

# Optimal Dividends with Backruptcy

Joscha Duchscherer

## Abstract

We study the optimal dividend policy for an insurance company with shareholders exhibiting constant absolute risk aversion (CARA). The firm faces bankruptcy risk, and we model its reserves using a Cramér–Lundberg process, approximated by Brownian motion. We solve the resulting stochastic control problem numerically and analyze the probability of ruin under optimal strategies.

## 1 The Model

We consider an insurance company operating over a time interval  $[0, T]$ . The company's reserve process  $R_t$  follows a classical Cramér–Lundberg risk process

$$dR_t = \mu dt - b dN_t. \quad (1)$$

Here,  $b \geq 0$  is the fixed size of each claim, and  $N_t$  is a Poisson process with intensity  $\lambda > 0$ , representing the number of claims. The drift  $\mu \geq 0$  reflects premium income, which is the positive cash flow earned by the insurance company through insurance contracts. The company pays out a dividend stream  $d = (d_t)_{0 \leq t \leq T}$  to its shareholders. The wealth process  $X$  of the insurance company evolves as

$$dX_t = \mu dt - d_t dt - b dN_t.$$

Bankruptcy occurs if  $X_t < 0$ . We then define the time of ruin  $\tau$  by

$$\tau = \inf\{t \geq 0 : X_t < 0\}.$$

The insurance company wishes to maximize the shareholder value of the dividends, given the shareholders have the utility function

$$u(d) = -\frac{1}{\gamma} e^{-\gamma d}. \quad (2)$$

The corresponding value function of the shareholders is then given by

$$V(t, x) = \sup_{d \in \mathcal{A}} \mathbb{E} \left[ \int_t^{\tau \vee T} e^{-\delta(s-t)} u(d_s) ds + e^{-\delta(T-t)} u(X_T) \right].$$

Here,  $\mathcal{A}$  denotes the set of admissible strategies. Following Palmowski & Baran (2017),  $(d_t)_{0 \leq t \leq T}$  is admissible if it is non-negative and adapted cadlag process, where no dividends are paid after a ruin occurs, i.e.,  $d_t = 0$  for all  $t \geq \tau$ . To simplify, we approximate the Poisson process using the Brownian motion  $(W_t)_{0 \leq t \leq T}$ . Using Iglehart (1969), the wealth process evolves then according to

$$dX_t = (\mu - b\lambda)dt - d_t dt + b\sqrt{\lambda} dW_t. \quad (3)$$

Given this setup the value function solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{d \geq 0} \{ V_t + (\mu - d - b\lambda)V_x + 0.5b^2\lambda V_{xx} - \delta V + u(d) \} \quad (4)$$

with terminal condition

$$V(T, x) = u(x).$$

The terminal condition reflects that all remaining wealth at maturity is transferred to the shareholders. Using (2), we obtain from the First Order Condition

$$d^* = \begin{cases} -\ln(V_x) & \text{for } V_x \geq 1 \\ 0 & \text{else.} \end{cases} \quad (5)$$

Further, we can derive a boundary condition at the lower boundary  $x = 0$ . Ruin occurs immediately at  $X_t = 0$  due to the oscillation of the Brownian motion, and no further dividends are paid, i.e.,  $d_s = 0$  for all  $s \geq t$ . We then obtain:

$$V(t, 0) = \int_0^T e^{-\delta s} u(0) ds = - \int_t^T \frac{e^{-\delta(s-t)}}{\gamma} ds = -\frac{1}{\gamma\delta} [1 - e^{-\delta(T-t)}] - \frac{1}{\gamma} e^{-\delta(T-t)}.$$

## 2 Numerical Method

### 2.1 Computing the Value Function

We solve the HJB equation (4) using an explicit finite differences scheme and set up an equally spaced grid in the  $(t, x)$ -space using the grid points

$$\{(t_n, x_i) | n = 0, 1, \dots, N, i = 0, 1, \dots, I\},$$

where  $t_n = n\Delta t$  and  $x_i = i\Delta x$ , with  $\Delta t$  and  $\Delta x$  being the respective grid spacings in each direction. In the code, we set the upper grid point in each direction, which implies grid spacings of  $\Delta t = \frac{T}{N_t}$  and  $\Delta x = \frac{x_{max}}{N_x}$ , respectively. Given the terminal condition, we compute the value function approximately at each grid point by iteratively stepping backward through the grid. At any time  $t_n$  with  $n < N$ , the value of  $V_{n,i} \triangleq V(t_n, x_i)$ , for  $1 < i < I - 1$ , can be approximated out of the known values at  $t_{n+1}$  using the discretized HJB equation:

$$\begin{aligned} 0 = & \frac{V_{n+1,i} - V_{n,i}}{\Delta t} + u(d_{n+1,i}^*) - \delta V_{n+1,i} + \frac{V_{n+1,i+1} - V_{n+1,i-1}}{2\Delta x} (\mu - d_{n+1,i}^* - b\lambda) \\ & + \frac{b^2\lambda}{2} \frac{V_{n+1,i+1} - 2V_{n+1,i} + V_{n+1,i-1}}{(\Delta x)^2}, \end{aligned} \quad (6)$$

where we have used first- and second-order central difference to approximate the corresponding derivatives. For the explicit scheme we evaluate the derivatives at the time step  $n + 1$ . Since optimal policies typically vary slowly, we evaluate the optimal policy  $d_{n+1,i}^* \triangleq d^*(t_{n+1}, x_i)$  at  $t_{n+1}$ . Reordering the terms gives the equation

$$V_{n,i} = a_{n+1,i} V_{n+1,i-1} + b_{n+1,i} V_{n+1,i} + c_{n+1,i} V_{n+1,i+1} + \Delta t u(d_{n+1,i}^*)$$

with the coefficients given by

$$\begin{aligned} a_{n+1,i} &= \frac{\Delta t}{2} \left[ -\frac{\mu - d_{n+1,i}^* - b\lambda}{\Delta x} - \frac{b^2\lambda}{(\Delta x)^2} \right], \\ b_{n+1,i} &= 1 + \Delta t \left[ \frac{b^2\lambda}{(\Delta x)^2} - \delta \right], \\ c_{n+1,i} &= \frac{\Delta t}{2} \left[ \frac{\mu - d_{n+1,i}^* - b\lambda}{\Delta x} - \frac{b^2\lambda}{(\Delta x)^2} \right], \end{aligned}$$

At each iteration step, we first calculate the value function using the equation above. Next, we compute the boundary values. At the lower boundary  $x_{min} = 0$ , we apply the boundary condition

$$V_{n,0} = -\frac{1}{\gamma\delta} \left[ 1 - e^{-\delta\Delta t(N_t-n)} \right] - \frac{1}{\gamma} e^{-\delta\Delta t(N_t-n)}.$$

The primary reason for using an explicit scheme is the lack of boundary conditions at the upper bound. Therefore, we use linear extrapolation to estimate the value at  $x = x_{max}$ . Finally, we compute the optimal dividend policy by discretizing equation (7) using central differences

$$d_{n,i}^* = \begin{cases} -\ln \left( \frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x} \right) & \text{for } \frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x} \geq 1 \\ 0 & \text{else.} \end{cases} \quad (7)$$

We also set  $d_{n,0}^* = 0$  and use linear extrapolation to determine  $d_{n,N_x}^*$ .

### Computing the Ruin Probability

We calculate the probability of ruin  $P(\tau \leq t)$  using Monte Carlo simulations. To do this, we simulate  $N_{MC}$  paths up to time  $T$  by discretizing the wealth equation (3). Given a value  $X_t$ , the next value can be approximated by

$$X_{t+1} = X_t + (\mu - b\lambda)\Delta t - d_t^*(t, X_t)\Delta t + b\sqrt{\lambda}\sqrt{\Delta t}\varepsilon_t,$$

where  $\varepsilon_t$  is a standard normally distributed random variable. The values for  $d_t^*(t, X_t)$  are obtained by linear interpolation of the policy function on the grid if  $X_t$  is within the grid, by linear extrapolation for  $X_t > x_{max}$ , and set to zero for  $X_t < 0$ . Once the path value drops below zero, i.e.,  $X_t < 0$ , all subsequent values are set to zero. The probability of ruin is then estimated by counting the number of paths where  $X_t = 0$  at time  $t$ , and dividing by the total number of simulated paths

$$P(\tau \leq t) = \frac{\text{Number of paths with } X_t = 0}{N_{MC}}.$$

## 3 Results

In the attached code, we solve for the value function using the following parameter values:

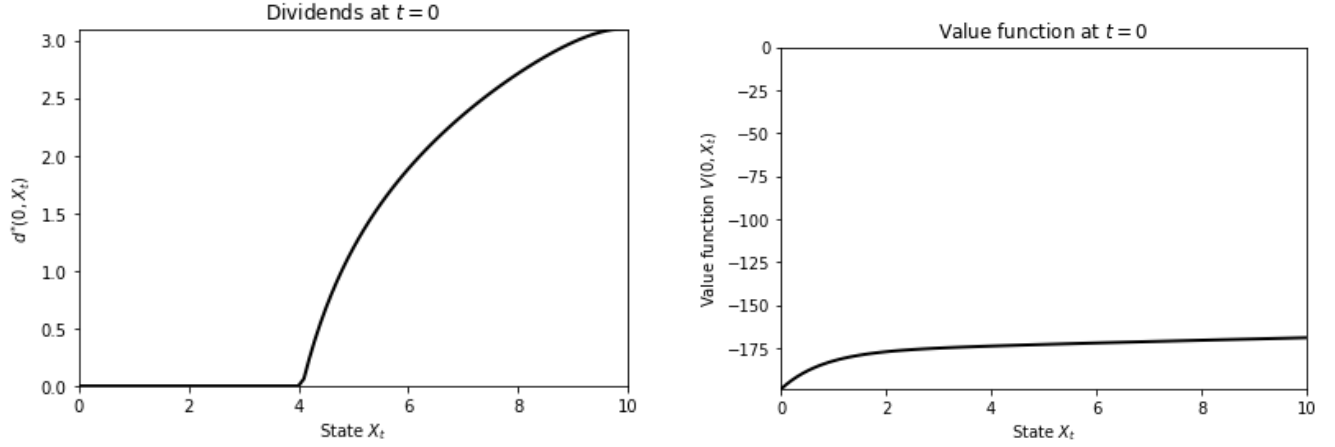
$$\mu = 2, \quad \gamma = 0.1, \quad \delta = 0.05, \quad \lambda = 0.1, \quad \text{and} \quad b = 5.$$

For the grid, we use:

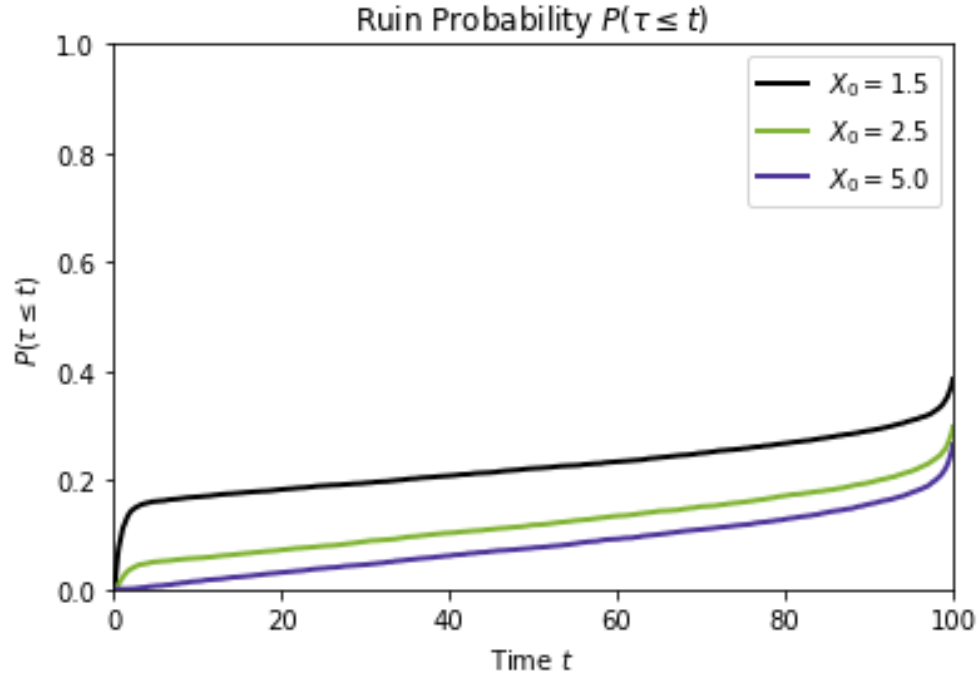
$$T = 100, \quad N_t = 100.000, \quad x_{max} = 10, \quad \text{and} \quad N_x = 100,$$

which implies a grid spacing of  $\Delta t = 0.001$  and  $\Delta x = 0.1$ . For the Monte Carlo simulations, we simulate a total of  $N_{MC} = 10.000$  paths. The value function and the policy function at  $t = 0$  are depicted in the Figure 1. The probability of ruin for three different starting values is depicted in Figure 2. The black line represents the probability of ruin for  $X_0 = 1.5$ , the green line for  $X_0 = 2.5$ , and the purple line for  $X_0 = 5$ .

**Figure 1: Optimal Dividends and Value Function.** The left figure depicts the optimal dividend policy  $d^*(0, X_t)$  at  $t = 0$ . The right figure shows the corresponding value function  $V(0, X_t)$ . We set the parameters to  $\mu = 2$ ,  $\gamma = 0.1$ ,  $\delta = 0.05$ ,  $\lambda = 0.1$  and  $\beta = 5$ .



**Figure 2: Probability of Ruin** The picture depicts the probability of ruin for three different starting values. The black line depicts the probability of ruin for  $X_0 = 1.5$ , the green line for  $X_0 = 2.5$ , and the purple line for  $X_0 = 5$ . We set the parameters to  $\mu = 2$ ,  $\gamma = 0.1$ ,  $\delta = 0.05$ ,  $\lambda = 0.1$  and  $\beta = 5$ .



## References

- Iglehart, D. L. (1969), 'Diffusion approximations in collective risk theory', *Journal of Applied Probability* **6**(2), 285–292.
- Palmowski, Z. & Baran, S. (2017), 'Optimizing expected utility of dividend payments for a cramer-lundberg risk process'.