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Quantum hypergraph states

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Quantum hypergraph states

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Abstract. We introduce a class of multiqubit quantum states which generalizes graph states. These states correspond to an underlying mathematical hypergraph, i.e. a graph where edges connecting more than two vertices are considered. We derive a generalized stabilizer formalism to describe this class of states. We introduce the notion of k -uniformity and show that this gives rise to classes of states which are inequivalent under the action of the local Pauli group. Finally we disclose a one-to-one correspondence with states employed in quantum algorithms, such as Deutsch–Jozsa’s and Grover’s.

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1. Introduction

Quantum algorithms constitute one of the main applications of modern quantum information theory. They offer computational speed-up, that provably no classical system could ever exhibit [1]. The crucial quantum property for such a speed-up remains a heavily debated open question up to date (see e.g. [2–8]). A famous way of implementing quantum algorithms is the measurement-based approach, where the computations are performed through the preparation of a highly entangled particular type of graph state (namely, a cluster state), which is subsequently processed via local measurements (introduced in [9]).

Some of the most prominent algorithms, however, are often phrased in the circuit model (e.g. Grover’s algorithm [10] or Deutsch–Jozsa’s algorithm [11]), where the algorithm is usually formulated in terms of an initialization of a real equally weighted (REW) pure state (i.e. a superposition of all basis states with real amplitudes and equal probabilities), on which quantum gates subsequently act and a final measurement concludes the computation. Thus, this family of states plays a central role in several quantum algorithms. From the construction of graph states, as reviewed below, it is obvious that they are special cases of such REW states. Due to the special properties of graph states it can also easily be seen that in a many-body system they can be created using only particular two-body interactions. The first question we address in this work is whether all REW states can be created using the two-body interactions occurring in graph states. We show that this is not the case, i.e. the set of REW states is strictly larger than the set of graph states.

From a physical point of view, where the n -qubits are a composite system of interacting spin 1/2 particles, it is then interesting to ask what kind of interactions are necessary to create all possible REW states. In this paper we answer this question by introducing hypergraph states, i.e. quantum states created by using up to n -body interactions of a given kind. We show that these states indeed cover all possible REW states, by providing an explicit simple procedure to find the associated hypergraph to a given REW state, and that they have an illustrative graph representation. We also find that they are stabilized by generalizations of the stabilizers of graph

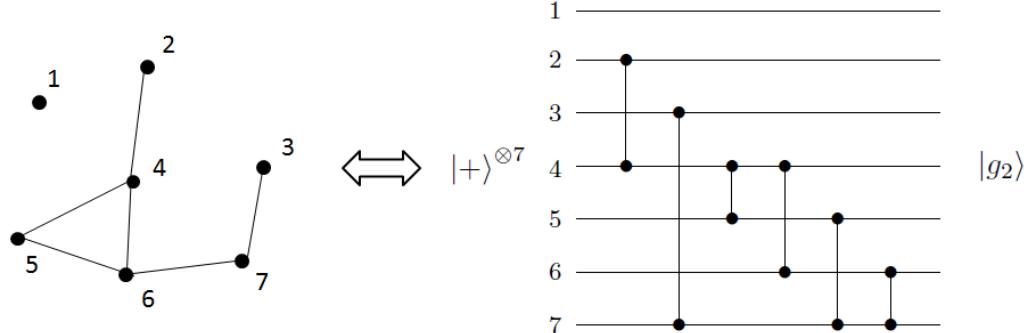


Figure 1. Correspondence between a mathematical graph and the quantum state associated to it. Since controlled- Z gates are symmetric, C^2Z gates are here depicted as two dots connected by a vertical line. This notation will also be used for general controlled- Z gates in the next figures.

states, and that, by introducing the notion of k -uniformity, they can be shown to constitute a set of different entanglement classes under local Pauli operations.

The present work is structured as follows. In section 2, we briefly review concepts related to standard graph states. We introduce and mathematically define k -uniform hypergraph states in section 3 and general hypergraph states in section 4. In section 5 the equivalence and connection with REW states employed in quantum algorithms is proven and discussed. We finally summarize our results in section 6 and discuss possible ways to extend our work.

2. Standard graph states

We will here briefly review some basic concepts related to graph states (following Hein *et al* [12]), for fixing the notation and introducing concepts that will be useful later. Given a mathematical graph $g_2 = \{V, E\}$, i.e. a set of n vertices V and a set of edges E , one can find the corresponding quantum graph state as follows. First, assign to each vertex a qubit and initialize each qubit as the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, so that the initial n -qubit state is given by $|+\rangle^{\otimes n}$. Then, perform controlled- Z operations between any two qubits that are connected by an edge. By performing the operation $C^2Z_{i_1 i_2} = \text{diag}(1, 1, 1, -1)$ for any two connected qubits i_1 and i_2 , we get the corresponding quantum graph state

$$|g_2\rangle = \prod_{\{i_1, i_2\} \in E} C^2Z_{i_1 i_2}|+\rangle^{\otimes n}, \quad (1)$$

where $\{i_1, i_2\} \in E$ means that the two vertices i_1 and i_2 are connected by an edge. This procedure is sketched for an example in figure 1, which clearly points out the correspondence. In this work we will denote by $\mathbf{1}$, X , Y and Z the identity and the three Pauli matrices σ_x , σ_y and σ_z , respectively. We will also denote by $C^k Z_{i_1 i_2 \dots i_k}$, the general controlled- Z gate acting on the k qubits, labelled by $i_1 i_2 \dots i_k$. Notice that k is an integer in the interval $1 \leq k \leq n$, and by definition we take $C^1 Z_{i_1} \equiv Z_{i_1}$. The gate $C^k Z_{i_1 i_2 \dots i_k}$ introduces a minus sign to the input state $|11\dots 1\rangle_{i_1 i_2 \dots i_k}$, i.e. $C^k Z_{i_1 i_2 \dots i_k} |11\dots 1\rangle_{i_1 i_2 \dots i_k} = -|11\dots 1\rangle_{i_1 i_2 \dots i_k}$, and leaves all the other components of the computational basis unchanged. Hence the action of the controlled- Z gate is invariant under permutations of the n qubits in the computational basis, thus any of the k

qubits on which $C^k Z_{i_1 i_2 \dots i_k}$ acts can be thought of as the target qubit. We will take $C^0 Z \equiv -1$, the reason will be clear later.

We call the set of all graph states G_2 , and denote a general state taken from this set by $|g_2\rangle$. The subscript 2 indicates that only two-body interactions, represented by $C^2 Z$, are involved. It is easy to show, by counting all possible edge configurations, that the number of graph states is $2^{B(n,2)}$, where $B(n, 2)$ is the binomial coefficient ‘ n choose 2’.

Graph states can alternatively be defined by exploiting the stabilizer formalism: given the graph g_2 , one defines a set of operators $\{K_i^{(2)}\}$ for which the state $|g_2\rangle$ is a simultaneous eigenvector with eigenvalue one. Explicitly, for any vertex i the correlation operation $K_i^{(2)}$ is given as follows:

$$K_i^{(2)} = X_i \otimes Z_{N(i)} = X_i \bigotimes_{j \in N(i)} Z_j, \quad (2)$$

where $N(i) = \{j | \{i, j\} \in E\}$ is the neighbourhood of the vertex i , namely the vertices j which are connected to i by an edge. Again, the index 2 refers to a two-body interaction, i.e. an edge between two vertices. Thus, the set of n operators $\{K_i^{(2)}\}_{i=1,2,\dots,n}$ uniquely defines the graph state $|g_2\rangle$ associated to the graph g_2 , according to

$$K_i^{(2)}|g_2\rangle = |g_2\rangle \quad \text{for every } i = 1, 2, \dots, n. \quad (3)$$

It can be shown that the set $\{K_i^{(2)}\}$ gives rise to a commutative subgroup called stabilizer (as each element of the group stabilizes the state $|g_2\rangle$, see equation (3)) of the Pauli group on n qubits, generated by the tensor product of the Pauli matrices X , Y and Z . The definitions of graph states based on the explicit procedure involving $C^2 Z$ gates and on the stabilizer formalism can be shown to be equivalent [12].

3. k -uniform hypergraph states

In this section, we generalize the notion of graph states allowing interactions which involve more than two parties. The mathematical tools needed to achieve this are k -uniform hypergraphs. A k -uniform hypergraph $g_k = \{V, E\}$ is a set of n vertices V with a set of edges E , where each edge connects exactly k vertices, and is called k -hyperedge (thus, a connected graph in the common sense is a two-uniform hypergraph).

Given a k -uniform hypergraph, by following a similar procedure as before, one can find the corresponding k -uniform quantum hypergraph state as follows. Assign to each vertex a qubit and initialize each qubit in the state $|+\rangle$. Wherever there is a k -hyperedge, perform a controlled- Z operation between the k connected qubits. Formally, if the qubits i_1, i_2, \dots, i_k are connected, then perform the operation $C^k Z_{i_1 i_2 \dots i_k}$. In this way we arrive at the state

$$|g_k\rangle = \prod_{\{i_1, i_2, \dots, i_k\} \in E} C^k Z_{i_1 i_2 \dots i_k} |+\rangle^{\otimes n}, \quad (4)$$

where $\{i_1, i_2, \dots, i_k\} \in E$ means that the k vertices are connected by a k -hyperedge. In figure 2, we show an explicit example of a three-uniform hypergraph and the circuit implementation of the corresponding quantum hypergraph state.

For a fixed k with $1 \leq k \leq n$, we call the class of k -uniform hypergraph states G_k and denote the quantum state associated with the general k -uniform hypergraph g_k as $|g_k\rangle$. The case $k = 1$ can be simply thought of as Z gates acting locally on single qubits, while the case $k = n$ is

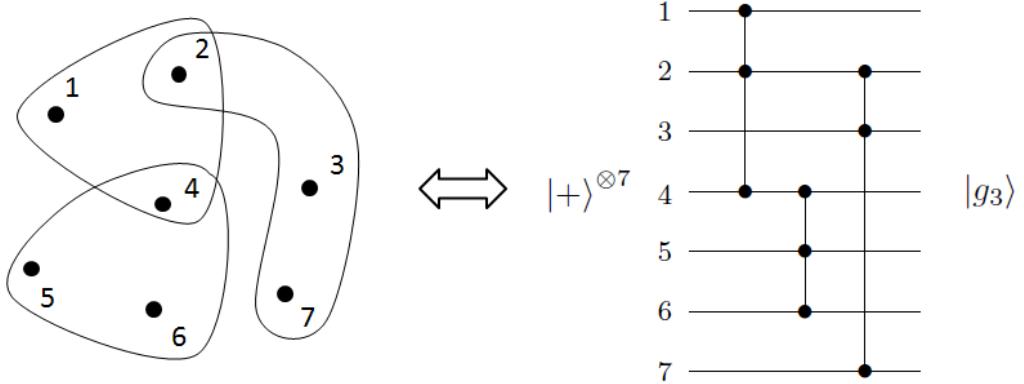


Figure 2. Correspondence between a mathematical three-uniform hypergraph and the quantum state associated to it. A hyperedge is visualized by a closed curve around a set of vertices.

the only one involving interactions among all n qubits, namely $C^n Z_{i_1 i_2 \dots i_n}$. Obviously, by setting $k = 2$ we recover the class of graph states. By the same counting argument used above, for fixed k the number of possible k -uniform hypergraphs is given by $2^{B(n,k)}$. We will now show that k -uniform hypergraph states can be described in terms of a generalized stabilizer formalism. Given a k -uniform hypergraph g_k , for each vertex $i = 1, 2, \dots, n$ we define the correlation operator

$$K_i^{(k)} = X_i \otimes C^{k-1} Z_{N(i)} = X_i \otimes \prod_{(i_1, i_2, \dots, i_{k-1}) \in N(i)} C^{k-1} Z_{i_1 i_2 \dots i_{k-1}}, \quad (5)$$

where the neighbourhood $N(i)$ of the vertex i is given by $N(i) = \{(i_1, i_2, \dots, i_{k-1}) \mid \{i, i_1, i_2, \dots, i_{k-1}\} \in E\}$, namely all $k - 1$ -tuples $(i_1, i_2, \dots, i_{k-1})$ of vertices connected to i via a k -hyperedge. Notice that, if k -body interactions are involved, then the stabilizers are defined in terms of controlled- Z gates acting on $k - 1$ qubits. Hence, the generalized stabilizers for general k no longer belong to the Pauli group, except in the case of graph states where we recover the stabilizer operators given by local Pauli matrices. These operators nevertheless can be shown to ‘stabilize’ the regarded state as follows.

The k -uniform hypergraph state $|g_k\rangle$ corresponding to g_k is then defined as the unique eigenvector with eigenvalues one of the n operators $\{K_i^{(k)}\}$, namely

$$K_i^{(k)} |g_k\rangle = |g_k\rangle \quad \text{for every } i = 1, 2, \dots, n. \quad (6)$$

The set of the operators generated by $\{K_i^{(k)}\}_{i=1,2,\dots,n}$ is an Abelian group (see appendix B). This Abelian group can be thought of as a subgroup of a generalized Pauli group which contains, besides the tensor product of Pauli matrices, also $C^{k-1} Z$ gates acting on any $k - 1$ -tuple of qubits as generators. Furthermore, as for standard graph states, the definition following the generalized stabilizer is completely equivalent to the constructive procedure involving controlled- $C^k Z$ operations. The equivalence can be explicitly derived in the more general case of non-uniform hypergraphs, that will be considered in the following, of which the k -uniform hypergraphs are a strict subset.

The classification induced by k -uniformity allows us to prove that two sets G_k and $G_{k'}$ cannot be connected by local Pauli operators for $k \neq k'$ (apart from the trivial separable state $|+\rangle^{\otimes n}$ which corresponds to the empty graph and thus is already contained in every class).

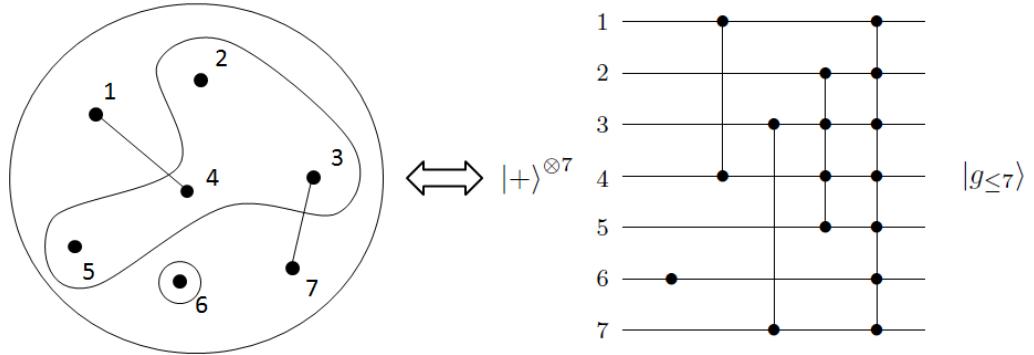


Figure 3. Correspondence between a mathematical hypergraph and the quantum hypergraph state. The circle around vertex 6 stands for a local Z gate, corresponding to a hyperedge of order $k = 1$, while the big circle around all vertices corresponds to a full-body interaction.

Therefore, each set G_k gives rise to an inequivalent class under the action of local Pauli group of n qubits (see appendix A). It is, however, an open question whether two sets G_k and $G_{k'}$ with $k \neq k'$ are inequivalent under the action of the general local unitaries. An affirmative answer to this question would imply a corresponding multipartite entanglement classification. Notice nevertheless that for the class of standard graph states, i.e. G_2 , it is known that there exist states which are local unitary equivalent, but not local Clifford equivalent [13]. This finally suggests that the local unitary inequivalence of G_k and $G_{k'}$ might be a hard problem to solve.

4. Hypergraph states

We are now ready to define a general hypergraph state as follows. A hypergraph $g_{\leq n} = \{V, E\}$ is a set of n vertices V with a set of hyperedges E of any order k (thus k is no longer fixed but may range from 1 to n). Given a mathematical hypergraph, the corresponding quantum state can be found by following the three steps: assign to each vertex a qubit and initialize each qubit as $|+\rangle$ (the total initial state is then given by $|+\rangle^{\otimes n}$). Wherever there is a hyperedge, perform a controlled- Z operation between all connected qubits. Formally, if the qubits i_1, i_2, \dots, i_k are connected by a k -hyperedge, then perform the operation $C^k Z_{i_1 i_2 \dots i_k}$. Hence, eventually, we obtain the quantum state

$$|g_{\leq n}\rangle = \prod_{k=1}^n \prod_{\{i_1, i_2, \dots, i_k\} \in E} C^k Z_{i_1 i_2 \dots i_k} |+\rangle^{\otimes n}, \quad (7)$$

where $\{i_1, i_2, \dots, i_k\} \in E$ means that the k vertices are connected by a k -hyperedge. Notice that the product of $k = 1, 2, \dots, n$ accounts for different types of hyperedges in the hypergraph.

We illustrate the correspondence with an example in figure 3. There, some hyperedges connecting one, two, four and seven vertices appear, and thus controlled- Z operations acting on one, two, four and seven qubits must be considered.

We denote the set of all general hypergraph states for graphs with n vertices as $G_{\leq n}$, indicating that hyperedges connecting up to n vertices are present. An element of this set, corresponding to the particular graph $g_{\leq n}$, is called $|g_{\leq n}\rangle$. Each set of k -uniform hypergraphs,

for fixed k , is obviously a subset of all possible hypergraphs. In order to count the number of hypergraph states, we exploit any possible combination of k -uniform hypergraphs. Since the number of the latter ones for fixed k is given by $2^{B(n,k)}$, the total number of hypergraph states for n vertices turns out to be $\prod_{k=1}^n 2^{B(n,k)} = 2^{\sum_{k=1}^n B(n,k)} = 2^{2^n - 1}$.

We will now describe general hypergraph states in a generalized stabilizer formalism. Given a general hypergraph, for any vertex i we define the following correlation operator:

$$K_i = X_i \otimes \prod_{k=1}^n C^{k-1} Z_{N(i)} = X_i \otimes \prod_{k=1}^n \prod_{(i_1, i_2, \dots, i_{k-1}) \in N(i)} C^{k-1} Z_{i_1 i_2 \dots i_{k-1}}, \quad (8)$$

where the product over k takes into account all kinds of hyperedges that appear. For any value of k , the neighbourhood $N(i)$ of the vertex i is still defined as $N(i) = \{(i_1, i_2, \dots, i_{k-1}) | \{i, i_1, i_2, \dots, i_{k-1}\} \in E\}$. Different kinds of neighbourhoods can obviously appear in this scenario (single vertices, couples and in general $k-1$ -tuples), depending on the order k of the hyperedges that connect the vertex i to other vertices. For instance, in figure 3 the neighbourhood $N(4)$ of vertex 4 consists of the tuples 1, (2, 3, 5) and (1, 2, 3, 5, 6, 7), since the vertex is connected via the depicted hyperedges of order 2, 4 and 7. We introduce a generalized stabilizer group, being generated by the set $\{K_i\}$, which stabilizes the corresponding hypergraph state. We show in appendix B that this group is Abelian. The unique hypergraph state corresponding to the set $\{K_i\}$ is then defined as the unique eigenvector with eigenvalues one of any generator K_i , i.e.

$$K_i |g_{\leq n}\rangle = |g_{\leq n}\rangle \quad \text{for every } i = 1, 2, \dots, n. \quad (9)$$

Furthermore, one can show (see appendix C) that the definition according to the generalized stabilizer formalism is equivalent to the one given above in terms of controlled-Z gates.

5. Real equally weighted states and their equivalence with hypergraph states

Let us now introduce a different set G_{\pm} of n -qubit states, namely the REW pure states, defined as

$$|f\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n - 1} (-1)^{f(x)} |x\rangle, \quad (10)$$

where $|x\rangle$ are the computational basis states, while $f(x)$ is a Boolean function, i.e. $f : \{0, 1\}^n \rightarrow \{0, 1\}$. The state $|f\rangle$ is uniquely defined by the function f via the signs (either plus or minus) in front of each component x of the computational basis. According to this, we can count the number of REW states, which turns out to be $2^{2^n - 1}$. These states are employed in many quantum protocols, and in particular in the well-known quantum algorithms of Deutsch–Jozsa and Grover. Notice that a more general class of equally weighted states, with generic phase factors in front of each computational basis state, and an explicit method to generate them was analysed in [14].

Is there any relation between REW states and graph states or hypergraph states? From the construction in equation (1) it is clear that the graph states are REW states, since the action of $C^2 Z$ can only produce some minus signs. Thus, $G_2 \subseteq G_{\pm}$ holds. But is the reverse also true, i.e. are all REW states graph states? A first hint that this is not the case comes from the fact that the number of REW states is exponentially larger than the number of graph states, i.e. $2^{2^n - 1}$

versus $2^{B(n,2)}$, see above. In order to prove that not every REW state is a graph state, we provide a counterexample, given by the state

$$|f\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle). \quad (11)$$

It is easy to show that the geometric measure of genuine multipartite entanglement [15] of the state above is $E_2(|f\rangle) = 1/4$ [7], however, every connected graph state has a multipartite geometric measure $E_2(|g_2\rangle) \geq 1/2$ [16].

Notice that, by construction, graph states involve only particular two-body interactions, which are not sufficient to achieve all REW states. We will now investigate the relation between REW states and the wider class of hypergraph states and state our main result: *the set G_{\pm} of REW states and the set $G_{\leq n}$ of hypergraph states coincide*. We prove this statement as follows: the inclusion $G_{\leq n} \subseteq G_{\pm}$ is trivial, since any $|g_{\leq n}\rangle$ is obtained from $|+\rangle^{\otimes n}$ by applying controlled-Z gates. The opposite inclusion $G_{\pm} \subseteq G_{\leq n}$ can be proved by the following constructive approach. Suppose we are given a REW state $|f\rangle$, then the following procedure leads to the underlying hypergraph. Firstly, erase all the minus signs of the states with one excitation, i.e. of the form $|0\dots01,0\dots0\rangle$, by applying local Z_j gates. Notice that, by doing this, we might create unnecessary minus signs in front of states with more than one excitation. Secondly, apply C^2Z gates in order to erase the negative signs in front of the components with two excitations (either coming from the original state $|f\rangle$ or as by-products of the previous step). Observe that, since C^2Z acts non-trivially only on states with more than one excitation, the minus signs previously erased will remain untouched. As a general rule, apply C^kZ operations, from $k=1$ until n , erasing successively the minus signs in front of the components of the computational basis. In general, at the step k , we have erased the minus signs in front of the states with up to k excitations. The set of gates that are needed to transform $|f\rangle$ back to $|+\rangle^{\otimes n}$ provides the underlying hypergraph for the REW state under examination. Notice that, since the procedure is uniquely defined according to the REW state from which we start, the underlying hypergraph is unique. Therefore, the correspondence between the sets G_{\pm} and $G_{\leq n}$ is one-to-one.

As an explicit example consider the three-qubit REW state

$$|f\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle). \quad (12)$$

It is straightforward to see that the sequence of transformations $Z_1, C^2Z_{23}, C^3Z_{123}$ applied to the above state leads to the initial state $|+\rangle^{\otimes 3}$. Therefore the hypergraph corresponding to (12) is the one depicted in figure 4.

Other interesting examples are REW states employed in quantum algorithms. In Grover's algorithm for instance, REW states with only one minus sign appear, such as the state (11) for three qubits (the minus sign marks the single solution of the search problem). It is easy to see that, when the number of minus signs is odd, the REW state always involves a controlled-Z gate acting on all the qubits, therefore involving n -body interactions. On the contrary, for REW states employed in Deutsch–Jozsa's algorithm, such a gate is never needed, since the function f is either constant or balanced (balanced means that the number of minus signs equals the number of plus signs), while the application of a controlled Z gate acting on all qubits would change just one sign in the n -qubit state, therefore necessarily leading to an unbalanced function. An explicit example of a balanced state is given by modifying the state in (12) such that a plus sign is in front of the component $|111\rangle$. Such a state would be generated by a sequence of the Z_1 and the C^2Z_{23} gates, without the application of C^3Z_{123} .

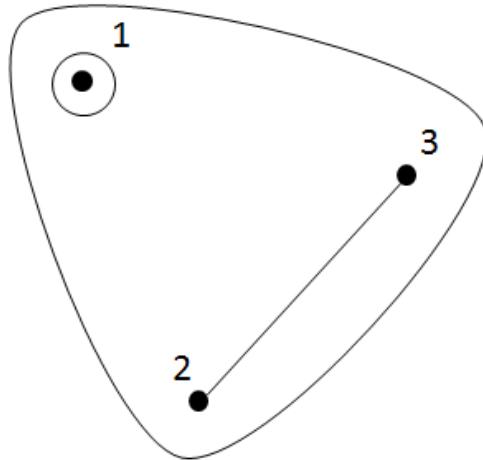


Figure 4. Hypergraph corresponding to the REW state given by equation (12).

6. Conclusions

In conclusion, we have introduced the class of quantum hypergraph states, which are associated to corresponding mathematical hypergraphs and are stabilized by non-local observables. We introduced the notion of k -uniformity and proved that this gives rise to classes of states which are inequivalent under the action of the local Pauli group. We showed that there is a one-to-one correspondence between the set of hypergraph states and REW states, which are essential for quantum algorithms. A constructive method was introduced which allows us to generate the hypergraph underlying a given REW state, i.e. a quantum state encoding a given Boolean function f . We have discussed the types of many-body interactions needed to generate general hypergraph states in a Hamiltonian description, going beyond the two-body interaction that characterizes the graph states.

For future studies, since the class of hypergraph states naturally generalizes the class of graph states, it will be of great interest to ask whether some of the many results about the latter, such as for instance measurement-based quantum computing [9], entanglement witnessing [16] and quantum error correcting techniques [17, 18], can be extended to the former. Some achievement in this sense already exists, mainly related to purification protocols [19]. Furthermore, this larger class of states may enable even more applications and quantum protocols, especially in connection to already existing algorithms employing hypergraphs, as e.g. the 3-SAT problem [20].

While finishing this paper, we learnt about related work [21] which contains a similar analysis. Subsequent works by some of the same authors address the issues of the characterization of three-qubit hypergraph states [22], and the relationship among hypergraph states, locally maximally entangleable states and W states [23].

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Appendix A. Inequivalence of k -uniform hypergraph states under the local Pauli group

In this appendix we prove that every k -uniform hypergraph state cannot be transformed to any other k' -uniform hypergraph with $k \neq k'$, by the only action of local Pauli operators, namely X , Y and Z .

Let us rewrite a general hypergraph state in the more convenient form

$$|g_{\leq n}\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} c_{\alpha_l^x} |x\rangle, \quad (\text{A.1})$$

where α_l^x denotes the set of cardinality l of subsystems of the state x that are in the state $|1\rangle$. In other words, given the state $|x\rangle$, α_l^x represents the set of indices corresponding to qubits where the excitations are. Then, having in mind that a general k -uniform hypergraph can be created from $|+\rangle^{\otimes n}$ using $\prod_j C^k Z_{\alpha_k^j}$ (α_k^j are index sets of cardinality k referring to the vertices on which the controlled Z operations act, for a given hyperedge j), it is easy to see that for any k -uniform state there is at least one c_{α_k} negative (condition C1) and all $c_{\alpha_{k'}}$ with $k' < k$ are positive (condition C2).

In the following we prove that, starting from a $k' < k$ -uniform hypergraph state, it is not possible to transform it into a k -uniform one by only using local X and Z . Notice that, as $Y = iXZ$, the Y operations are already considered. Furthermore, since X and Z anticommute, it is not restrictive to apply always Z before X . As a result the two following cases describe the most general strategy we could apply.

Case 1. We just use any $k' < k$ controlled Z operations (which includes local Z ’s when $k' = 1$). This nevertheless fails always because to make c_{α_k} negative you generate at least one $c_{\alpha_{k'}}$ with $k' < k$ which is negative as well which contradicts C2.

Case 2. We apply arbitrary $k' < k$ controlled Z operations and then include any number of X gates anywhere. We will now show that this procedure will fail again. Let us denote as c_{β_l} the coefficient that will afterwards be transformed to the negative coefficient c_{α_k} (l is of course arbitrary). We then need to apply X in $\gamma_l \equiv (\alpha_k \cup \beta_l) \setminus (\alpha_k \cap \beta_l)$ (such that $c_{\beta_l} \rightarrow c_{\alpha_k}$ and C1 holds). Now, since the action of X ’s cannot change the sign of the coefficient c_{β_l} , the number of $C^{k'} Z_{\alpha_{k'}}$ operations we apply in the set β_l must be odd (thus the number of different subsets $\alpha_{k'}$ must be odd as well). Let us denote this number as N_{β_l} , and the following N_S will always denote the number of sets $\alpha_{k'}$ (coming from $C^{k'} Z_{\alpha_{k'}}$ operations) included in the general set of indices S . We can then distinguish four different cases that may happen, summarized in table A.1.

By N_{cr} we mean the subsets $\alpha_{k'}$ that lie across the border of the set $\beta_l \setminus \alpha_k$ and the intersection $\beta_l \cap \alpha_k$ (see figure A.1 for a comprehensible explanation). Notice that $N_{\beta_l} = N_{\beta_l \setminus \alpha_k} + N_{\text{cr}} + N_{\beta_l \cap \alpha_k}$ must be odd from the hypothesis, while the number of sets $\alpha_{k'}$ in $\alpha_k \setminus \beta_l$, namely $N_{\alpha_k \setminus \beta_l}$, is instead not determined, and might be either odd or even. Notice that the number of subsets $\alpha_{k'}$ in γ_l is given by $N_{\gamma_l} = N_{\beta_l \setminus \alpha_k} + N_{\alpha_k \setminus \beta_l}$.

Table A.1. All possible cases for index sets—for an explanation, see main text.

$N_{\beta_l \setminus \alpha_k}$	N_{cr}	$N_{\beta_l \cap \alpha_k}$	$N_{\alpha_k \setminus \beta_l}$	$N_{\gamma_l'}$
Odd	Odd	Odd	Even	Odd (1)
			Odd	Even (2)
			Even	Even (3)
Even	Even	Odd	Odd	Odd (4)
			Even	Even (5)
			Odd	Odd (6)
Even	Odd	Even	Even	Odd (7)
			Odd	Even (8)
			Odd	Even (8)

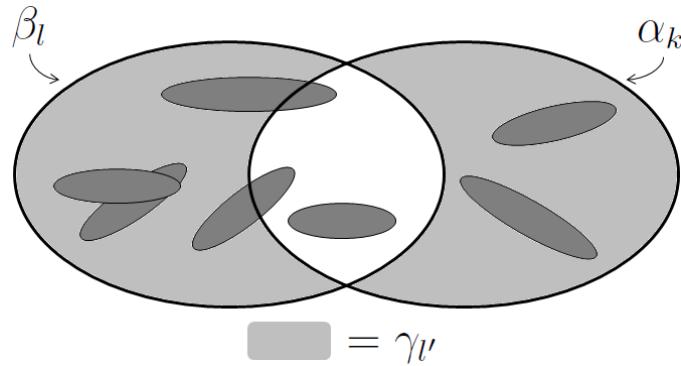


Figure A.1. Drawing showing an example for possible index sets. Each dark grey circle represents a set $\alpha_{k'}$. In this case $N_{\beta_l \setminus \alpha_k} = 2$, $N_{cr} = 2$, $N_{\beta_l \cap \alpha_k} = 1$, $N_{\alpha_k \setminus \beta_l} = 2$ and $N_{\gamma_l'} = 4$. This is an example of case (3) in table A.1. Notice that we do not take into account the case where subsets $\alpha_{k'}$ cross the border between the set $\alpha_k \setminus \beta_l$ and the intersection, since it is easy to see that this case never affects our counting.

For the cases (1)–(4)–(6)–(7) the contradiction to C2 can be found by realizing that $c_{\gamma_l'} = -1$ (since $N_{\gamma_l'}$ is odd). This coefficient will be mapped into $c_{\emptyset} = -1$ (the coefficient of the state with all zeros) by the action of $X_{\gamma_l'}$, and thus showing a contradiction to C2.

For the cases (2)–(3) the contradiction to C2 is given by $c_{\gamma_l' \cup (\alpha_{k'} \in \beta_l \cap \alpha_k)} = -1$ (since $N_{\gamma_l'} + N_{\beta_l \cap \alpha_k}$ is odd), which becomes $c_{(\alpha_{k'} \in \beta_l \cap \alpha_k)} = -1$ after $X_{\gamma_l'}$. By $\gamma_l' \cup (\alpha_{k'} \in \beta_l \cap \alpha_k)$ we mean the union between the set γ_l' and the sets $\alpha_{k'}$ which belong to the intersection $\beta_l \cap \alpha_k$.

For the case (8), the contradiction to C2 is $c_{\gamma_l' \setminus (\alpha_{k'} \in \alpha_k \setminus \beta_l)} = -1$, since this coefficient is mapped by $X_{\gamma_l'}$ into $c_{(\alpha_{k'} \in \alpha_k \setminus \beta_l)} = -1$. By $\gamma_l' \setminus (\alpha_{k'} \in \alpha_k \setminus \beta_l)$ we mean the difference between the set γ_l' and the sets $\alpha_{k'}$ which belong to the set given by $\alpha_k \setminus \beta_l$.

Regarding the case (5), since N_{cr} is odd we can always find a subset θ_t in the intersection $\beta_l \cap \alpha_k$ with cardinality $t < k$ such that the coefficient $c_{(\beta_l \setminus \alpha_k) \cup \theta_t} = -1$. Therefore, when we apply $X_{\gamma_l'}$ this will be mapped into $c_{(\alpha_k \setminus \beta_l) \cup \theta_t} = -1$ which clearly shows a contradiction to C2 since $(\alpha_k \setminus \beta_l) \cup \theta_t$ is a subset of α_k with cardinality strictly smaller than k .

Appendix B. Group structure of the generalized stabilizer operators

We now prove that the operators $\{K_i\}_{i=1,2,\dots,n}$ defined in equation (8) generate an Abelian group. The group properties follow immediately: the closure is given by construction, the associativity by the matrix algebra, the identity and the inverse belong to the set since $K_i^2 = \mathbf{1}$ and $K_i = K_i^\dagger$ hold, respectively.

On the other hand, the commutativity can be proved as follows. Suppose we are given K_i and K_j with $i \neq j$, otherwise everything trivializes. Since the concept of neighbourhood is symmetric we can keep K_i fixed and see what happens for different K_j . If j is not in $N(i)$ then the stabilizer operators trivially commute. Therefore, the only situations we have to check is when $j \in N(i)$, namely when some of the CZ gates acting on $N(i)$ in the definition of K_i involves also the qubit j . Each of these gates takes the form $C^k Z_{j i_1 i_2 \dots i_{k-1}}$ (with k arbitrary) and generally does not commute with X_j defining K_j .

It is then easy to see that, in order to prove that $[K_i, K_j] = 0$, it is sufficient to show that

$$[(X_i \otimes C^k Z_{j i_1 i_2 \dots i_{k-1}}), (C^k Z_{i i_1 i_2 \dots i_{k-1}} \otimes X_j)] = 0, \quad (\text{B.1})$$

for any number of qubits $k - 1$ and vertices $i_1 i_2 \dots i_{k-1}$. This is because we can think to commute the two operators K_i and K_j by following a step-by-step procedure consisting of swapping each term $(X_i \otimes C^k Z_{j i_1 i_2 \dots i_{k-1}})$ of K_i with the corresponding term $(C^k Z_{i i_1 i_2 \dots i_{k-1}} \otimes X_j)$ of K_j .

In order to prove equation (B.1), we rewrite the general controlled Z gate acting on k qubits as

$$C^k Z_{j i_1 i_2 \dots i_{k-1}} = \mathbf{1}_j \otimes (\mathbf{1} - P)_{i_1 i_2 \dots i_{k-1}} + Z_j \otimes P_{i_1 i_2 \dots i_{k-1}}, \quad (\text{B.2})$$

where $P_{i_1 i_2 \dots i_{k-1}} = |11\dots1\rangle_{i_1 i_2 \dots i_{k-1}} \langle 11\dots1|$. Then, by exploiting the anti-commutativity of Pauli matrices, it follows that

$$\begin{aligned} & (X_i \otimes C^k Z_{j i_1 i_2 \dots i_{k-1}})(C^k Z_{i i_1 i_2 \dots i_{k-1}} \otimes X_j) \\ &= (X_i \otimes \mathbf{1}_j \otimes (\mathbf{1} - P)_{i_1 i_2 \dots i_{k-1}} + X_i \otimes Z_j \otimes P_{i_1 i_2 \dots i_{k-1}}) \\ &\quad \times (\mathbf{1}_i \otimes X_j \otimes (\mathbf{1} - P)_{i_1 i_2 \dots i_{k-1}} + Z_i \otimes X_j \otimes P_{i_1 i_2 \dots i_{k-1}}) \\ &= X_i \otimes X_j \otimes (\mathbf{1} - P)_{i_1 i_2 \dots i_{k-1}} + X_i Z_i \otimes Z_j X_j \otimes P_{i_1 i_2 \dots i_{k-1}} \\ &= X_i \otimes X_j \otimes (\mathbf{1} - P)_{i_1 i_2 \dots i_{k-1}} + Z_i X_i \otimes X_j Z_j \otimes P_{i_1 i_2 \dots i_{k-1}} \\ &= (C^k Z_{i i_1 i_2 \dots i_{k-1}} \otimes X_j)(X_i \otimes C^k Z_{j i_1 i_2 \dots i_{k-1}}). \end{aligned} \quad (\text{B.3})$$

Thus, since the commutativity relation stated in equation (B.1) holds for any $k - 1$ and qubits $i_1 i_2 \dots i_{k-1}$, the commutativity of any two stabilizers defined by (8) finally follows.

Appendix C. Equivalence of the circuital definition and the stabilizers description

In order to prove that the two definitions stated in the main paper are equivalent we will essentially follow Hein *et al* [12]. The proof is by induction on the number of hyperedges. The case with no hyperedges is trivially stabilized by the Pauli matrices $\{X_1, X_2, \dots, X_n\}$, since the corresponding graph state is given by $|+\rangle^{\otimes n}$. Suppose now a general hypergraph state $|g_{\leq n}\rangle$, corresponding to the hypergraph $g_{\leq n}$, is stabilized by K_i as defined in (8), namely $K_i |g_{\leq n}\rangle = |g_{\leq n}\rangle$. We want to show that if we apply $C^k Z_{i_1 i_2 \dots i_k}$ to $|g_{\leq n}\rangle$, the new hypergraph state $|g'_{\leq n}\rangle = C^k Z_{i_1 i_2 \dots i_k} |g_{\leq n}\rangle$ is stabilized by a new stabilizer generated by K'_i , derived from the hypergraph $g'_{\leq n}$ where the k -hyperedge $\{i_1, i_2, \dots, i_k\}$ is added (or removed).

Namely we want to prove that $K'_i|g'_{\leq n}\rangle = |g'_{\leq n}\rangle$, where K'_i is defined according to (8) for the new hypergraph $g'_{\leq n}$. If we consider $i \neq i_1, i_2, \dots, i_k$ then by definition we have $K'_i = K_i$ and, since $[K_i, C^k Z_{i_1 i_2 \dots i_k}] = 0$, the following holds:

$$K'_i|g'_{\leq n}\rangle = |g'_{\leq n}\rangle \text{ for } i \neq i_1, i_2, \dots, i_k. \quad (\text{C.1})$$

So as for the proof regarding the commutativity of the stabilizer group, we need to focus only on the operators $\{K'_{i_1}, K'_{i_2}, \dots, K'_{i_k}\}$, since the others are not affected by the action of $C^k Z_{i_1 i_2 \dots i_k}$. Keeping in mind the decomposition (B.2) of $C^k Z_{i_1 i_2 \dots i_k}$, it is then easy to show that for every $i = i_1, i_2, \dots, i_k$ the following relation holds

$$C^k Z_{i_1 i_2 \dots i_k} K_i C^k Z_{i_1 i_2 \dots i_k} = C^{k-1} Z_{i_2 \dots i_k} K_i = K'_i \quad \text{for } i = i_1, i_2, \dots, i_k. \quad (\text{C.2})$$

Therefore, by exploiting the equation above, we can easily show that the hypergraph state $|g'_{\leq n}\rangle$ is eigenstate of K'_i with eigenvalue one for vertices $i = i_1, i_2, \dots, i_k$. Hence, it follows that the hypergraph state $|g'_{\leq n}\rangle$ is stabilized by any K'_i with $i = i_1, i_2, \dots, i_n$, which are the correlation operators that can be defined according to the hypergraph $g'_{\leq n}$.

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