operational semantics has been explored in the past. It was studied in the case of categories by Seely [17], posets by Ghani and Lüth [11], and others, for various related purposes. Here we allow quite general enrichments, to incorporate these approaches in a common framework - but we focus attention on graph-enriched Lawvere theories, which have a clear connection to the original idea of operational semantics:

$$\begin{array}{rl} \text{sorts} & : \text{generating object } s \\ \text{term constructors} & : \text{generating morphisms } s^n \to s \\ \text{structural congruence} & : \text{commuting diagrams} \\ * \quad \text{rewrite rules} & : \text{generating hom-edges} \quad * \end{array}$$

There are many other useful enriching categories. Better yet, there are functors between them that allow the seamless *transition* between different kinds of operational semantics. There is a *spectrum* of enriching categories which forms a gradient of resolution for the semantics of term calculi. For an enriching category $\mathcal{V}$, a $\mathcal{V}$-theory is a $\mathcal{V}$-enriched Lawvere theory with natural number arities (see §4):

Graphs: Gph-theories represent "small-step" operational semantics
- a hom-graph edge represents a *single* term rewrite.
Categories: Cat-theories represent "big-step" operational semantics:
- identity and composition represent the *reflexive-transitive* closure of the rewrite relation.
Posets: Pos-theories represent "full-step" operational semantics:
- a hom-poset boolean represents the *existence* of a big-step rewrite.
Sets: Set-theories represent denotational semantics (provided the calculus is *confluent*, see [18]):
- a hom-set element represents an *equivalence class* of the symmetric closure of the big-step relation.

Operational semantics is unified by enriched Lawvere theories and canonical functors between enriching categories. This provides a more systematic categorical representation of computation.

A motivating example is "logic as a distributive law" [19], an algorithm for deriving a spatial-behavioral type system from a formal presentation of a computational calculus. Essential properties such as soundness be proven, a powerful query language is generated, and modalities can be "built in" to express principles of the system.

This idea impacts more than computation, and even though these concepts have been well-known for decades, the practical significance has not been fully realized. With the modern paradigm of "programs $\simeq$ proofs", this method applies to many subjects throughout mathematics. We demonstrate this idea with several examples, and consider future potential.

2 Lawvere Theories

* blurb about theories vs monads and their importance to computer science *

The "theory of monoids" can be defined without any reference to sets:

$$\begin{array}{rl} \text{an object} & M \\ \text{an identity element} & e : 1 \to M \\ \text{and multiplication} & m : M^2 \to M \\ \text{with associativity} & m \circ (m \times M) = m \circ (M \times m) \\ \text{and unitality} & e \circ M = M = M \circ e \end{array}$$

Lawvere theories formalize this idea. They were originally called *finite product* theories: a skeleton N of the category of finite sets FinSet is the free category with finite coproducts on 1 - every finite set is equal to the disjoint union of copies of $\{*\}$; conversely, N^{op} is the free category with finite *products* on 1. So, a category with finite products $\mathcal{T}$ equipped with a strictly product-preserving bijective-on-objects functor $\iota : \mathrm{N}^{\mathrm{op}} \to \mathcal{T}$ is essentially a category generated by one object $\iota(1) = M$ and n-ary operations $M^n \to M$, as well as the projection and diagonal morphisms of finite products. Lawvere theories form a category Law, with finite-product functors $f : \mathcal{T} \to \mathcal{T}'$ such that $f\iota = \iota'$.

The abstraction of this definition is powerful: the syntax encapsulates the algebraic theory, *independent* of semantics, and then one is free to realize M in almost any mathematical object. For another category with finite products $\mathcal{C}$, a **model** of the Lawvere theory in $\mathcal{C}$ is a product-preserving functor $\mu : \mathcal{T} \to \mathcal{C}$. By the "free" property above, this functor is determined by $\mu(\iota(1)) = \mu(M) = X \in \mathcal{C}$. The models of $\mathcal{T}$ in $\mathcal{C}$ form a category $[\mathcal{T}, \mathcal{C}]_{fp}$, in which the morphisms are natural transformations. The general theory can be thereby modelled in many useful ways. For example, ordinary groups are models $\mathcal{T}_{\mathrm{Grp}} \to \mathrm{Set}$, while functors $\mathcal{T}_{\mathrm{Grp}} \to \mathrm{Top}$ are topological groups.

Lawvere theories and *finitary monads* provide complementary representations of algebraic structures and computation, as discussed by Hyland and Power in [5], and they were proven to be equivalent by Linton in [9]. Let $\iota : \mathrm{N}^{\mathrm{op}} \to \mathcal{T}$ be a Lawvere theory and $\mathrm{Mod} = [\mathcal{T}, \mathrm{Set}]_{fp}$ be the category of models. There is an adjunction:

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Mod} \arrow[l, bend left, "U"] \end{tikzcd}$$

There is the *underlying set* functor $U : \mathrm{Mod} \to \mathrm{Set}$ which sends each model $\mu : \mathcal{T} \to \mathrm{Set}$ to the image of the generating object, $\mu(\iota(1)) = X$ in Set. There is the *free model* functor $F : \mathrm{Set} \to \mathrm{Mod}$ which sends each finite set n to the representable $\mathcal{T}(|n|, -) : \mathcal{T} \to \mathrm{Set}$, and in general a set X to the functor which sends n to the set of all n-ary operations on X: $\{f(x_1, ..., x_n) | f \in \mathcal{T}(n, 1), x_i \in X\}$ - this is the *filtered colimit* of representables indexed by the poset of finite subsets of X [16], which pertains to conditions of finitude in §3. These form the adjunction:

$$\mathrm{Mod}(F(n), \mu) \cong \mu(n) \cong \mathrm{Set}(n, U(\mu))$$

The left isomorphism is by the Yoneda lemma, and the right isomorphism is by the universal property of $(-)^n$ in Set. Essentially, these are opposite ways of representing the n-ary operations of a model.

This adjunction induces a monad T on Set, which sends each set X to the set of all terms in the theory on X up to equality - the integral symbol is a *coend*, essentially a coproduct quotiented by the equality the theory:

$$T(X) = \int^{n \in \mathrm{N}} \mathcal{T}(n, 1) \times X^n$$

Conversely, for a monad T on Set, its *Kleisli category* is the category of all *free algebras* of the monad. There is a "comparison" functor $k : \mathrm{Set} \to Kl(T)$ which is the identity on objects and preserves products, so restricting the domain of k to N forms the canonical Lawvere theory corresponding to the monad. This restriction is what limits the equivalence to *finitary* monads. There is a good explanation of all this in Milewski's categorical computation blog [12]. This generalizes to arbitrary *locally finitely presentable* modelling categories $\mathcal{C}$, which is discussed in §3.

The correspondence of Lawvere theories and finitary monads forms an equivalence of categories, as well as the categories of models and algebras for every corresponding pair $(\mathcal{T}, T)$:

$$\mathrm{Law} \cong \mathrm{Mnd}_f$$

$$\mathrm{Mod}(\mathcal{T}) \cong \mathrm{Alg}(T)$$

The aforementioned references suffice; we do not need further details.

3 Enrichment

We generalize *sets* of morphisms to *objects* of morphisms, to endow formal systems with operational information. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category [7], the "enriching" category.

A **$\mathcal{V}$-category** or $\mathcal{V}$-enriched category $\mathcal{C}$ is:

$$\begin{array}{rl} \text{a collection of objects} & Obj(\mathcal{C}) \\ \text{a hom-object function} & \mathcal{C}(-,-) : Obj(\mathcal{C}) \times Obj(\mathcal{C}) \to Obj(\mathcal{V}) \\ \text{composition morphisms} & \circ_{a,b,c} : \mathcal{C}(b,c) \otimes \mathcal{C}(a,b) \to \mathcal{C}(a,c) \quad \forall a,b,c \in Obj(\mathcal{C}) \\ \text{identity elements} & i_a : I \to \mathcal{C}(a,a) \quad \forall a \in Obj(\mathcal{C}) \end{array}$$

such that composition is associative and unital.

A **$\mathcal{V}$-functor** $F : \mathcal{C} \to \mathcal{D}$ is:

$$\begin{array}{rl} \text{a function} & F_0 : Obj(\mathcal{C}) \to Obj(\mathcal{D}) \\ \text{hom-functions} & F_{ab} : \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb) \quad \forall a,b \in \mathcal{C} \end{array}$$

such that F is compatible with composition and identity.

A $\mathcal{V}$**-natural transformation** $\alpha : F \Rightarrow G$ is:

$$\text{a family} \quad \alpha_a : I \to \mathcal{D}(Fa, Ga) \quad \forall a \in Obj(\mathcal{C})$$

such that α is "natural" in a. Hence there is a *2-category* $\mathcal{V}$Cat of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. See [6] for reference.

Let $\mathcal{V}$ be a **closed symmetric monoidal category**, providing

$$\begin{array}{rl} \text{internal hom} & [-,-] : \mathcal{V}^{\mathrm{op}} \otimes \mathcal{V} \to \mathcal{V} \\ \text{symmetry braiding} & \tau_{a,b} : a \otimes b \cong b \otimes a \quad \forall a, b \in Obj(\mathcal{C}) \\ \text{tensor-hom adjunction} & \mathcal{V}(a \otimes b, c) \cong \mathcal{V}(a, [b, c]) \quad \forall a, b, c \in Obj(\mathcal{V}) \end{array}$$

Then $\mathcal{V}$ is itself a $\mathcal{V}$-category, denoted $\tilde{\mathcal{V}}$, with internal hom as the hom-object function. The tensor-hom adjunction generalizes to an *action* of $\mathcal{V}$ on any $\mathcal{V}$-category $\mathcal{C}$: for $x \in Obj(\mathcal{V})$ and $a, b \in Obj(\mathcal{C})$, the **power** of b by x and the **copower** of a by x are objects of $\mathcal{C}$ which represent the adjunction:

$$\mathcal{C}(a \odot x, b) \cong \mathcal{V}(x, \mathcal{C}(a, b)) \cong \mathcal{C}(a, x \pitchfork b) \tag{1}$$

and $\mathcal{C}$ is $\mathcal{V}$-powered or copowered if all powers or copowers exist.

These are the two basic forms of enriched limit and colimit, which are not especially intuitive; but they are a direct generalization of a familiar idea in the category of sets. In Set, the power is the "exponential" function set and the copower is the product. To generalize this to an action on other Set-categories, note that:

$$\begin{array}{rcl} X \pitchfork Y = & Y^X & \cong \prod_{x \in X} Y \\ X \odot Y = & X \times Y & \cong \coprod_{y \in Y} X \end{array}$$

So, categories are canonically Set-powered or copowered by indexed products or coproducts of copies of an object, provided that these exist. Even though the definition of Lawvere theory seems to be all about products, it is actually about *powers*, because these constitute the *arities* of the operations. This is precisely what is generalized in the enriched form.

(We will use exponential notation $x \pitchfork b = b^x$, and denote the unit I by 1, because the enriching categories under consideration are cartesian.)

There are just a few more technicalities. Given a $\mathcal{V}$-category $\mathcal{C}$, one often considers the Yoneda embedding into the $\mathcal{V}$-presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, and it is important if certain subcategories are representable; generally, some properties of $\mathcal{C}$ depend on a condition of "finitude." [1] A category is **locally finitely presentable** if it is the category of models for a *sketch*, which is a generalization of Lawvere theory to finite limits, and an object is finitely presentable or **finite** if its representable functor is *finitary*, or preserves filtered colimits. A $\mathcal{V}$-category $\mathcal{C}$ is locally finitely presentable if the underlying category $\mathcal{C}_0$ is LFP, $\mathcal{C}$ has finite powers, and $(-)^x : \mathcal{C}_0 \to \mathcal{C}_0$ is finitary. The details are not crucial - all categories to be considered are locally finitely presentable. Denote by $\mathcal{V}_f$ the subcategory of $\mathcal{V}$ of finite objects - in Gph, these are simply graphs with finitely many vertices and edges.

4 Enriched Lawvere theories

All of these abstract definitions culminate in the central concept: for a symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$, a $\mathcal{V}$*-enriched Lawvere theory* à la Power [14] is a finitely-powered $\mathcal{V}$-category $\mathcal{T}$ equipped with a strictly power-preserving bijective-on-objects $\mathcal{V}$-functor $\iota : \mathcal{V}_f^{\mathrm{op}} \to \mathcal{T}$. A *model* of a $\mathcal{V}$-theory is a finite-power $\mathcal{V}$-functor $\mu : \mathcal{T} \to \mathcal{V}$, and $\mathcal{V}$-natural transformations between them form the $\mathcal{V}$-category of models $[\mathcal{T}, \mathcal{V}]_{fp}$. The monadic adjunction and equivalence of §2 generalize to $\mathcal{V}$-theories, as originally formulated by Power.

However, this requires $\mathcal{T}$ to have *all* powers of $\mathcal{V}_f$, i.e. the theory must have arities for every finite object of $\mathcal{V}$. It is certainly useful to include these generalized arities, but this introduces the question of how to *present* such a theory; this is not nearly as straightforward as n-ary operations, and to the authors' knowledge a general method of enriched presentation does not yet exist. However, this is not needed for our purposes - we only need *natural number* arities, while still retaining enrichment.

A very general and useful definition of enriched algebraic theory was introduced by Lucyshyn-Wright [10], which allows for theories to be parameterized by a **system of arities**, a full subcategory inclusion $j : \mathcal{J} \hookrightarrow \mathcal{V}$ containing the monoidal unit and closed under tensor.

Definition 4.1. A $\mathcal{V}$-enriched algebraic theory with j-arities or $\mathcal{J}$**-**$\mathcal{V}$ **theory** $(\mathcal{T}, \tau)$ is a $\mathcal{V}$-category $\mathcal{T}$ equipped with a $\mathcal{J}$-power preserving bijective-on-objects $\mathcal{V}$-functor $\tau : \tilde{\mathcal{J}}^{\mathrm{op}} \to \mathcal{T}$. A **model** of this theory in a $\mathcal{V}$-category $\mathcal{C}$ is a finite-power preserving $\mathcal{V}$-functor $\mathcal{T} \to \mathcal{C}$.

A $\mathcal{J}$-$\mathcal{V}$ theory is essentially a $\mathcal{V}$-category with objects being $\mathcal{J}$-powers s^J of a generating object s, for $J \in \mathcal{J}$ - note that s itself is s^1. In the same way that every $n \in \mathrm{N}^{\mathrm{op}}$ is a power of $1 \in \mathrm{Set}$, every $J \in \tilde{\mathcal{J}}$ is a power of the monoidal unit $1 \in \mathcal{V}$ (using equation 1):

$$\tilde{\mathcal{J}}(J \odot 1, J) \cong \mathcal{V}(1, \tilde{\mathcal{J}}(J, J)) \cong \tilde{\mathcal{J}}(J, J^1)$$

This is just the direct generalization of the usual isomorphisms $J \times 1 \simeq J \simeq J^1$. Since a τ preserves $\mathcal{J}$-powers, this implies that every object of $\mathcal{T}$ is a power of $s = \tau(1)$.

Of course, these form categories: $\mathcal{J}$-$\mathcal{V}$ theories and $\mathcal{J}$-power preserving $\mathcal{V}$-functors $f : \mathcal{T} \to \mathcal{T}'$ such that $f\tau = \tau'$ give the category $\mathcal{V}$Law; and for every theory $\mathcal{T}$ and every $\mathcal{V}$-category with $\mathcal{J}$-powers, there is the category $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$ of functors and $\mathcal{V}$-natural transformations between them. (Note: if $\mathcal{V}$ is a *cosmos*, i.e. complete and cocomplete, then $\mathcal{V}$Cat has enriched functor categories - hence $\mathcal{V}$Law and $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$ are actually $\mathcal{V}$-categories. This is potentially useful, and the "operational $\mathcal{V}$'s of this paper are indeed cosmoi.)

Here is an overview of the concepts:

$$\begin{array}{rcccl} j: & \mathcal{J} & \hookrightarrow & \mathcal{V} & \text{arities} \\ & & & \frown & \text{enrichment} \\ \tau: & \tilde{\mathcal{J}}^{\mathrm{op}} & \to & \mathcal{T} & \text{theory} \\ & & & (\downarrow) & \text{models} \\ & & & \mathcal{C} & \text{semantics} \end{array}$$

This subsumes existing formulations; for example, Power's definition is the case $\mathcal{J} = \mathcal{V}_f$. A system of arities is **eleutheric** if left Kan extensions along j exist and are preserved by $\mathcal{V}(K, -)$ for all $K \in Ob(\mathcal{J})$. This is what is needed to have the essential *monadicity* theorems: Lucyshyn-Wright proved that any $\mathcal{J}$-$\mathcal{V}$ theory for an eleutheric system of arities has a category of models for $\mathcal{C} = \mathcal{V}$ which is *monadic* over $\mathcal{V}$, and the induced $\mathcal{V}$-monad is "j-ary" in that it "conditionally preserves $\mathcal{J}$-flat colimits", i.e. can be thought of as a monad with $\mathcal{J}$ arities.

The usual kinds of arities were all proved to be eleutheric: in particular, finite cardinals. Hence, N-$\mathcal{V}$ theories have all of the nice relations with monads as ordinary Lawvere theories - now they have the rich "operational" information of $\mathcal{V}$, and even this $\mathcal{V}$ itself is adaptable.

5 Change of Base

We propose a general framework in which one can *transition* seamlessly between different forms of operational semantics: small-step, big-step, full-step, denotational:

$$\begin{tikzcd} \mathrm{Gph} \arrow[r, bend left, "F_C"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "U_G"] \arrow[r, bend left, "F_P"] \arrow[r, phantom, "\perp"] & \mathrm{Pos} \arrow[l, bend left, "U_C"] \arrow[r, bend left, "U_S"] \arrow[r, phantom, "\top"] & \mathrm{Set} \arrow[l, bend left, "F_P"] \end{tikzcd}$$

This is effected by a **monoidal functor** - a functor

$$(F, \lambda, \upsilon) : (\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}) \to (\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}})$$

which transfers the tensor and unit via the *laxor* and *unitor*

$$\begin{array}{rl} \lambda : & F(a) \otimes_{\mathcal{W}} F(b) \to F(a \otimes_{\mathcal{V}} b) \\ \upsilon : & I_{\mathcal{W}} \to F(I_{\mathcal{V}}) \end{array}$$

such that λ is natural in a, b and associative, and unital relative to υ.

This induces a **change of base** functor $F_* : \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ [3]. If $f : \mathcal{C} \to \mathcal{D} \in \mathcal{V}\mathrm{Cat}$ is a $\mathcal{V}$-functor, then $F_*(f)_{obj} = f_{obj}$ and $F_*(f)_{hom} = F \circ f_{hom}$, and $F_*(\mathcal{C})$ is defined:

$$\begin{array}{rl} \text{objects} & Obj(\mathcal{C}) \\ \text{hom-function} & F \circ \mathcal{C}(-,-) \\ \text{composition} & F(\circ_{a,b,c}) \circ \lambda \\ \text{identity} & F(i_a) \circ \upsilon \end{array}$$

The change of base operation forms a 2-functor

$$\begin{array}{ccc} \mathrm{MonCat} & \xrightarrow{(-)_*} & 2\mathrm{Cat} \\ (\mathcal{V} \to \mathcal{W}) & \mapsto & (\mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}) \end{array}$$

In particular, there is an important correspondence of adjunctions:

$$\begin{tikzcd} \mathrm{Set} \arrow[rr, bend left, "-\odot I"] \arrow[rr, phantom, "\perp"] & & \mathcal{V} \arrow[ll, bend left, "\mathcal{V}(I,-)"] \arrow[r, mapsto] & \mathrm{Cat} \arrow[rr, bend left, "(-\odot I)_*"] \arrow[rr, phantom, "\perp"] & & \mathcal{V}\mathrm{Cat} \arrow[ll, bend left, "(\mathcal{V}(I,-))_*"] \end{tikzcd}$$

Each set X is represented in $\mathcal{V}$ as the X-indexed coproduct of the unit object, and conversely, each object a of $\mathcal{V}$ has is represented in Set by the hom-set from the unit to a. This process induces the change of base whereby ordinary Set-categories are converted to $\mathcal{V}$-categories, denoted $\mathcal{C} \mapsto \tilde{\mathcal{C}}$.

This is precisely what is needed: the "arity" category N sits inside many enriching categories under various guises: as *finite discrete graphs*, *categories*, *posets*, etc. For each $\mathcal{V}$ we can define the arity subcategory $\mathrm{N}_{\mathcal{V}}$ to be the full subcategory of finite coproducts (copowers) of the unit object, and this remains essentially unchanged by the change-of-base above to the $\mathcal{V}$-category $\tilde{\mathrm{N}}_{\mathcal{V}}$.

We only need to show that everything is simplified by restricting to this particular $\mathcal{J}$.

6 Simplify with N-arities

Most of the enriched algebraic theory literature deals with generalized arities; these will be important in time, but for present applications, we would like the benefits of enrichment with the simplicity of natural number arities. Here we provide some lemmas for this simplification.

Let $(\mathcal{V}, \times, I_{\mathcal{V}})$ be a cartesian closed category with finite coproducts. Define $\mathrm{N}_{\mathcal{V}}$ to be the full subcategory of finite coproducts of the unit object:

$$n_{\mathcal{V}} = \coprod_{n \in \mathrm{N}} I_{\mathcal{V}}$$

which is the *copower* of $I_{\mathcal{V}}$ by a finite set $n \in \mathrm{N}$, characterized by the universal property

$$\mathcal{V}(n_{\mathcal{V}}, a) = \mathcal{V}(I_{\mathcal{V}} \odot n, a) \simeq \mathrm{Set}(n, \mathcal{V}(I_{\mathcal{V}}, a))$$

This is our "system of arities", the full monoidal subcategory $\mathcal{J} \hookrightarrow \mathcal{V}$. We will call N-$\mathcal{V}$ theories **$\mathcal{V}$-theories** for simplicity.

The point is that instead of thinking about fancy enriched powers, we just want to think about good old products:

Lemma 1. Let V and $\mathrm{N}_{\mathcal{V}}$ be as above. Then $n_{\mathcal{V}}$-powers in $\tilde{\mathcal{V}}$ are isomorphic to n-powers, i.e. n-fold products, in $\mathcal{V}$.

Proof.

$$\begin{array}{rcll} \mathcal{V}(a,[n_{\mathcal{V}},b]) & \simeq & \mathcal{V}(a \times n_{\mathcal{V}}, b) & \text{hom-tensor adjunction} \\ & = & \mathcal{V}(a \times (\coprod_n I_{\mathcal{V}}), b) & \text{definition of } n_{\mathcal{V}} \\ & \simeq & \mathcal{V}(\coprod_n (a \times I_{\mathcal{V}}), b) & \text{distributivity} \\ & \simeq & \mathcal{V}(\coprod_n a, b) & \text{unitality} \\ & \simeq & V(a, \prod_n b) & \text{co/continuity of hom} \end{array}$$

Note that each of these isomorphisms is *natural* in a; hence by the Yoneda lemma, this implies that $[n_{\mathcal{V}}, b] \simeq \prod_n b$. □

So, the *full* sub-$\mathcal{V}$-category $\tilde{\mathrm{N}}_{\mathcal{V}}$ has hom-objects which are essentially sets:

$$[n_{\mathcal{V}}, m_{\mathcal{V}}] \simeq \textstyle\prod_n (\coprod_m I_{\mathcal{V}}) \;\; (\text{“ } \simeq m^n \text{ ”})$$

In $\mathcal{V}$Cat, the objects of the theory $\mathcal{T}$ are $n_{\mathcal{V}}$-powers of a generating object s. Alas, we cannot simply say that “$s^{n_{\mathcal{V}}} \simeq \prod_n s$”, because the latter does not make sense in the $\mathcal{V}$-category $\mathcal{T}$; products are characterized by a Set-enriched universal property. However, we only need that homs into $s^{n_{\mathcal{V}}}$ are equivalent to n homs into s:

Lemma 2. Let $\mathcal{T}$ be a $\mathcal{V}$-category with $\mathrm{N}_{\mathcal{V}}$-powers.

$$\begin{array}{rcll} \mathcal{T}(a, s^{n_{\mathcal{V}}}) & \simeq & [n_{\mathcal{V}}, \mathcal{T}(a,s)] & \\ & = & [\coprod_n I_{\mathcal{V}}, \mathcal{T}(a,s)] & \\ & \simeq & \prod_n [I_{\mathcal{V}}, \mathcal{T}(a,s)] & \simeq \quad \prod_n \mathcal{T}(a,s) \end{array}$$

If the functor $F : \mathcal{V} \to \mathcal{W}$ induces a change of base $F_* : \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ which preserves $\mathcal{V}$-theories - i.e. every $\mathcal{V}$-theory $\tau_{\mathcal{V}}$ corresponds to an $\mathcal{W}$-theory $\tau_{\mathcal{W}}$ - then F is a *change of semantics*. Since the powers $s^{n_{\mathcal{V}}}$ are the only objects of $\mathcal{T}$, it suffices to determine when the above universal property is preserved. Because the homs of base change are defined

$$F_*(\mathcal{T})(a, s^{n_{\mathcal{V}}}) = F(\mathcal{T}(a, s^{n_{\mathcal{V}}}))$$

we only need F to preserve finite products:

Lemma 3. Let $F : \mathcal{V} \to \mathcal{W}$ be a product-preserving functor, and let $\mathrm{N}_{\mathcal{V}}$, $\mathrm{N}_{\mathcal{W}}$ be defined as above. If $f : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-functor which preserves $\mathrm{N}_{\mathcal{V}}$-powers, then $F_*(f) : F_*(\mathcal{C}) \to F_*(\mathcal{D})$ is a $\mathcal{W}$-functor which preserves $\mathrm{N}_{\mathcal{W}}$-powers. *****

Proof.

$$F_*(f)(\mathcal{D}(a, s^{n_{\mathcal{V}}})) = F(f(\mathcal{D}(a, s^{n_{\mathcal{V}}}))) \simeq F(f(\textstyle\prod_n \mathcal{D}(a,s))) \simeq \prod_n F(\mathcal{D}(a,s)) = \prod_n F_*(\mathcal{D})(a,s) \simeq F_*(\mathcal{D})(a, s^{n_{\mathcal{W}}})$$

□

Then finally, we simply use $F_*(\tau)$ and the isomorphism $N : \mathrm{N}_{\mathcal{V}} \simeq \mathrm{N}_{\mathcal{W}}$ to construct a $\mathcal{W}$-functor which precisely fits the definition of an N-$\mathcal{W}$ theory:

Theorem 4. Let $\mathcal{V}$, $\mathcal{W}$ be cartesian closed categories with finite coproducts; let $F : \mathcal{V} \to \mathcal{W}$ be a product-preserving functor, and denote by $\tilde{N} : \tilde{\mathrm{N}}_{\mathcal{W}}^{\mathrm{op}} \to F_*(\tilde{\mathrm{N}}_{\mathcal{V}}^{\mathrm{op}})$ the isomorphism which sends $n_{\mathcal{W}} \mapsto n_{\mathcal{V}}$ and is the identity on morphisms. Then F is a **change of semantics**; i.e. for every N-$\mathcal{V}$ theory $\tau_{\mathcal{V}} : \mathrm{N}_{\mathcal{V}}^{\mathrm{op}} \to \mathcal{T}$, the $\mathcal{W}$-functor $\tau_{\mathcal{W}} := F_*(\tau_{\mathcal{V}}) \circ \tilde{N} : \mathrm{N}_{\mathcal{W}}^{\mathrm{op}} \to F_*(\mathcal{T})$ is an N-$\mathcal{W}$ theory. Moreover, F preserves *models*, i.e. for every N-$\mathcal{V}$-power preserving $\mu : \mathcal{T} \to \mathcal{C}$, $F_*\mu$ is $\mathrm{N}_{\mathcal{W}}$-power preserving.

Proof. The $\mathcal{W}$-functor $\tau_\mathcal{W}$ is clearly bijective-on-objects, because τ and $\tilde{N}$ are. It preserves $\mathrm{N}_\mathcal{W}$-powers by the previous lemma and:

$$\begin{array}{rcll} \tau_\mathcal{W}(\tilde{\mathrm{N}}^{\mathrm{op}}_\mathcal{W}(I_\mathcal{W}^{m_\mathcal{W}}, I_\mathcal{W}^{n_\mathcal{W}})) & \simeq & F_*(\mathcal{T})(F_*(\tau_\mathcal{V})(\tilde{N}(I_\mathcal{W}^{m_\mathcal{W}})), F_*(\tau_\mathcal{V})(\tilde{N}(I_\mathcal{W}^{n_\mathcal{W}}))) & \text{definition of } \tau_\mathcal{W} \\ & = & F_*(\mathcal{T})(F_*(\tau_\mathcal{V})(I_\mathcal{V}^{m_\mathcal{V}}), F_*(\tau_\mathcal{V})(I_\mathcal{V}^{n_\mathcal{V}})) & \text{isomorphism } \tilde{\mathrm{N}} \\ & \simeq & F(\mathcal{T}(\tau_\mathcal{V}(I_\mathcal{V})^{m_\mathcal{V}}, \tau_\mathcal{V}(I_\mathcal{V})^{n_\mathcal{V}})) & \tau_\mathcal{V} \text{ preserves products} \\ & \simeq & \prod_n F(\mathcal{T}(\tau_\mathcal{V}(I_\mathcal{V})^{m_\mathcal{V}}, \tau_\mathcal{V}(I_\mathcal{V}))) & \text{Lemma 2 and 3} \\ & \simeq & F_*(\mathcal{T})(\tau_\mathcal{V}(I_\mathcal{V})^{m_\mathcal{V}}, \tau_\mathcal{W}(I_\mathcal{W})^{n_\mathcal{W}}) & \text{Lemma 1} \end{array}$$

This preservation is *strict* because $F_*(\mathcal{T})$ has the same objects as $\mathcal{T}$, so this isomorphism implies that $\tau_\mathcal{W}(I_\mathcal{W}^{n_\mathcal{W}}) = \tau_\mathcal{W}(I_\mathcal{W})^{n_\mathcal{W}}$. The preservation of models follows from the previous lemma. □

Hence, any product-preserving functor between cartesian closed categories constitutes a "change of semantics" - this is a simple, ubiquitous condition, which provides for a method of transitioning formal systems between various *modus operandi*.

7 Putting It All Together

In addition to change-of-base, there are two other natural and useful transitions for these theories. Let $\mathcal{V}$Law be the category of $\mathcal{V}$-theories, and let $f : \mathcal{T} \to \mathcal{T}'$ be a morphism of theories; this induces a "change-of-theory" functor between the respective categories of models

$$f^* : \mathcal{V}\mathrm{Mod}(\mathcal{T}', \mathcal{C}) \to \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C})$$

given by precomposition by f. Similarly, given an N-power preserving functor $g : \mathcal{C} \to \mathcal{C}'$, this induces a "change of model" functor

$$g_* : \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \to \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C}')$$

given by postcomposition by g.

All of this can be packaged up nicely using the **Grothendieck construction**: given a (pseudo)functor $F : \mathcal{D} \to \mathrm{Cat}$, there is a *fibration* $\tilde{F} : \int F \to \mathcal{D}$ which encapsulates all of the categories in the image of F - the category $\int F$ consists of pairs $(d, x) : d \in \mathcal{D}, x \in F(d)$ and morphisms $(f : d \to d', a : F(f)(x) \to x')$. (Although we noted after Definition 4.1 that $\mathcal{V}$Law and $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$ are $\mathcal{V}$-categories when $\mathcal{V}$ is nice, we will focus on the nonenriched case for simplicity and generality.)

This idea allows us to bring together *all* of the different enrichments, theories, and models into *one* big category. For every enriching category $\mathcal{V}$, let $\mathcal{V}\mathrm{Cat}_{fp}$ be the subcategory of $\mathcal{V}$Cat of $\mathcal{V}$-categories with finite powers and finite-power preserving functors; then there is a functor

$$\mathcal{V}\mathrm{Mod} : \mathcal{V}\mathrm{Law}^{\mathrm{op}} \times \mathcal{V}\mathrm{Cat}_{fp} \to \mathrm{Cat}$$

which sends $(\mathcal{T}, \mathcal{C}) \mapsto \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C})$. Functoriality characterizes the contravariant change-of-theory and the covariant change-of-model above. Utilizing the construction, there is a category

$$\int \mathcal{V}\mathrm{Mod}$$

with objects and morphisms

$$((f, g), \alpha) : ((\mathcal{T}, \mathcal{C}), \mu) \to ((\mathcal{T}', \mathcal{C}'), \mu')$$

being $\mathcal{V}$-functors $f : \mathcal{T} \to \mathcal{T}'$, $g : \mathcal{C} \to \mathcal{C}'$, and $\mathcal{V}$-natural transformation $\alpha : \mathcal{V}\mathrm{Mod}(f, g)(\mu) \to \mu'$.

Moreover, the assignment $\mathcal{V} \mapsto \int \mathcal{V}\mathrm{Mod}$ is itself a functor from cartesian closed categories to categories:

$$\mathrm{th} : \mathrm{CCC} \to \mathrm{Cat}$$

Given $F : \mathcal{V} \to \mathcal{W}$, base change $F_* : \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ is a 2-functor, thereby inducing the functor $\bar{F} : \mathcal{V}\mathrm{Mod} \to \mathcal{W}\mathrm{Mod}$ which sends a morphism $((f,g),\alpha)$ to $((F_*(f), F_*(g)), F_*(\alpha))$. Thus, we can utilize the construction *again* to encompass even the enrichment:

$$\mathrm{Thy} := \int \mathrm{th}$$

with objects and morphisms

$$(F, ((f,g),\alpha)) : (\mathcal{V}, ((\mathcal{T},\mathcal{C}),\mu)) \to (\mathcal{W}, ((\mathcal{T}',\mathcal{C}'),\mu'))$$

being product-preserving functor $F : \mathcal{V} \to \mathcal{W}$ (hence a change of semantics), $\mathcal{W}$-functors

This category assimilates a whole lot of useful information. Most importantly, there are morphisms between objects of "different kinds", something which we consider often but is normally not possible in category theory. For example, let $\mathcal{V}$ be Set, let $\mathbb{Z}$ be the ring of integers and let $\mathbb{R}$ be the topological group of real numbers. These are the objects $((\mathcal{T}_{\mathrm{Ring}}, \mathrm{Set}), \mathbb{Z})$ and $((\mathcal{T}_{\mathrm{Grp}}, \mathrm{Top}), \mathbb{R})$.

8 Applications

8.1 Combinatory Calculi

The λ-calculus is the elegant formal language which is the foundation of functional computation, the model of intuitionistic logic, and the internal logic of cartesian closed categories - this is the Curry-Howard-Lambek correspondence [2]. Despite its simplicity, there are subtle complications regarding *substitution*, or evaluation of functions. * If $f(x) = x(y)$, the x is *bound* by whatever is "plugged in", while y is *free*, meaning it can refer to some other constant or function. But if $f(y)$ is to be evaluated, one must *rename* the reference to another variable z, otherwise it will be "captured" as $y(y)$. *

This problem was noticed early in the history of mathematical foundations, even before the λ-calculus, and so Moses Schönfinkel invented *combinatory logic* [15] - a basic form of logic without the red tape of *variable binding*, hence without functions. The SKI-calculus is the *variable-free* representation of the λ-calculus; λ-terms are translated via *abstraction elimination* into strings of combinators and applications. This is an important method for programming languages to minimize the subtleties of variables.

The key insight here is that Lawvere theories are by definition free of variables. When representing a computational calculus as an N-Gph theory, the general rewrite rules are simply edges in the hom-graphs $t^n \to t$, with the object t serving in place of the variable. Below is the theory of the SKI-combinator calculus:

$$\boxed{\mathrm{Th}(SKI)}$$

Sorts	t	
Term Constructors	S	$: 1 \to t$
	K	$: 1 \to t$
	I	$: 1 \to t$
	$(-\ -)$	$: t^2 \to t$
Structural Congruence	n/a	
Rewrites	σ	$: (((S\ x)\ y)\ z) \Rightarrow ((x\ z)\ (y\ z))$
	κ	$: ((K\ y)\ z) \Rightarrow y$
	ι	$: (I\ z) \Rightarrow z$

These denote rewrites for arbitrary subterms x, y, z without any variable binding involved, by using the cartesian monoidal structure of the category. They are simply edges with vertices:

$$
\begin{aligned}
(((S\ x)\ y)\ z) \quad &: t^3 \xrightarrow{l^{-1}\times t^2} 1\times t^3 \xrightarrow{S\times t^3} t^4 \xrightarrow{(-\ -)\times t^2} t^3 \xrightarrow{(-\ -)\times t} t^2 \xrightarrow{(-\ -)} t \\
((x\ z)\ (y\ z)) \quad &: t^3 \xrightarrow{t^2\times\Delta} t^4 \xrightarrow{t\times\tau\times t} t^4 \xrightarrow{(-\ -)\times(-\ -)} t^2 \xrightarrow{(-\ -)} t
\end{aligned}
$$

$$
\begin{aligned}
((K\ y)\ z) \quad &: t^2 \xrightarrow{l^{-1}\times t} 1\times t^2 \xrightarrow{K\times t^2} t^3 \xrightarrow{(-\ -)\times t} t^2 \xrightarrow{(-\ -)} t \\
y \quad &: t^2 \xrightarrow{t\times !} t\times 1 \xrightarrow{r} t
\end{aligned}
$$

$$
\begin{aligned}
(I\ z) \quad &: t \xrightarrow{l^{-1}} 1\times t \xrightarrow{I\times t} t^2 \xrightarrow{(-\ -)} t \\
z \quad &: t \xrightarrow{t} t
\end{aligned}
$$

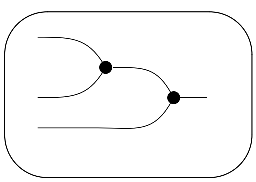
(Exposition of model, example of computation)
(Change of base encapsulates the change of semantics reviewed in every term calculi paper)

8.2 String Diagrams

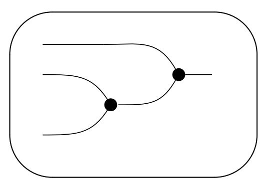
Pseudomonoids [4] are important in higher category theory. A monoidal category is "just a pseudomonoid in Cat!" The tensor is only associative and unital *up to isomorphism*; these are now reified as 2-morphisms in a Cat-enriched Lawvere theory, and can be understood as invertible rewrites of string diagrams:

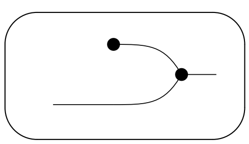
$\boxed{\text{Th(PsMon)}}$

$$M^2 \xrightarrow{m} M \xleftarrow{e} 1$$

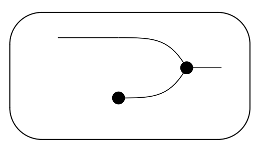


$$
\begin{tikzcd}
M^3 \arrow[r, "M\times m"] \arrow[d, "m\times M"'] \arrow[dr, phantom, "\overset{a}{\Longrightarrow}"] & M^2 \arrow[d, "m"] \\
M^2 \arrow[r, "m"'] & M
\end{tikzcd}
$$





$$
\begin{tikzcd}
M \arrow[r, "e\times M"] \arrow[d, "M\times e"'] \arrow[dr, equal] & M^2 \arrow[d, "m"] \arrow[dl, phantom, "\Swarrow l"] \\
M^2 \arrow[r, "m"'] \arrow[ur, phantom, "r \Nearrow"] & M
\end{tikzcd}
$$



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