Optimal Dividends with Backruptcy

Joscha Duchscherer

Abstract

We study the optimal dividend policy for an insurance company with shareholders exhibiting constant absolute risk aversion (CARA). The firm faces bankruptcy risk, and we model its reserves using a Cramér–Lundberg process, approximated by Brownian motion. We solve the resulting stochastic control problem numerically and analyze the probability of ruin under the optimal strategy.

1   The Model

We consider an insurance company operating over a time interval $[0, T]$. The company's reserve process R_t follows a classical Cramér–Lundberg risk process

$$dR_t = \mu dt - b dN_t. \tag{1}$$

Here, $b \geq 0$ is the fixed size of each claim, and N_t is a Poisson process with intensity $\lambda > 0$, representing the number of claims. The drift $\mu \geq 0$ reflects premium income, which is the positive cash flow earned by the insurance company through insurance contracts. The company pays out a dividend stream $d = (d_t)_{0 \leq t \leq T}$ to its shareholders. The wealth process X of the insureance company evolves as

$$dX_t = \mu dt - d_t dt - b dN_t.$$

Bankruptcy occurs if $X_t < 0$. We then define the time of ruin τ by

$$\tau = \inf\{t \geq 0 : X_t < 0\}.$$

The insurance company wishes to maximize the shareholder value of the dividends, given the sharholders have the utility function

$$u(d) = -\frac{1}{\gamma} e^{-\gamma d}. \tag{2}$$

The corresponding value function of the shareholders is then given by

$$V(t, x) = \sup_{d \in \mathcal{A}} \mathrm{E}\left[\int_t^{\tau \vee T} e^{-\delta(s-t)} u(d_s) ds + e^{-\delta(T-t)} u(X_T) \right].$$

Here, $\mathcal{A}$ denotes the set of admissible strategies. Following Palmowski & Baran (2017), $(d_t)_{0 \leq t \leq T}$ is admissible if it is non-negative and adapted cadlag process, where no dividends are paid after a ruin occurs, i.e., $d_t = 0$ for all $t \geq \tau$. To simplify, we approximate the Poisson process using the Brownian motion $(W_t)_{0 \leq t \leq T}$. Using Iglehart (1969), the wealth process evolves then according to

$$dX_t = (\mu - b\lambda)dt - d_t dt + b\sqrt{\lambda}\, dW_t. \tag{3}$$

Given this setup the value function solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{d \geq 0} \left\{ V_t + (\mu - d - b\lambda)V_x + 0.5 b^2 \lambda V_{xx} - \delta V + u(d) \right\} \tag{4}$$

with terminal condition
$$V(T,x) = u(x).$$

The terminal condition reflects that all remaining wealth at maturity is transferred to the shareholders. Using (2), we obtain from the first-order condition

$$d^* = \begin{cases} -\ln(V_x) & \text{for} \quad V_x \geq 1 \\ 0 & \text{else.} \end{cases} \tag{5}$$

Further, we can derive a boundary condition at the lower boundary $x = 0$. Ruin occurs immediately at $X_t = 0$ due to the oscillation of the Brownian motion, and no further dividends are paid, i.e., $d_s = 0$ for all $s \geq t$. We then obtain:

$$V(t,0) = \int_0^T e^{-\delta s} u(0) ds = - \int_t^T \frac{e^{-\delta(s-t)}}{\gamma} ds = -\frac{1}{\gamma\delta}\left[1 - e^{-\delta(T-t)}\right] - \frac{1}{\gamma}e^{-\delta(T-t)}.$$

2 Numerical Method

2.1 Computing the Value Function

We solve the HJB equation (4) using an explicit finite differences scheme and set up an equally spaced grid in the (t,x)-space using the grid points

$$\{(t_n, x_i)|\ n = 0, 1, \ldots, N,\ i = 0, 1, \ldots, I\},$$

where $t_n = n\Delta t$ and $x_i = i\Delta x$, with Δt and Δx being the respective grid spacings in each direction. In the code, we set the upper grid point in each direction, which implies grid spacings of $\Delta t = \frac{T}{N}$ and $\Delta x = \frac{x_{max}}{I}$, respectively. Given the terminal condition, we compute the value function approximately at each grid point by iteratively stepping backward through the grid. At any time t_n with $n < N$, the value of $V_{n,i} \triangleq V(t_n, x_i)$, for $1 \leq i \leq I-1$, can be approximated out of the known values at t_{n+1} using the discretized HJB equation:

$$\begin{aligned} 0 = &\frac{V_{n+1,i} - V_{n,i}}{\Delta t} + u(d^*_{n+1,i}) - \delta V_{n+1,i} + \frac{V_{n+1,i+1} - V_{n+1,i-1}}{2\Delta x}\left(\mu - d^*_{n+1,i} - b\lambda)\right) \\ &+ \frac{b^2\lambda}{2}\frac{V_{n+1,i+1} - 2V_{n+1,i} + V_{n+1,i-1}}{(\Delta x)^2}, \end{aligned} \tag{6}$$

where we have used first- and second-order central difference to approximate the corresponding derivatives. For the explicit scheme we evaluate the derivatives at the time step $n+1$. Since optimal policies typically vary slowly, we evaluate the optimal policy $d^*_{n+1,i} \triangleq d^*(t_{n+1}, x_i)$ at t_{n+1}. Reordering the terms gives the equation

$$V_{n,i} = a_{n+1,i}V_{n+1,i-1} + b_{n+1,i}V_{n+1,i} + c_{n+1,i}V_{n,i+1} + \Delta t u(d^*_{n+1,i})$$

with the coefficients given by

$$a_{n+1,i} = \frac{\Delta t}{2}\left[-\frac{\mu - d^*_{n+1,i} - b\lambda}{\Delta x} + \frac{b^2\lambda}{(\Delta x)^2}\right],$$

$$b_{n+1,i} = 1 + \Delta t\left[\frac{b^2\lambda}{(\Delta x)^2} - \delta\right],$$

$$c_{n+1,i} = \frac{\Delta t}{2}\left[\frac{\mu - d^*_{n+1,i} - b\lambda}{\Delta x} + \frac{b^2\lambda}{(\Delta x)^2}\right].$$

At each iteration step, we first calculate the value function using the equation above. Next, we compute the boundary values. At the lower boundary $x_{min} = 0$, we apply the boundary condition

$$V_{n,0} = -\frac{1}{\gamma\delta}\left[1 - e^{-\delta\Delta t(N_t - n)}\right] - \frac{1}{\gamma}e^{-\delta\Delta t(N_t - n)}.$$

The primary reason for using an explicit scheme is the lack of boundary conditions at the upper bound. Therefore, we use linear extrapolation to estimate the value at $x = x_{max}$. Finally, we compute the optimal dividend policy by discretizing (5) using central differences

$$d_{n,i}^* = \begin{cases} -\ln\left(\frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x}\right) & \text{for} \quad \frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x} \geq 1 \\ 0 & \text{else.} \end{cases}$$

Further, we set $d_{n,0}^* = 0$ and use linear extrapolation to determine $d_{n,I}^*$.

2.2 Computing the Ruin Probability

We calculate the probability of ruin $P(\tau \leq t)$ using Monte Carlo simulations. To do this, we simulate N_{MC} paths up to time T by discretizing the wealth equation (3). Given a value X_t, the next value can be approximated by

$$X_{t+1} = X_t + (\mu - b\lambda)\Delta t - d_t^*(t, X_t)\Delta t + b\sqrt{\lambda}\sqrt{\Delta t}\,\varepsilon_t,$$

where ε_t is a standard normally distributed random variable. The values for $d_t^*(t, X_t)$ are obtained by linear interpolation of the policy function on the grid if X_t is within the grid, by linear extrapolation for $X_t > x_{max}$, and set to zero for $X_t < 0$. Once the path value drops below zero, i.e., $X_t < 0$, all subsequent values are set to zero. The probability of ruin is then estimated by counting the number of paths where $X_t = 0$ at time t, and and dividing by the total number of simulated paths

$$P(\tau \leq t) = \frac{\text{Number of paths with } X_t = 0}{N_{MC}}.$$

3 Results

In the attached code, we solve for the value function using the following parameter values:

$$\mu = 2, \quad \gamma = 0.1, \quad \delta = 0.05, \quad \lambda = 0.1, \quad \text{and} \quad b = 5.$$

For the grid, we use:

$$T = 100, \quad N = 100.000, \quad x_{max} = 10, \quad \text{and} \quad I = 100,$$

which implies a grid spacing of $\Delta t = 0.001$ and $\Delta x = 0.1$. For the Monte Carlo simulations, we simulate a total of $N_{MC} = 10.000$ paths. The value function and the policy function at $t = 0$ are depict in the Figure 1. The probability of ruin for three different starting values is depicted in Figure 2. The black line depicts the probability of ruin for $X_0 = 1.5$, the green line for $X_0 = 2.5$, and the purple line for $X_0 = 5$.

Figure 1: Optimal Dividends and Value Function. The left figure depicts the optimal dividend policy $d^*(0, X_t)$ at $t = 0$. The right figure shows the corresponding value function $V(0, X_t)$. We set the parameters to $\mu = 2$, $\gamma = 0.1$, $\delta = 0.05$, $\lambda = 0.1$, $b = 5$ and $T = 100$.
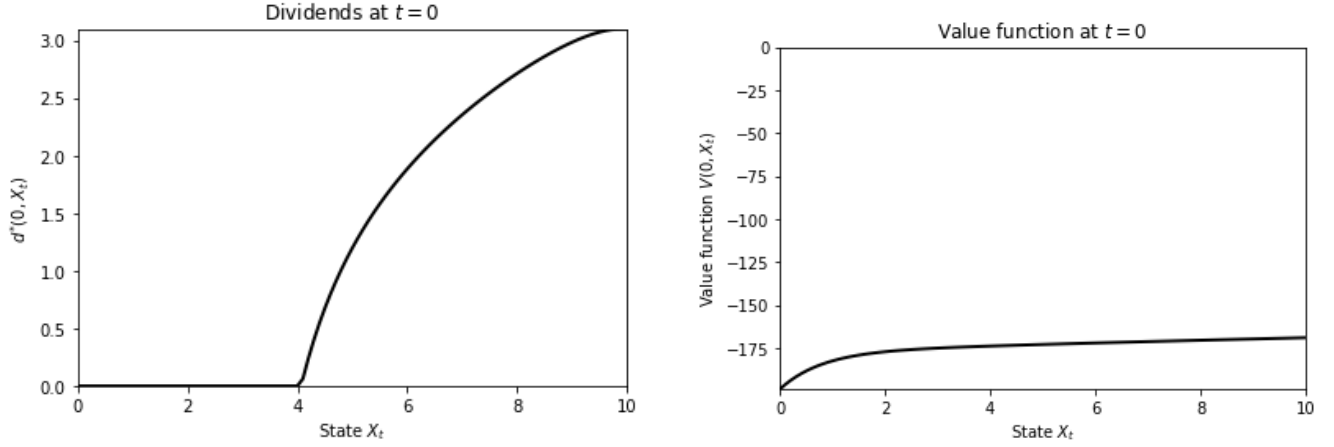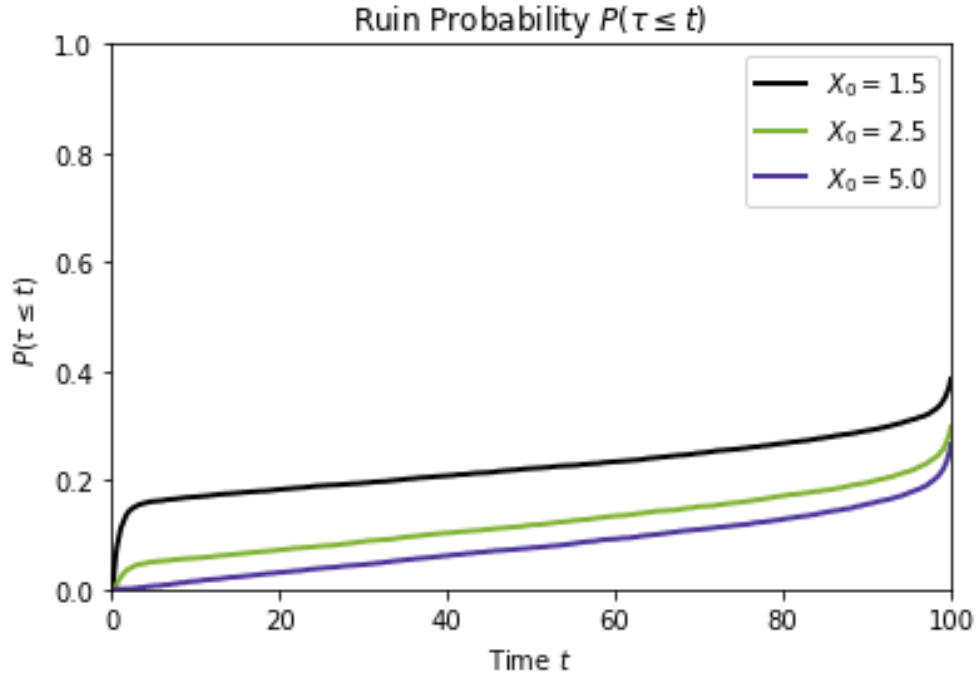

Figure 2: Probability of Ruin The picture depicts the probability of ruin for three different starting values. The black line depicts the probability of ruin for $X_0 = 1.5$, the green line for $X_0 = 2.5$, and the purple line for $X_0 = 5$. We set the parameters to $\mu = 2$, $\gamma = 0.1$, $\delta = 0.05$, $\lambda = 0.1$, $b = 5$ and $T = 100$.

References

Iglehart, D. L. (1969), 'Diffusion approximations in collective risk theory', *Journal of Applied Probability* **6**(2), 285–292.

Palmowski, Z. & Baran, S. (2017), 'Optimizing expected utility of dividend payments for a cramer-lundberg risk process'.