Optimal Dividends with Backruptcy

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Abstract

We study the optimal dividend policy for an insurance company with shareholders exhibiting constant absolute risk aversion (CARA). The firm faces bankruptcy risk, and we model its reserves using a Cramér–Lundberg process, approximated by Brownian motion. We solve the resulting stochastic control problem numerically and analyze the probability of ruin under optimal strategies.

1 The Model

We consider an insurance company operating over a time interval $[0, T]$. The company's reserve process R_t follows a classical Cramér–Lundberg risk process

$$dR_t = \mu dt - bdN_t. \tag{1}$$

Here, $b \geq 0$ is the fixed size of each claim, and N_t is a Poisson process with intensity$\lambda > 0$, representing the number of claims. The drift $\mu \geq 0$ reflects premium income, which is the positive cash flow earned by the insurance company through insurance contracts. The company pays out a dividend stream $d = (d_t)_{0 \leq t \leq T}$ to its shareholders. The wealth process X of the insureance company evolves as

$$dX_t = \mu dt - d_t dt - bdN_t.$$

Bankruptcy occurs if $X_t < 0$. We then define the time of ruin τ by

$$\tau = \inf\{t \geq 0 : X_t < 0\}.$$

The insurance company wishes to maximize the shareholder value of the dividends, given the sharholders have the utility function

$$u(d) = -\frac{1}{\gamma} e^{-\gamma d}. \tag{2}$$

The corresponding value function of the shareholders is then given by

$$V(t, x) = \sup_{d \in \mathcal{A}} \mathrm{E}\left[\int_t^{\tau \vee T} e^{-\delta(s-t)} u(d_s) ds + e^{-\delta(T-t)} u(X_T)\right].$$

Here, $\mathcal{A}$ denotes the set of admissible strategies. Following Palmowski & Baran (2017), $(d_t)_{0 \leq t \leq T}$ is admissible if it is non-negative and adapted cadlag process, where no dividends are paid after a ruin occurs, i.e., $d_t = 0$ for all $t \geq \tau$. To simplify, we approximate the Poisson process using the Brownian motion $(W_t)_{0 \leq t \leq T}$. Using Iglehart (1969) , the wealth process evolves then according to

$$dX_t = (\mu - b\lambda)dt - d_t dt + b\sqrt{\lambda}\, dW_t. \tag{3}$$

Given this setup the value function solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{d \geq 0} \left\{V_t + (\mu - d - b\lambda)V_x + 0.5b^2\lambda V_{xx} - \delta V + u(d)\right\} \tag{4}$$

with terminal condition

$$V(T,x) = u(x).$$

The terminal condition reflects that all remaining wealth at maturity is transferred to the shareholders. Using (2), we obtain from the First Order Condition

$$d^* = \begin{cases} -\ln(V_x) & \text{for} \quad V_x \geq 1 \\ 0 & \text{else.} \end{cases} \tag{5}$$

Further, we can derive a boundary condition at the lower boundary $x = 0$. Ruin occurs immediately at $X_t = 0$ due to the oscillation of the Brownian motion, and no further dividends are paid, i.e., $d_s = 0$ for all $s \geq t$. We then obtain:

$$V(t,0) = \int_0^T e^{-\delta s} u(0) ds = -\int_t^T \frac{e^{-\delta(s-t)}}{\gamma} ds = -\frac{1}{\gamma\delta}\left[1 - e^{-\delta(T-t)}\right] - \frac{1}{\gamma} e^{-\delta(T-t)}.$$

2 Numerical Method

2.1 Computing the Value Function

We solve the HJB equation (4) using an explicit finite differences scheme and set up an equally spaced grid in the (t,x)-space using the grid points

$$\{(t_n, x_i)|\ n = 0, 1, \dots, N,\ i = 0, 1, \dots, I\},$$

where $t_n = n\Delta t$ and $x_i = i\Delta x$, with Δt and Δx being the respective grid spacings in each direction. In the code, we set the upper grid point in each direction, which implies grid spacings of $\Delta t = \frac{T}{N_t}$ and $\Delta x = \frac{x_{max}}{N_x}$, respectively. Given the terminal condition, we compute the value function approximately at each grid point by iteratively stepping backward through the grid. At any time t_n with $n < N$, the value of $V_{n,i} \triangleq V(t_n, x_i)$, for $1 < i < I - 1$, can be approximated out of the known values at t_{n+1} using the discretized HJB equation:

$$\begin{aligned} 0 =& \frac{V_{n+1,i} - V_{n,i}}{\Delta t} + u(d^*_{n+1,i}) - \delta V_{n+1,i} + \frac{V_{n+1,i+1} - V_{n+1,i-1}}{2\Delta x}\left(\mu - d^*_{n+1,i} - b\lambda)\right) \\ &+ \frac{b^2\lambda}{2} \frac{V_{n+1,i+1} - 2V_{n+1,i} + V_{n+1,i-1}}{(\Delta x)^2}, \end{aligned} \tag{6}$$

where we have used first- and second-order central difference to approximate the corresponding derivatives. For the explicit scheme we evaluate the derivatives at the time step $n+1$. Since optimal policies typically vary slowly, we evaluate the optimal policy $d^*_{n+1,i} \triangleq d^*(t_{n+1}, x_i)$ at t_{n+1}. Reordering the terms gives the equation

$$V_{n,i} = a_{n+1,i} V_{n+1,i-1} + b_{n+1,i} V_{n+1,i} + c_{n+1,i} V_{n,i+1} + \Delta t u(d^*_{n+1,i})$$

with the coefficients given by

$$\begin{aligned} a_{n+1,i} &= \frac{\Delta t}{2}\left[-\frac{\mu - d^*_{n+1,i} - b\lambda}{\Delta x} - \frac{b^2\lambda}{(\Delta x)^2}\right], \\ b_{n+1,i} &= 1 + \Delta t\left[\frac{b^2\lambda}{(\Delta x)^2} - \delta\right], \\ c_{n,i} &= \frac{\Delta t}{2}\left[\frac{\mu - d^*_{n+1,i} - b\lambda}{\Delta x} - \frac{b^2\lambda}{(\Delta x)^2}\right],]. \end{aligned}$$

At each iteration step, we first calculate the value function using the equation above. Next, we compute the boundary values. At the lower boundary $x_{min} = 0$, we apply the boundary condition

$$V_{n,0} = -\frac{1}{\gamma\delta}\left[1 - e^{-\delta\Delta t(N_t - n)}\right] - \frac{1}{\gamma}e^{-\delta\Delta t(N_t - n)}.$$

The primary reason for using an explicit scheme is the lack of boundary conditions at the upper bound. Therefore, we use linear extrapolation to estimate the value at $x = x_{max}$. inally, we compute the optimal dividend policy by discretizing equation (8) using central differences

$$d^*_{n,i} = \begin{cases} -\ln\left(\frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x}\right) & \text{for} \quad \frac{V_{n,i+1} - V_{n,i-1}}{2\Delta x} \geq 1 \\ 0 & \text{else.} \end{cases}. \tag{7}$$

We also set $d^*_{n,0} = 0$ aand use linear extrapolation to determine d^*_{n,N_x}.

Computing the Ruin Probability

We calculate the probability of ruin $P(\tau \leq t)$ using Monte Carlo simulations. To do this, we simulateN_{MC} paths up to time T by discretizing the wealth equation (3). Given a value X_t, the next value can be approximated by

$$X_{t+1} = X_t + (\mu - b\lambda)\Delta t - d^*_t(t, X_t)\Delta t + b\sqrt{\lambda}\sqrt{\Delta t}\,\varepsilon_t,$$

where $\varepsilon_t$is a standard normally distributed random variable. The values for $d^*_t(t, X_t)$ are obtained by linear interpolation of the policy function on the grid if X_t is within the grid, by linear extrapolation for $X_t > x_{max}$, and set to zero for $X_t < 0$. Once the path value drops below zero, i.e., $X_t < 0$, all subsequent values are set to zero. The probability of ruin is then estimated by counting the number of paths where $X_t = 0$ at time t, and and dividing by the total number of simulated paths

$$P(\tau \leq t) = \frac{\text{Number of paths with } X_t = 0}{N_{MC}}.$$

3 Results

In the attached code, we solve for the value function using the following parameter values:

$$\mu = 2, \quad \gamma = 0.1, \quad \delta = 0.05, \quad \lambda = 0.1, \quad \text{and} \quad b = 5.$$

For the grid, we use:

$$T = 100, \quad N_t = 100.000, \quad x_{max} = 10, \quad \text{and} \quad N_x = 100,$$

which implies a grid spacing of $\Delta t = 0.001$ and $\Delta x = 0.1$. For the Monte Carlo simulations, we simulate a total of $N_{MC} = 10.000$ paths. The value function and the policy function at $t = 0$ are depict in the Figure 1. The probability of ruin for three different starting values is depicted in Figure 2. The black line represents the probability of ruin for $X_0 = 1.5$, the green line for $X_0 = 2.5$, and the purple line for $X_0 = 5$.

Figure 1: Optimal Dividends and Value Function. The left figure depicts the optimal dividend policy $d^*(0, X_t)$ at $t = 0$. The right figure shows the corresponding value function $V(0, X_t)$. We set the parameters to $\mu = 2$, $\gamma = 0.1$, $\delta = 0.05$, $\lambda = 0.1$ and $\underline{=}5$.

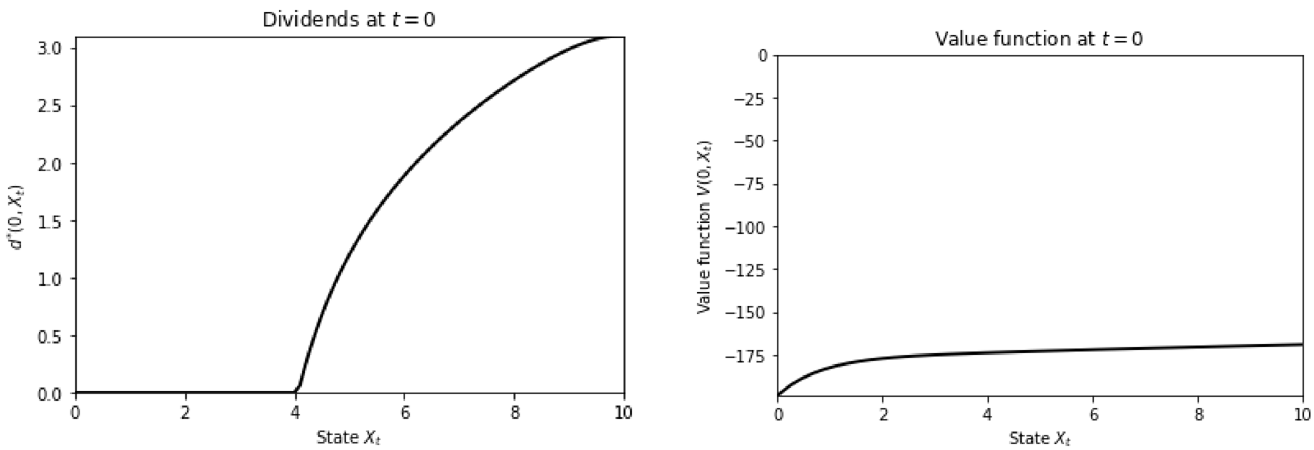
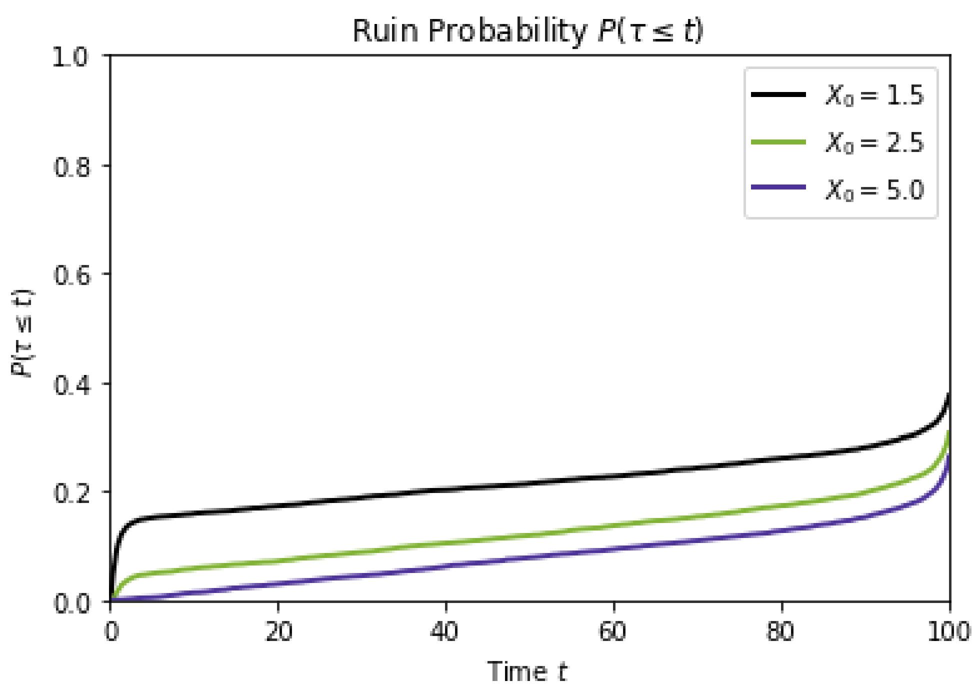


Figure 2: Probability of Ruin The picture depicts the probability of ruin for three different starting values. The black line depicts the probability of ruin for $X_0 = 1.5$, the green line for $X_0 = 2.5$, and the purple line for $X_0 = 5$. We set the parameters to $\mu = 2$, $\gamma = 0.1$, $\delta = 0.05$, $\lambda = 0.1$ and $\underline{=}5$.



References

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