Optimal Consumption and Bond Investment

Joscha Duchscherer

Abstract

This paper addresses the optimal consumption and bond investment problem under Epstein-Zin preferences in the presence of stochastic interest rates. We employ a finite difference method to numerically solve the corresponding Hamilton-Jacobi-Bellman (HJB) equation and examine how varying the elasticity of intertemporal substitution (EIS) influences investor behavior.

1 The Model

We consider an investor over a time interval $[0, T]$, who can invest either in a bond P_t or in a money market account that earns the risk-free rate, i.e., the short rate r_t.THe short rate r_t is stochastic. Following Vasicek (1977), we assume that the short rate evolves under the risk-neutral measure according to

$$dr_t = \kappa(\vartheta - r_t)dt + \beta dW_t^{\mathbb{Q}}, \tag{1}$$

where $W^{\mathbb{Q}} = (W_t^{\mathbb{Q}})_{0 \leq t \leq T}$ denotes the correspongin Browinan motion under the risk-neutral measure. The short rate process is mean-reverting: $\vartheta > 0$ is the long-term mean, $\kappa > 0$ is the speed of mean reversion, and $\beta > 0$ determines the volatility of r_t. The investor can only invest in a zero-coupon bond P_t with maturity T_P. Under the dynamics in (1), the bond price takes the form (see, e.g., Björk (2009))

$$P(t, r_t) = e^{A(t,T_P) - B(t,T_P)r_t},$$

with

$$\begin{aligned} B(t, T_P) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T_P - t)}\right) \\ A(t, T_P) &= \frac{(B(t, T_P) - T_P + t)\left(\kappa^2\vartheta - 0.5\beta^2\right)}{\kappa^2} - \frac{\beta^2 B(t, T_P)^2}{4\kappa}. \end{aligned}$$

The bond price dynamics under the risk-neutral measure are then given by

$$dP_t = P\left[r_t dt - B\beta dW_t^{\mathbb{Q}}\right]. \tag{2}$$

In addition to investing, the investor also consumes. Based on (2), the wealth dynamics under the risk-neutral measure are

$$dX_t = X_t[r_t dt - \pi_t B\beta dW_t^{\mathbb{Q}}] - c_t dt \tag{3}$$

where π_t denotes the bond portfolio share, and c_t the consumption rate. We assume the investor has continuous-time Epstein-Zin preferences. With a relative risk aversion parameter $\gamma > 1$, the optimization problem is given by

$$V(t, x, r) = \sup_{(c,\pi)\in\mathcal{A}} \mathrm{E}^{\mathbb{P}}\left[\int_t^T f(c_s, V_s)ds + \frac{\alpha}{1-\gamma}X_T^{1-\gamma}\right],$$

where the aggregator f is defined by

$$f(c, v) = \begin{cases} \delta\theta v\left[\left(\frac{c}{([1-\gamma]v)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right] & \text{for} \quad \psi \neq 1 \\ (1-\gamma)\delta v\ln(c) - \delta v\ln([1-\gamma]v) & \text{for} \quad \psi = 1 \end{cases} \tag{4}$$

and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Here, $\psi > 0$ represents the EIS, and $\delta > 0$ is the time preference-rate. The parameter α caputres the strength of the bequest motive at maturity T. Following Kraft et al. (2022), we set

$$\alpha = \epsilon^{\frac{1-\gamma}{\psi-1}} \delta^{\frac{1}{\theta}}, \tag{5}$$

which implies that the consumption-wealth ratio at maturity satisfies

$$cw(T, y) = \frac{1}{\epsilon}.$$

Thus, ϵ refers to the fraction of terminal wealth consumed, while the reminaing wealth is left as a bequest. Note that the investor maximizes expected utility under the physical measure $\mathbb{P}$, not the risk-neutral measure $\mathbb{Q}$. Hence, before deriving the corresponding HJB equation, we transform the dynamics of r_t and X_t to the physical measure using

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda(t, r_t)dt,$$

where $\lambda(t, r_t)$ denotes the market price of risk. Although $\lambda(t, r_t)$ can depend on time and the short rate, we later assume it is constant. Under the physical measure, the dynamics becomes

$$\begin{aligned} dr_t &= \kappa \left(\vartheta - r_t + \frac{\beta\lambda(t, r_t)}{\kappa} \right) dt + \beta dW_t^{\mathbb{P}}, \\ dX_t &= X_t[(r - D_t\beta\lambda)dt - D_t\beta dW_t^{\mathbb{P}}] - c_t dt, \end{aligned}$$

where we defined the portfolio duration by

$$D_t \triangleq \pi_t B(t, T_P).$$

We use D_t instead of π_t as a control variable. The value function $V(t, x, r)$ satisfied the HJB equation:

$$\begin{aligned} 0 = \sup_{(c,D)} \{ & V_t + [(r - D\beta\lambda(t, r))x - c] V_x + 0.5D^2\beta^2x^2V_{xx} - D\beta^2xV_{xr} + \kappa \left(\theta - r + \frac{\beta\lambda(t, r)}{\kappa} \right) V_r \\ & + 0.5\beta^2 V_{rr} + f(c, V) \} \end{aligned} \tag{6}$$

with terminal condition

$$V(T, x, r) = \frac{\alpha}{1-\gamma} x^{1-\gamma}.$$

To reduce the HJB equation, we assume that $V(t, x, r)$ has the following form

$$V(t, x, r) = \frac{x^{1-\gamma}}{1-\gamma} g(t, r). \tag{7}$$

Instead of controlling consumption directly, we introduce the consumption-wealth ratio $cw = \frac{c}{X}$. Substituting (7) into the HJB equation (6) yields the reduced-form HJB equation for g:

$$\begin{aligned} 0 = \sup_{(cw,D)} \{ & g_t + (1-\gamma) \left[r - D\beta\lambda(t, r) - \gamma 0.5D^2\beta^2 - cw \right] g - (1-\gamma)D\beta^2 g_r \\ & + \kappa \left(\theta - r + \frac{\beta\lambda(t, r)}{\kappa} \right) g_r + 0.5\beta^2 g_{rr} + f(cw, g) \} \end{aligned} \tag{8}$$

with terminal condition

$$g(T, r) = \alpha.$$

Using the aggregator defined in (4), the first-order conditions yield the optimal policies:

$$\begin{aligned} cw^*(t, r) &= \begin{cases} \delta^\psi (g(t, r))^{-\frac{\psi}{\theta}} & \text{for} \quad \psi \neq 1 \\ \delta & \text{for} \quad \psi = 1, \end{cases} \\ D^*(t, r) &= -\frac{\lambda(t, r)}{\beta\gamma} - \frac{1}{\gamma} \frac{g_r(t, r)}{g(t, r)} \end{aligned} \tag{9}$$

Using these optimal policies, we solve the reduced-from HJB equation (8) numerically.

2 Numerical Method

2.1 Computing the Reduced-Form Value Function $g(t, y)$

We solve (8) using an explicit finite differences scheme and set up an equally spaced grid in the (t, r)-space using the grid points

$$\{(t_n, r_i)|\ n = 0, 1, \ldots, N,\ i = 0, 1, \ldots, I\},$$

where $t_n = n\Delta t$ and $r_i = r_{min} + i\Delta r$, with Δt and Δr denoting the grid spacings in in the time and interest-rate directions, respectively. In the code, we specify the upper bounds of the gird and the number of grid points, implying $\Delta t = \frac{T}{N}$ and $\Delta r = \frac{r_{max} - r_{min}}{I}$. Given the terminal condition, we compute g approximately at each grid point by iteratively stepping backward through the grid. At any time t_n with $n < N$, the value of $g_{n,i} \triangleq g(t_n, r_i)$ for $1 \le i \le I - 1$, can be approximated out of the known values at t_{n+1} using the discretized reduced-form HJB equation:

$$\begin{aligned}
0 =& \frac{g_{n+1,i} - g_{n,i}}{\Delta t} + (1-\gamma)(r_i - D^*_{n+1,i}\beta\lambda_{n+1,i} - \gamma 0.5(D^*_{n+1,i})^2\beta^2 - cw^*_{n+1,i})g_{n+1,i} + f(cw^*_{n+1,i}, g_{n+1,i}) \\
&+ \frac{g_{n+1,i+1} - g_{n+1,i-1}}{2\Delta r}\left(-(1-\gamma)\beta^2 D^*_{n+1,i} + \kappa\left(\vartheta - r_i + \frac{\beta\lambda_{n+1,i}}{\kappa}\right)\right) \\
&+ \frac{\beta^2}{2}\frac{g_{n+1,i+1} - 2g_{n+1,i} + g_{n+1,i-1}}{(\Delta r)^2}.
\end{aligned}$$

We use central differences to approximate the first- and second-order derivatives and evaluate them at time step $n+1$. Since optimal policies typically vary slowly, we evaluate the controls $cw^*_{n+1,i} \triangleq cw^*(t_{n+1}, r_i)$ and $D^*_{n+1,i} \triangleq D^*(t_{n+1}, r_i)$ at t_{n+1}. Further, we also evaluate the market price of risk $\lambda_{n+1,i} =\triangleq \lambda(t_{n+1}, r_i)$ at t_{n+1}. Reordering the equation gives:

$$g_{n,i} = a_{n+1,i}g_{n+1,i-1} + b_{n+1,i}g_{n+1,i} + c_{n+1,i}g_{n,i+1} + \Delta t f(cw^*_{n+1,i}, g_{n+1,i}),$$

where the coefficients are given by

$$\begin{aligned}
a_{n+1,i} &= \frac{\Delta t}{2}\left[(1-\gamma)\beta^2\frac{D^*_{n+1,i}}{\Delta r} - \frac{\kappa\left(\vartheta - r_i + \frac{\beta\lambda_{n+1,i}}{\kappa}\right)}{\Delta r} + \frac{\beta^2}{(\Delta r)^2}\right], \\
b_{n+1,i} &= 1 + \Delta t\left[(1-\gamma)(r_i - D^*_{n+1,i}\beta\lambda_{n+1,i} - \gamma 0.5(D^*_{n+1,i})^2\beta^2 - cw^*_{n+1,i}) - \frac{\beta^2}{(\Delta r)^2}\right], \\
c_{n+1,i} &= \frac{\Delta t}{2}\left[-(1-\gamma)\beta^2\frac{D^*_{n+1,i}}{\Delta r} + \frac{\kappa\left(\vartheta - r_i + \frac{\beta\lambda_{n+1,i}}{\kappa}\right)}{\Delta r} + \frac{\beta^2}{(\Delta r)^2}\right], .
\end{aligned}$$

At each iteration step, we first update the value function using the expression above. Next, we compute the boundary values. Since boundary conditions at $r = r_{min}$ and $r = r_{max}$ are not explicitly given, we use linear extrapolation to estimate them. Finally, we compute the optimal consumption and investment strategy by discretizing (10) using central differences

$$\begin{aligned}
cw^*_{n,i} &= \begin{cases} \delta^\psi (g_{n,i})^{-\frac{\psi}{\theta}} & \text{for} \quad \psi \neq 1 \\ \delta & \text{for} \quad \psi = 1 \end{cases} \\
D^*_{n,i} &= -\frac{\lambda_{n,i}}{\beta\gamma} - \frac{1}{\gamma}\frac{g_{n,i+1} - g_{n,i-1}}{2\Delta r}\frac{1}{g_{n,i}}
\end{aligned} \tag{10}$$

We also apply linear extrapolation to compute $D^*_{n,0}$ and $D^*_{n,I}$.

3 Results

In the attached code, we follow Chan et al. (1992) and solve the model using the following parameter values:

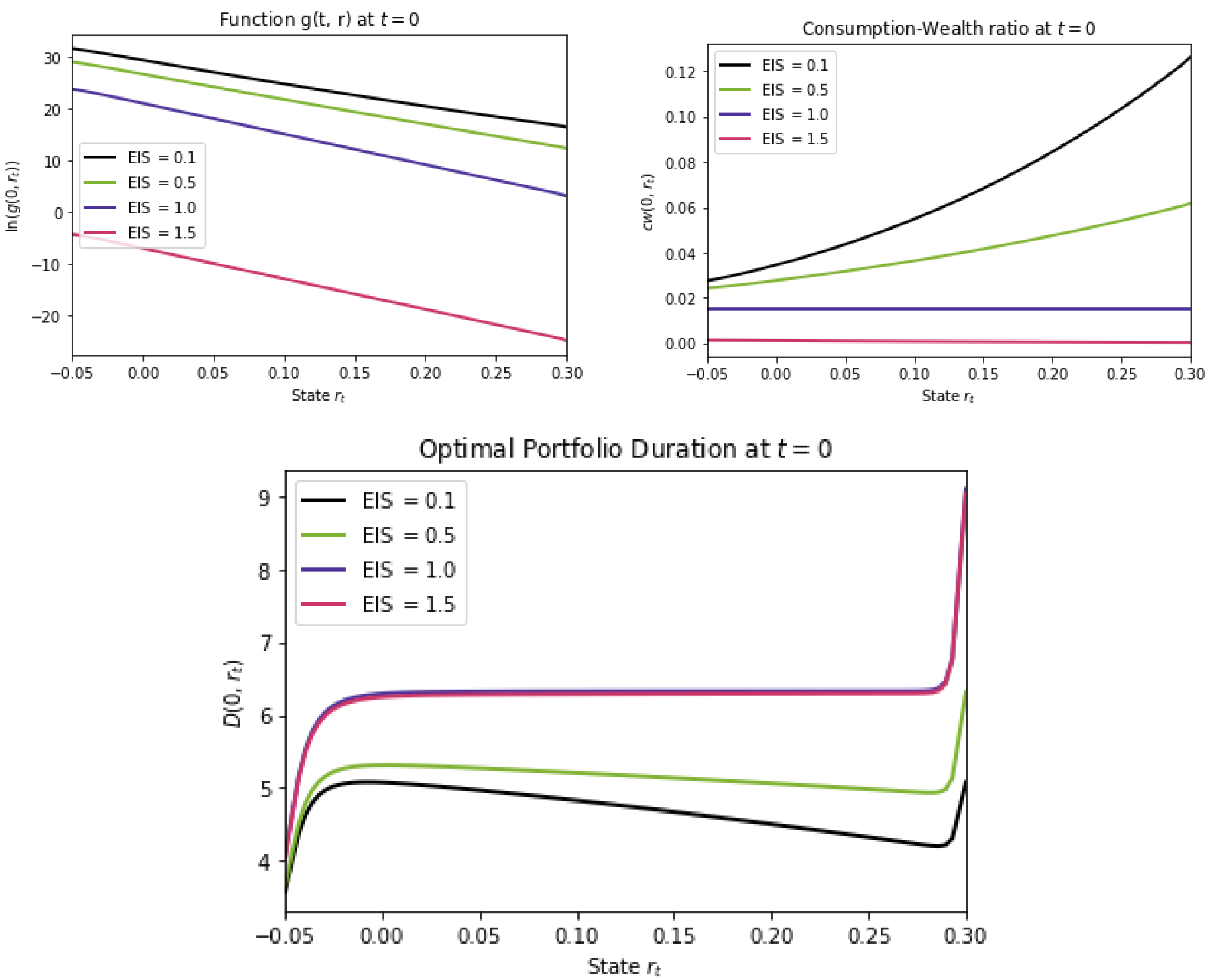
$$\gamma = 10, \quad \delta = 0.015, \quad \lambda = -0.05, \quad \kappa = 5, \quad \vartheta = 0.04, \quad \beta = 0.015, \quad \epsilon = 2, \quad \text{and} \quad T_P = 100$$

For the grid, we set:

$$T = 50, \quad N = 1.000.000, \quad r_{min} = -0.02, \quad , r_{max} = 0.3 \quad \text{and} \quad I = 100,$$

which implies a grid spacing of $\Delta t = 0.00005$ and $\Delta r = 0.0035$. First, we calculate the value function and optimal strategies for four different values of the EIS. Figure 1 presents the value function and policy functions at $t = 0$ for $\psi = 0.1$ (black), $\psi = 0.5$ (green), $\psi = 1.0$ (purple), and $\psi = 1.5$ (red), where the first case corresponds to time-additive utility.

Figure 1: Value function, Optimal Consumption-Wealth Ratio and Optimal Portfolio Durationfor a varying EIS. The left figure depicts the logarithm of $g(0, r_t)$ at $t = 0$ for four different values of EIS, $\psi = 0.1$ (black), $\psi = 0.5$ (green), $\psi = 1.0$ (purple), and $\psi = 1.5$ (red). The right figure displays the corresponding optimal consumption-wealth ratio $cw(0, r_t)$ and the figure below the corresponding optimal portfolio duration $D(0, r_t)$. We set the parameters to $\gamma = 10$, $\delta = 0.015$, $\lambda = -0.05$, $\kappa = 5$, $\vartheta = 0.04$, $\beta = 0.015$, $\epsilon = 2$, $T_P = 100$ and $T = 50$.



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