Consumption Portfolio Problem with Stochastic Volatility

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Abstract

We study an optimal consumption and investment problem under Epstein-Zin preferences in a stochastic volatility setting. Using a finite difference method, we numerically solve the associated Hamilton-Jacobi-Bellman (HJB) equation and analyze the impact of the elasticity of intertemporal substitution (EIS) and return-volatility correlation on optimal strategies.

1 The Model

We consider an investor over a time interval $[0, T]$. The investor can allocate wealth between a risky asset, represented by the stock S_t, which exhibits stochastic volatility, and a risk-free asset offering a constant return rate r. Following Heston (1993), the dynamics of the stock and its volatility process Y_t are given by

$$\begin{aligned} dS_t &= S_t[(r + \lambda Y_t)dt + \sqrt{Y_t}dW_t] \\ dY_t &= \kappa(\vartheta - Y_t)dt + \beta\sqrt{Y_t}d\hat{W}_t, \end{aligned} \tag{1}$$

with

$$d\langle W, \hat{W}\rangle_t = \rho,$$

where $\rho \in [-1, 1]$ determines the correlation between the two Brownian motions $W = (W_t)_{0\leq t\leq T}$ and $\hat{W} = (\hat{W}_t)_{0\leq t\leq T}$. The parameter λ governs the state-dependent market price of risk, i.e., the excess return $\lambda\sqrt{Y_t}$ associated with the Brownian motion W_t. The volatility process $Y = (Y_t)_{0\leq t\leq T}$ determines not only the volatility of S_t but also its excess return. Hence, periods of high volatility correspond to higher expected returns, and vice versa. The process Y_t is mean-reverting: $\vartheta > 0$ is the long-term mean, $\kappa > 0$ is the speed of mean reversion, and $\beta > 0$ controls the volatility of Y_t. In addition to investing, the investor also consumes. Given the dynamics in (1), the investor's wealth process $X = (X_t)_{0\leq t\leq T}$ evolves as

$$dX_t = X_t[(r + \pi_t\lambda Y_t)dt + \pi_t\sqrt{Y}_t dW_t] - c_t dt, \tag{2}$$

where π_t is the proportion of wealth invested in the risky asset, the portfolio share, and c_t the consumption rate. We assume the investor has continuous-time Epstein-Zin preferences. With a relative risk aversion parameter $\gamma > 1$, the optimization problem is given by

$$V(t, x, y) = \sup_{(c,\pi)\in\mathcal{A}} \mathrm{E}\left[\int_t^T f(c_s, V_s)ds + \frac{\alpha}{1-\gamma}X_T^{1-\gamma}\right],$$

where the aggregator f is defined by

$$f(c, v) = \begin{cases} \delta\theta v\left[\left(\frac{c}{([1-\gamma]v)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right] & \text{for} \quad \psi \neq 1 \\ (1-\gamma)\delta v\ln(c) - \delta v\ln([1-\gamma]v) & \text{for} \quad \psi = 1 \end{cases} \tag{3}$$

and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Here, $\psi > 0$ represents the EIS, and $\delta > 0$ is the time preference-rate. The parameter α caputres the strength of the bequest motive at maturity. Following Kraft et al. (2022), we set

$$\alpha = \epsilon^{\frac{1-\gamma}{\psi-1}} \delta^{\frac{1}{\theta}}, \tag{4}$$

which implies that the consumption-wealth ratio at maturity satisfies

$$cw(T, y) = \frac{1}{\epsilon}.$$

Thus, ϵ refers to the fraction of terminal wealth consumed, while the reminaing wealth is left as a bequest. Given this setup, the value function $V(t, x, y)$ solves the following HJB equation:

$$0 = \sup_{(c,\pi)} \{V_t + [(r + \pi\lambda y)x - c]V_x + 0.5\pi^2 yx^2 V_{xx} + \beta\pi\rho xyV_{xy} + \kappa(\vartheta - y)V_y + 0.5\beta^2 yV_{yy} + f(c, V)\} \tag{5}$$

with terminal condition

$$V(T, x, y) = \frac{\alpha}{1-\gamma} x^{1-\gamma}.$$

To reduce the HJB equation, we assume that $V(t, x, y)$ has the following form

$$V(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} g(t, y). \tag{6}$$

Instead of controlling consumption directly, we introduce the consumption-wealth ratio $cw = \frac{c}{X}$. Substituting (6) into the HJB equation (5) yields the reduced-form HJB equation for g:

$$0 = \inf_{(cw,\pi)} \{g_t + (1-\gamma)(r + \pi\lambda y - \gamma 0.5\pi^2 y - cw)g + (1-\gamma)\beta\pi\rho y g_y + \kappa(\vartheta - y)g_y + 0.5\beta^2 y g_{yy} + f(cw, g)\} \tag{7}$$

with terminal condition

$$g(T, y) = \alpha.$$

Using the aggregator defined in (3), the first-order conditions yield the optimal policies:

$$\begin{aligned} cw^*(t, y) &= \begin{cases} \delta^{\psi}(g(t, y))^{-\frac{\psi}{\theta}} & \text{for} \quad \psi \neq 1 \\ \delta & \text{for} \quad \psi = 1 \end{cases} \\ \pi^*(t, y) &= \frac{\lambda}{\gamma} + \frac{\beta\rho}{\gamma} \frac{g_y(t, y)}{g(t, y)} \end{aligned} \tag{8}$$

Using these optimal policies, we solve the reduced-from HJB equation (7) numerically.

2 Numerical Method

2.1 Computing the Reduced-Form Value Function $g(t, y)$

We solve (7) using an explicit finite differences scheme and set up an equally spaced grid in the (t, y)-space using the grid points

$$\{(t_n, y_i)|\ n = 0, 1, \ldots, N,\ i = 0, 1, \ldots, I\},$$

where $t_n = n\Delta t$ and $y_i = y_{min} + i\Delta y$, with Δt and Δy denoting the respective grid spacings in each direction. In the code, we specify the upper bounds of the gird and the number of grid points, implying $\Delta t = \frac{T}{N}$ and $\Delta y = \frac{y_{max}-y_{min}}{I}$. Given the terminal condition, we compute g approximately at each grid point by iteratively stepping backward through the grid. At any time t_n with $n < N$, the value of $g_{n,i} \triangleq g(t_n, y_i)$ for $1 \le i \le I-1$, can be approximated out of the known values at t_{n+1} using the discretized reduced-form HJB equation:

$$\begin{aligned}
0 =& \frac{g_{n+1,i} - g_{n,i}}{\Delta t} + (1-\gamma)(r + \pi^*_{n+1,i}\lambda y_i - \gamma 0.5(\pi^*_{n+1,i})^2 y_i - cw^*_{n+1,i})g_{n+1,i} + f(cw^*_{n+1,i}, g_{n+1,i}) \\
&+ \frac{g_{n+1,i+1} - g_{n+1,i-1}}{2\Delta y}\big((1-\gamma)\beta\rho y_i \pi^*_{n+1,i} + \kappa(\vartheta - y_i)\big) + \frac{\beta^2 y_i}{2}\frac{g_{n+1,i+1} - 2g_{n+1,i} + g_{n+1,i-1}}{(\Delta y)^2}.
\end{aligned}$$

We use central differences to approximate the first- and second-order derivatives and evaluate them at time step $n+1$. Since optimal policies typically vary slowly, we evaluate the controls $cw^*_{n+1,i} \triangleq cw^*(t_{n+1}, y_i)$ and $\pi^*_{n+1,i} \triangleq \pi^*(t_{n+1}, y_i)$ at t_{n+1}. Reordering the equation gives:

$$g_{n,i} = a_{n+1,i}g_{n+1,i-1} + b_{n+1,i}g_{n+1,i} + c_{n+1,i}g_{n,i+1} + \Delta t f(cw^*_{n+1,i}, g_{n+1,i}),$$

where the coefficients are given by

$$\begin{aligned}
a_{n+1,i} &= \frac{\Delta t}{2}\left[-(1-\gamma)\beta\rho y_i \frac{\pi^*_{n+1,i}}{\Delta y} - \frac{\kappa(\vartheta - y_i)}{\Delta y} + \frac{\beta^2 y_i}{(\Delta y)^2}\right], \\
b_{n+1,i} &= 1 + \Delta t\left[(1-\gamma)(r + \pi^*_{n+1,i}\lambda y_i - \gamma 0.5(\pi^*_{n+1,i})^2 y_i - cw^*_{n+1,i}) - \frac{\beta^2 y_i}{(\Delta y)^2}\right], \\
c_{n+1,i} &= \frac{\Delta t}{2}\left[(1-\gamma)\beta\rho y_i \frac{\pi^*_{n+1,i}}{\Delta y} + \frac{\kappa(\vartheta - y_i)}{\Delta y} + \frac{\beta^2 y_i}{(\Delta y)^2}\right].
\end{aligned}$$

At each iteration step, we first update the value function using the expression above. We then compute the boundary values. Since boundary conditions at $y = y_{min}$ and $y = y_{max}$ are not explicitly given, we use linear extrapolation to estimate them. Finally, we compute the optimal consumption and investment strategy by discretizing (8) using central differences

$$\begin{aligned}
cw^*_{n,i} &= \begin{cases} \delta^\psi (g_{n,i})^{-\frac{\psi}{\theta}} & \text{for} \quad \psi \neq 1 \\ \delta & \text{for} \quad \psi = 1 \end{cases} \\
\pi^*_{n,i} &= \frac{\lambda}{\gamma} + \frac{\beta\rho}{\gamma}\frac{g_{n,i+1} - g_{n,i-1}}{2\Delta y}\frac{1}{g_{n,i}}
\end{aligned}$$

We also apply linear extrapolation to compute $\pi^*_{n,0}$ and $\pi^*_{n,I}$.

3 Results

In the attached code, we follow Liu & Pan (2003) and solve the model using the following parameter values:

$$r = 0.02, \quad \gamma = 10, \quad \delta = 0.015, \quad \lambda = 4, \quad \kappa = 5, \quad \vartheta = 0.0169, \quad \beta = 0.25, \quad \text{and} \quad \epsilon = 2.$$

For the grid, we set:

$$T = 100, \quad N = 1.000.000, \quad y_{min} = 0.0025, \quad y_{max} = 0.5\text{and} \quad I = 100,$$

which implies a grid spacing of $\Delta t = 0.0001$ and $\Delta y = 0.0049975$. First, we calculate the value function and optimal strategies for three different values of the EIS. Figure 1 presents the value function and policy functions at $t = 0$ for $\psi = 0.5$ (black), $\psi = 1.0$ (green), and $\psi = 1.5$ (purple), with the correlation coefficient fixed $\rho = -0.4$. Next, we analyze the impact of different correlation coefficients ρ on the results. Figure 2 displays the corresponding value function and policy functions at $t = 0$ for $\rho = -0.4$ (black), $\rho = 0$ (green), and $\rho = 0.4$ (purple), while holding $\psi = 0.5$.

Figure 1: Value function, Optimal Consumption-Wealth Ratio and Optimal Portfolio Share for a varying EIS. The left figure depicts the logarithm of $g(0, Y_t)$ at $t = 0$ for three different values of EIS, $\psi = 0.5$ (black), $\psi = 1.0$ (green), and $\psi = 1.5$ (purple). The right figure shows the corresponding optimal consumption-wealth ratio $cw(0, Y_t)$ and the figure below the corresponding optimal portfolio share $\pi(0, Y_t)$. All outputs are normalized to the corresponding value at $Y_t = 0$. We set the parameters to $r = 0.02$, $\gamma = 10$, $\delta = 0.015$, $\lambda = 4$, $\kappa = 5$, $\vartheta = 0.0169$, $\beta = 0.25$, $\epsilon = 2$, $T = 100$ and $\rho = -0.4$.

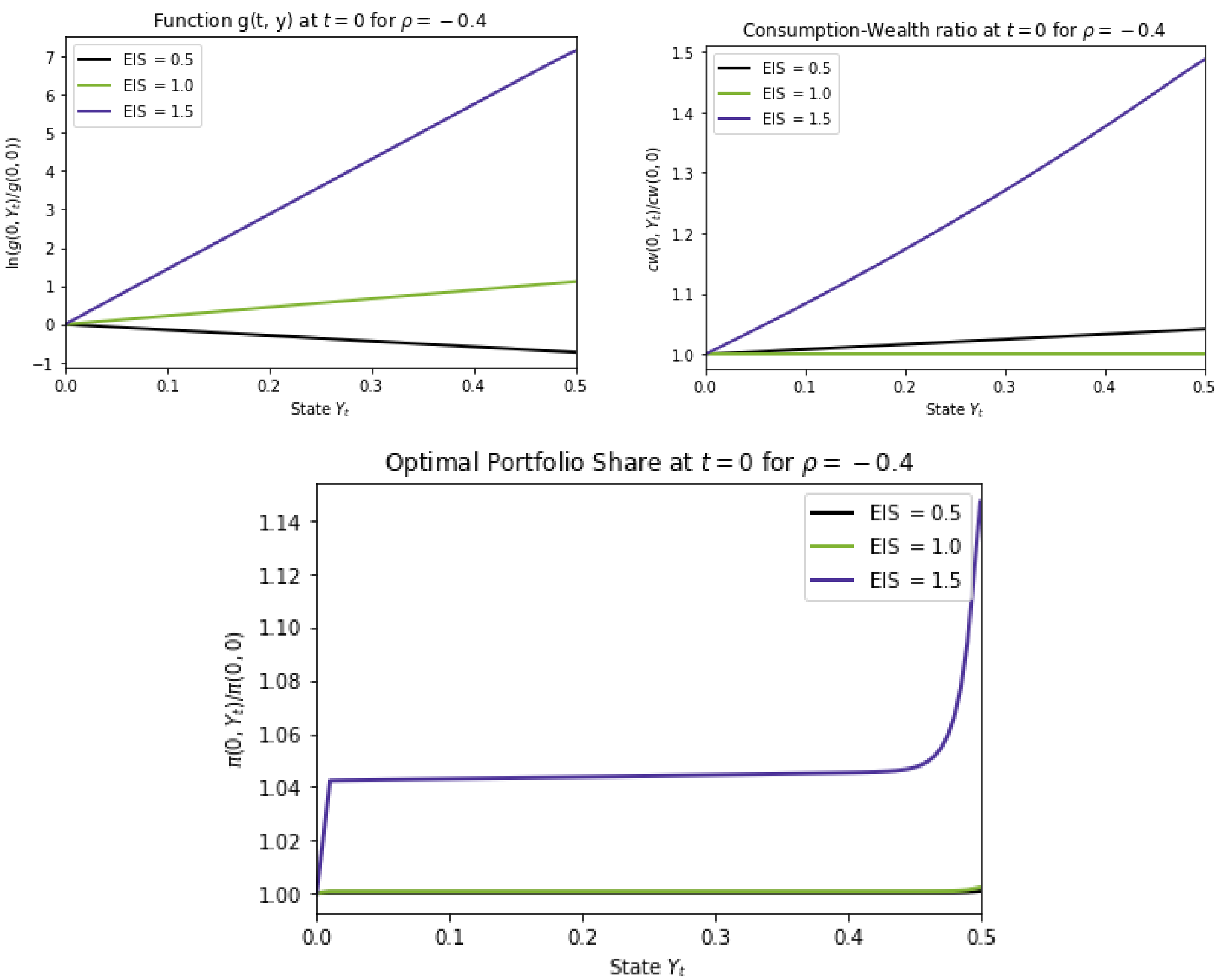
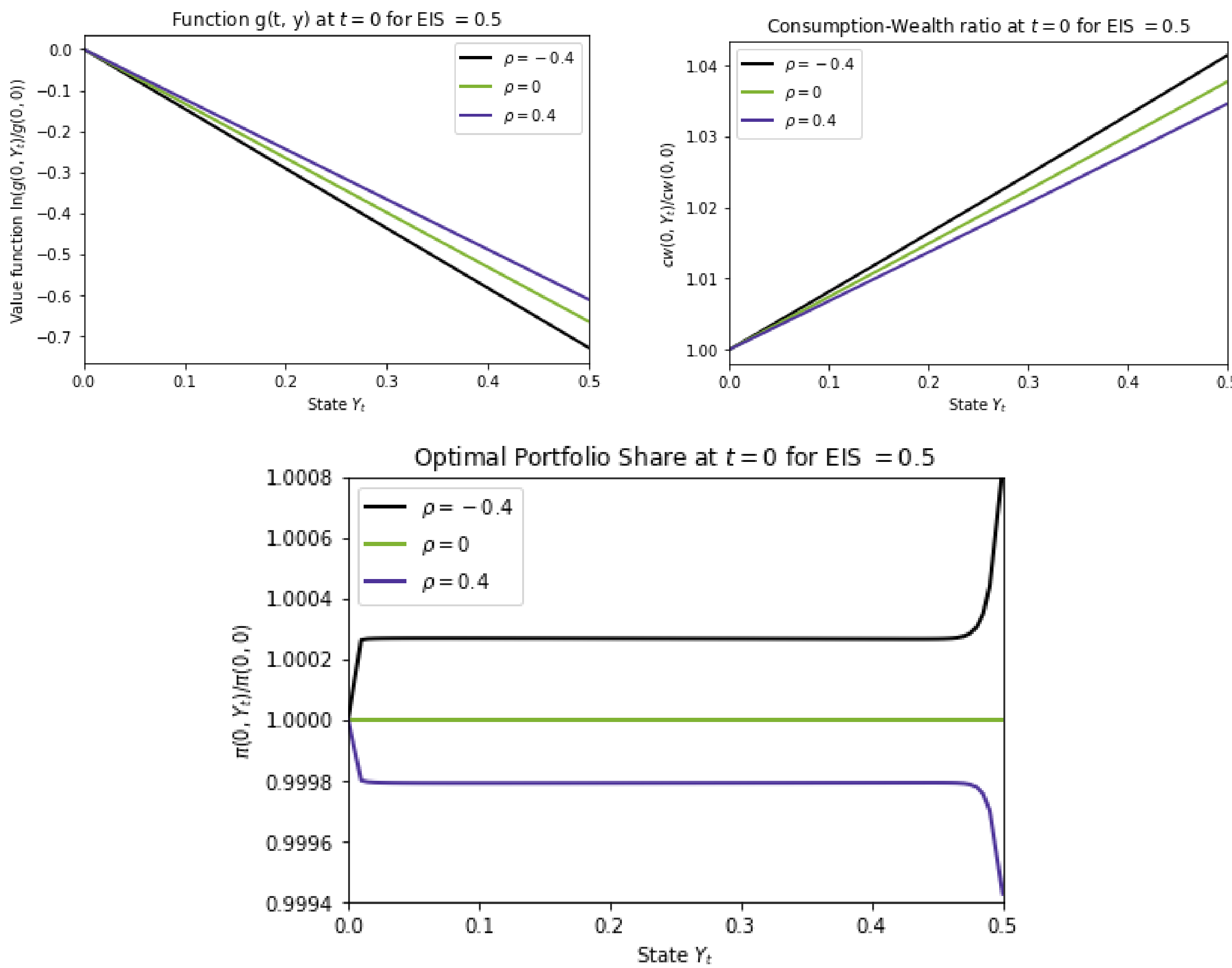


Figure 2: Value function, Optimal Consumption-Wealth Ratio and Optimal Portfolio Share for a varying ρ. The left figure depicts the logarithm of $g(0, Y_t)$ at $t = 0$ for three different values of ρ, $\psi = -0.4$ (black), $\rho = 0$ (green), and $\rho = 0.4$ (purple). The right figure shows the corresponding optimal consumption-wealth ratio $cw(0, Y_t)$ and the figure below the corresponding optimal portfolio share $\pi(0, Y_t)$. All outputs are normalized to the corresponding value at $Y_t = 0$. We set the parameters to $r = 0.02$, $\gamma = 10$, $\delta = 0.015$, $\lambda = 4$, $\kappa = 5$, $\vartheta = 0.0169$, $\beta = 0.25$, $\epsilon = 2$, $T = 100$ and $\psi = 0.5$.



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