Replication Sannikov (2008)

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Abstract

This study replicates the continuous-time principal–agent model of Sannikov (2008). We formulate the incentive-compatible optimal contracting problem with moral hazard in infinite horizon and solve it numerically using an implicit finite-difference method applied to the associated Hamilton–Jacobi–Bellman (HJB) equation. The replication confirms the original model's results for the baseline case (optimal retirement of the agent at an endogenously determined threshold) and demonstrates how outcomes change when the agent can quit, be replaced at a cost, or be trained (promoted) to higher productivity. Under each scenario, we compute the principal's value function and optimal effort and consumption policies.

1 The Model

1.1 The Benchmark Setting

Let $a = \{a_t \in A, 0 \leq t < \infty\}$ denote the agent's effort process (or effort strategy), with A being the set of feasible effort levels. Given an effort strategy a, the total output process $X = \{X_t, 0 \leq t < \infty\}$ driven by a standard Brownian motion $W = \{W_t, 0 \leq t < \infty\}$ is defined as

$$dX_t = a_t dt + \sigma dW_t,$$

where we assume a constant volatility σ. Applying effort imposes a cost on the agent, which is measured by the function $h(a_t)$. As compensation, the agent receives non-negative consumption payments c_t from the principal, which provides the utility $u(c_t)$ to him. Following Sannikov (2008), we assume that the cost of effort and the utility of consumption are measured in the same units. The principal faces the challenge of moral hazard, as she can only observe the output process and not the agent's effort. The principal must indirectly influence the agent's behaviour through contract design based on her observation of output. A contract contingent on the history of output is offered from the principal to the agent before he starts working. In essence, the principal–agent problem is the difficulty of finding a contract, based on the output history, that maximizes the principal's utility while ensuring that the agent chooses an effort strategy that maximizes his own utility and guarantees at least the reservation utility R_0 at $t = 0$.

To formalize the problem, we introduce the agent's and principal's continuation values for a given admissible consumption contract and effort strategy (c, a), denoted by W_t^A and W_t^P respectively. For the principal, we have

$$\begin{aligned} W_t^P &= \mathbb{E}_t \left[\delta \int_t^\infty e^{-\delta(s-t)} dX_s - \delta \int_t^\infty e^{-\delta(s-t)} c_s ds \right] \\ &= \mathbb{E}_t \left[\delta \int_t^\infty e^{-\delta(s-t)} (a_s - c_s) ds \right], \end{aligned}$$

and for the agent,

$$W_t^A = \mathbb{E}_t \left[\delta \int_t^\infty e^{-\delta(s-t)} (u(c_s) - h(a_s)) ds \right], \tag{1}$$

where we used $\mathbb{E}_t\left[\int_t^\infty e^{-\delta(s-t)}dW_s\right] = 0$. The principal has a risk-neutral utility function and thus only cares about the discounted expected cash flow (output minus the promised consumption to the agent). The agent's utility function is assumed to be risk-averse and separable. Following Sannikov (2008), we rescale the expected flow of utility with the discount rate δ for convenience and normalize it such that total pay-offs are of the same scale as flow pay-offs. In what follows, we sometimes refer to the set (c, a) simply as the consumption contract for readability. We also sometimes refer to the principal's continuation value as the principal's profit, since she is risk-neutral.

Mathematically, the principal–agent problem can be expressed as a Stackelberg non-zero-sum game (see, e.g., Possamaï and Touzi (2022)), which reflects a bi-level optimization problem. At the first level, the principal optimizes her utility at $t = 0$:

$$U_0^P \triangleq \sup_{c\in\mathcal{C}} W_0^P(c),$$

subject to the agent's optimization problem

$$U_0^A \triangleq \sup_{a\in\mathcal{A}} W_0^A(c,a) \geq R_0$$

where U_t^P and U_t^A denote the principal's and agent's optimal continuation utilities, respectively. The incentive compatibility constraint enters through the agent's maximization over admissible effort strategies $\mathcal{A}$ that satisfy the participation constraint (ensuring an continuation value of at least R_0). $\mathcal{C}$ denotes the set of implementable consumption contracts, meaning contracts for which at least one optimal effort strategy solving the agent's problem exists. The necessary technical requirements for a solution to this principal–agent problem are (see Sannikov (2008)):

- $A \subset \mathbb{R}$ is compact, with 0 being the smallest element.
- The agent's utility function $u : [0, \infty) \to [0, \infty)$ is at least C^2 with $u' > 0$ and $u'' < 0$. Additionally, we assume $\lim\limits_{c\to\infty} u'(c) \to 0$ and $u(0) = 0$.
- The agent's cost function $h : A \to \mathbb{R}$ is continuous, increasing, and convex. Moreover, $h(0) = 0$, and there exists a $\gamma_0 > 0$ such that $h(a) \geq \gamma_0 a$ for all $a \in A$.

(Weaker technical conditions can be found in Cvitanić and Zhang (2012), also treat the finite-horizon case and a risk-averse principal. Using a duality approach, they can even incorporate more complex participation constraints that cannot be handled by a dynamic programming approach.)

1.2 The Agent's Problem

Consider a fixed consumption contract c and an admissible effort strategy a. Using the martingale representation theorem we can express the stochastic differential equation (SDE) for the agent's continuation value as (see Sannikov (2008))

$$dW_t^A = \delta\left(W_t^A - u(c_t) + h(a_t)\right)dt + \delta Y_t(dX_t - a_t dt), \tag{2}$$

where $Y = \{Y_t, 0 \leq t < \infty\}$ is some progressively measurable process. The martingale representation theorem guarantees the existence of such a representation, but it provides no insight into how to find the process Y. Since W_t^A depends on (c, a), so does Y (we suppress this dependence for readability). Following Sannikov (2013), we proof the statement by defining the process

$$V_t^A = \mathbb{E}_t\left[\delta\int_0^\infty e^{-\delta s}(u(c_s) - h(a_s))ds\right] = \delta\int_0^t e^{-\delta s}(u(c_s) - h(a_s))ds + e^{-\delta t}W_t^A, \tag{3}$$

which represents the agent's total expected pay-off given the information at time t. By the law of iterated expectations, V_t^A is a martingale. Using the martingale representation theorem, we can express V_t^A as

$$dV_t^A = \delta Y_t e^{-\delta t}\sigma dZ_t = \delta Y_t e^{-\delta t}(dX_t - a_t dt). \tag{4}$$

On the other hand, using the expression (3) and applying Itô's lemma, we obtain

$$dV_t^A = \delta e^{-\delta t}(u(c_t) - h(a_t))dt - \delta e^{-\delta t} W_t^A dt + e^{-\delta t} dW_t^A. \tag{5}$$

Combing (4) and (5) yields (2). Additionally, under the above assumptions on the agent's utility and cost functions, Sannikov (2013) uses the dominated convergence theorem to prove that the following transversality condition holds for the agent's continuation value:

$$\lim_{s\to\infty} \mathbb{E}_t[e^{-rs}W_{t+s}] = 0 \quad \text{almost everywhere.} \tag{6}$$

The transversality condition ensures uniqueness: among all processes that satisfy SDE (2) for various starting values and different choices of Y, only one process satisfies (6). That process can then be identified with the agent's continuation value. If the value W_t^A promised by the principal to the agent does not align with the actual delivered value, the transversality condition is violated. Mathematically, this follows from considering any process $\widetilde{W}_t^A$ that satisfies (2) and (6). Exploiting the martingale property of V_t^A, one can show that if the transversality condition holds, any such $\widetilde{W}_t^A$ equals the agent's continuation value process (1).

The term δY_t represents the sensitivity of the agent's continuation value to output and thus influences the agent's incentives (see Sannikov (2008, 2013)). In discrete time, the one-shot deviation principle states that the agent's strategy a^* is optimal if and only if at each time t the agent maximizes the expected impact of effort on his continuation value minus the cost of effort. In continuous time, following Sannikov (2008), the analogous condition is that the agent at each instant maximizes

$$\delta Y_t a_t - \delta h(a_t).$$

Formally, given a contract c, an effort strategy a^* is optimal for the agent (i.e. incentive compatible) if for all $\tilde{a} \in A$ and all $t \geq 0$ (see Sannikov (2008, 2013))

$$\forall \tilde{a} \in \mathcal{A},\ t \in [0,\infty),\ Y_t a_t^* - h(a_t^*) \geq Y_t \tilde{a}_t - h(\tilde{a}_t) \tag{7}$$

holds.

The proof of this result utilizes the process V_t^A. Let $\widetilde{V}_t^A$ denote the total expected pay-off process following a deviation strategy $\tilde{a}$ (where the agent follows $\tilde{a}$ until time t and then reverts to a^* thereafter). The dynamics of $\widetilde{V}_t^A$ are (see Sannikov (2008))

$$\begin{aligned}
d\tilde{V}_t^A &= \delta e^{-\delta t}(u(c_t) - h(a_t))dt - \delta e^{-\delta t} W_t^A dt + e^{-\delta t} dW_t^A \\
&= \delta e^{-\delta t}(u(c_t) - h(a_t))dt - \delta e^{-\delta t}(u(c_t) - h(a_t^*))dt + \delta e^{-\delta t} Y_t (dX_t - a_t^* dt) \\
&= \delta e^{-\delta t}(h(a_t^*) - h(a_t))dt + \delta e^{-\delta t} Y_t (\tilde{a}_t - a_t^*)dt + \delta e^{-\delta t} Y_t \sigma dZ_t
\end{aligned}$$

where we have expressed the output process under the measure corresponding to effort $\tilde{a}$. If the incentive compatibility condition (7) holds, then the drift of $\widetilde{V}_t^A$ is non-positive, and it follows that

$$\begin{aligned}
W_0^A(c,\tilde{a}) = \mathbb{E}_0\left[\tilde{V}_t^A\right] &= \tilde{V}_0^A + \mathbb{E}_0\left[\int_0^t \delta e^{-\delta s} \underbrace{((h(a_s^*) - h(a_s)) + Y_s(\tilde{a}_s - a_s^*))}_{\leq 0} ds\right] \\
&\leq \tilde{V}_0^A = W_0^A(c, a^*).
\end{aligned}$$

Here we used again the fact that the expectation of the stochastic integral is zero. Given the transversality condition, this result holds as $t \to \infty$, implying that for any t the deviating strategy $\tilde{a}$ can never yield a higher payoff than a^*. Conversely, if condition (7) fails on a set of positive measures, then we choose a^* to be (see Sannikov (2008, 2013))

$$a_t^* = \underset{a_t \in \mathcal{A}}{\operatorname{argmax}} \{Y_t a_t - h(a_t)\} \tag{8}$$

and obtain an incentive-compatible effort strategy. Here we dropped the index on Y_t, given that we will consider Y_t as an additional control variable later. The reasoning behind this approach is motivated by the solution method, in which we propose a consumption contract c along with a recommended effort strategy a^{rec}. The recommended effort strategy is designed such that the agent will indeed follow it, thereby achieving incentive compatibility and satisfying the participation constraint. Instead of considering the control set $\{c_t, a_t \mid t \geq 0\}$, we can equivalently consider the controls $\{c_t, a_t, Y_t \mid t \geq 0\}$. The primary benefit of this is that we can treat Y_t as given and then choose $a(Y)$ such that incentive compatibility is satisfied. Given that $h(a)$ is strictly convex, there exists a unique effort level $a_t(Y_t)$ that satisfies (8). If we further assume that $h(a)$ is twice continuously differentiable, condition (8) yields

$$Y_t = h'(a_t^*). \tag{9}$$

The crucial step in this approach is the application of the martingale representation theorem: by introducing Y_t as an additional control variable, the incentive compatibility constraint depends only on variables at the same time t, making a dynamic programming approach viable.

1.3 The Principal's Problem

The most intuitive formulation of the principal–agent problem is that the principal offers a contract c to the agent, with contract payments at time t contingent on the history of output up to time t. The agent then reacts by selecting an effort strategy that maximizes his utility. As mentioned above, this setup resembles a two-step sequential game, usually solved by backward induction. The difficulty in this setup lies in the fact that one is effectively optimizing over a set of stochastic processes. The first step toward a solution is to introduce the recommended effort strategy into the optimization problem. This approach is common in the principal–agent literature and also appears in discrete time (see, e.g., Phelan and Townsend (1991)). As long as the recommended effort strategy yields a higher utility for the agent than any other strategy, the agent will adopt it. Equivalently, for a given consumption contract, the recommended effort strategy will be accepted if it is incentive compatible (and technically the participation constraint must also be satisfied). In discrete time, the complexity is reduced by using the agent's continuation value as a state variable (see Spear and Srivastava (1987)). The crux is that the agent's continuation value summarizes the past output history in a way that is relevant for the agent's incentives. The intuition is that the agent's incentives remain unchanged if one replaces the continuation contract with a different one, as long as both deliver the same continuation value. A solution to the principal–agent problem would then be characterized by a value function for the principal $F(w)$ and policy functions for the consumption $c(w)$ and recommended effort $a(w)$, all as functions of the agent's remaining utility w. At this stage we no longer distinguish between the recommended effort strategy and the effort actually chosen by the agent, since we will always assume incentive compatibility. In an infinite-horizon setting, w is identical to the agent's initial continuation value if he agrees to the proposed contract. Thus, the principal chooses a contract that maximizes her value function $F(w)$ under the constraint $w \geq R_0$. (In practice, the participation constraint can be checked at the end, after finding the value function and optimal policies as functions of w.)

In the risk-neutral setup, the output process does not need to be treated as a state variable. If the principal is risk-averse, however, the output process would become a second state variable alongside the agent's continuation value. Only in the risk-neutral infinite-horizon case can a reduced-form solution (with a single state variable w) be considered. Due to the infinite horizon, the value function is independent of time. Moving forward, we define the deterministic function $F(W_t^A)$ representing the principal's value (given the agent's continuation value W_t^A) by

$$F(W_t^A) = U_t^P$$

where is U_t^P is the principals optimal remaining utility analogue to equation (1.1). Following the arguments at the end of the last section, we consider the principal's set of controls to be $\{c_t, a_t, Y_t \mid t \geq 0\}$, implying that the principal controls the agent's continuation value process determined by the SDE

$$dW_t^A = \delta \left(W_t^A - u(c_t) + h(a_t) \right) dt + \delta Y_t dZ_t.$$

Following Sannikov (2008) the corresponding HJB equation is given by

$$\max_{a>0,\ c} \left\{ \delta(a-c) + \delta F' \left(w - u(c) + h(a) \right) + \frac{\delta^2 \sigma^2}{2} (h'(a))^2 F'' - \delta F \right\} = 0. \tag{10}$$

Incentive compatibility is ensured here by choosing $Y_t = h'(a_t)$ for $a_t > 0$. (The case $a_t = 0$ falls under a boundary condition and so need not be considered in the HJB interior.) Because of the infinite horizon, the HJB lacks an explicit time derivative term. Thus, boundary conditions are needed to have a unique solution. Various scenarios can be implemented by imposing different boundary conditions. The lower boundary at the agent's minimum continuation value w_L is obtained as follows. If the agent's lowest utility level is reached, he will exert zero effort and will be paid the minimal consumption $c_{\min}$ (see Cvitanić and Zhang (2012)). If at w_L the contract still specifies more than the minimal payment, the agent can choose zero effort forever, yielding himself a higher utility than the supposed minimum, contradicting the assumption that w_L is minimal. Thus, hitting w_L is effectively equivalent to "retiring" the agent by providing minimal consumption for the remainder of the horizon and allowing zero effort. Since the minimal consumption level in Sannikov's model is zero, we directly obtain the boundary condition at the lower bound:

$$F(w_L) = 0.$$

More complicated boundary conditions will be discussed in detail in the next section.

Using the HJB equation (10), we can already gain some insights about the optimal consumption contract and the optimal effort strategy. The optimal consumption at any state w maximizes the expression (see Sannikov (2008))

$$-c - F'(w)u(c).$$

Due to our assumptions on $u(c)$, we find that $c = 0$ as long as $F(w)$ is increasing. Differentiating the expression (21), we obtain the first-order condition (FOC) for optimal consumption when $F'(w) \geq 0$:

$$\frac{1}{u'(c)} = -F'(w).$$

The left-hand side is the marginal cost of providing the agent with utility w through current consumption, and the right-hand side is the marginal cost of providing the same utility through the principal's continuation payoff. Both costs must be equal under the optimal contract. Only if there are no costs (i.e. if $h(a) = 0$) will $F'(w) \geq 0$ hold, in which case zero consumption is optimal—maximizing the drift of the agent's continuation value in (2) and moving the state away from the inefficient retirement point w_L (see Sannikov (2008)).

The optimal effort maximizes the expression (see Sannikov (2008))

$$\delta a + \delta h(a) F'(w) + \frac{\delta^2 \sigma^2}{2} (h'(a))^2 F''(w),$$

implying the FOC

$$1 = F'(w)h'(a) + \delta\sigma^2 h'(a)h''(a)F''(w).$$

The first term in (1.3) reflects the expected flow of output; the second term is related to the cost of compensating the agent for his effort; and the third term is related to the cost of exposing the agent to income uncertainty. These two types of costs can act in opposite directions. We will later see that in all scenarios considered, $F''(w) < 0$, while $F'(w)$ may change sign, leading to a more complex effort profile. The exact behavior depends on the scenario and parameter choices.

1.4 The Different Scenarios

The baseline scenario considered by Sannikov (2008) is retirement. etirement occurs as soon as it becomes more beneficial for the principal to retire the agent by providing him his current continuation value via a constant consumption stream (while allowing zero effort). Sannikov (2008) assumes that at this moment the output process is stopped, turning the problem into an optimal stopping scenario. Even after the output process is stopped, the agent continues to receive a constant retirement payment c_{ret} until the end of the horizon. This retirement payment is calibrated to maintain the agent's current continuation value while he exerts zero effort. With the retirement payment fixed, its magnitude can be computed using the expression for the continuation value:

$$W^A_{ret,t} = \delta \int_t^\infty e^{-\delta(s-t)} u(c_{ret}) ds = u(c_{ret}),$$

where we omitted expectation operators due to the deterministic consumption stream. If the agent is to retire at a continuation value w_{ret}, a constant consumption stream of size $c_{ret} = u^{-1}(w_{ret})$ must be provided. The principal's retirement profit from retiring the agent at a general continuation value w is denoted $F_{ret}(w)$ and can be calculated as

$$F_{ret}(w) = -\delta \int_t^\infty e^{-\delta(s-t)} c_{ret} = -c_{ret} = -u^{-1}(w).$$

Given the principal's ability to retire the agent at any time, the principal's value function $F(w)$ must satisfy

$$F(w) \geq F_{ret}(w). \tag{11}$$

This implies a boundary condition $F(w_{ret}) = F_{ret}(w_{ret})$ for some point $w_{ret} \geq 0$. The threshold w_{ret} is endogenously determined, implying that we are dealing with a free-boundary problem. In the infinite-horizon setting, one imposes smooth-pasting conditions to ensure a smooth transition of the principal's value function from the non-retirement region to the retirement region. Specifically (see Sannikov (2008)) :

$$F(w_{ret}) = F_{ret}(w_{ret}) \quad \text{and} \quad F'(w_{ret}) = F'_{ret}(w_{ret}). \tag{12}$$

These two conditions jointly determine the solution for $F(w)$ and the retirement point w_{ret}. When using an ordinary differential equation (ODE) approach to solve the model, one would adjust the initial slope $F'(0)$ such that the smooth-pasting conditions are satisfied. Hence, the smooth-pasting conditions effectively translate into a condition on the initial slope. Alternatively, if one employs a partial differential equation (PDE) approach, the retirement scenario can be implemented by imposing the condition (11) directly.

As a first extension, Sannikov (2008) allows the agent the possibility to quit working for the principal at any time, in which case the agent receives an outside continuation value $\widetilde{W}^A_t > 0$. This $\widetilde{W}^A_t$ can be interpreted as the utility derived from some outside option (e.g. the value of a new employment opportunity minus any transition costs). If the agent quits, the output process is stopped, similarly to the retirement case. This introduces a new lower bound on the agent's continuation value, $\widetilde{W}_t^A$, which is exogenously given. In contrast to the retirement scenario, the principal is not obliged to pay any further consumption to the agent after he quits. The principal's profit, given that the agent quits, is then zero, since no further output is produced and no further payments are made. Incorporating this scenario into the model removes the lower retirement point (w_L becomes $\widetilde{w}$ instead of 0). To consider only reasonable configurations, we assume $\widetilde{W}_t^A \leq R_0$ (see Cvitanić and Zhang (2012)); otherwise, the agent would have an incentive to quit before he even starts working. More precisely, $\widetilde{W}_t^A \geq R_0$ only makes sense in cases where the principal promises the agent a higher initial continuation value than his reservation utility (something that, as we will see, can occur in Sannikov's model). However, in this scenario we restrict ourselves to $\widetilde{W}_t^A < R_0$. Let $\widetilde{w}$ denote the numerical value of the agent's outside option. The lower boundary condition for the principal's value function then becomes

$$F(\tilde{w}) = 0.$$

Thus, the lower boundary is now at $w_L = \widetilde{w}$ instead of 0. However, there still exists a higher retirement threshold w_{ret}, satisfying the same smooth-pasting conditions as in (12).

Sannikov (2008) also studies a scenario in which the agent cannot quit freely. Instead, the agent can only be replaced by a new one at a fixed replacement cost C. In this scenario, the principal retires the old agent and gains a profit D from replacing him. The value of D is determined endogenously under the assumption that the market is full of identical agents with the same reservation utility R_0. Under this assumption, D is determined by

$$D = F(w_0) - C, \tag{13}$$

where w_0 is the initial utility promised by the principal to any new agent with reservation utility R_0, and w_0 is not known a priori. Equation (30) simply reflects that the value of replacing the agent is the principal's profit from employing a new agent minus the cost of hiring him. In this scenario, it is essential that the principal is still obliged to pay the retirement benefit to the old agent. If not, it would be possible for the principal to increase her profit without bound by replacing the agent infinitely often. The value w_0 will be determined such that the principal maximizes her profit on the interval $[R_0, w_{ret}]$ (see Cvitanić and Zhang (2012)). To solve this, one can fix an initial guess for D, compute the principal's value function $F(w)$ using the boundary condition $F(0) = D$ and the smooth-pasting conditions

$$F(w_{ret}) = F_{ret}(w_{ret}) + D \quad \text{and} \quad F'(w_{ret}) = F'_{ret}(w_{ret}).$$

Then one can calculate the implied w_0 and $F(w_0)$ providing an updated on D via (13). If the computed D matches the initial guess, the problem is solved; otherwise, one adjusts the guess and repeats. Sannikov (2008) exploits the fact that different costs C only shift D in an affine way and do not affect the maximization; therefore, he works directly with a numerical value of D, assuming it is attained for some suitable C.

Lastly, Sannikov (2008) considers the scenario of promotion. The principal can train the agent at a cost K. Once the agent is trained his productivity is permanently increased, as an instantaneous increase in the effectiveness of effort, changing the output drift from a to θa with $\theta > 1$.. Simultaneously, Sannikov (2008) assumes that a trained agent has access to an outside option $\widetilde{W}^P \geq 0$, since the agent's new skills might be valuable to other firms. We solve this scenario by calculating the principal's profit arising from a trained agent—meaning we solve the HJB equation with adjusted productivity and apply the corresponding smooth-pasting conditions. Let $F_P(w)$ denote the solution of the adjusted HJB and let $\widetilde{w}_P$ be the numerical value of the outside option given promotion. Then $F_P(w)$ must satisfy the conditions

$$F_P(\tilde{w}_P) = 0, \quad F_P(w_{ret}) = F_{ret}(w_{ret}) \quad \text{and} \quad F'(w_{ret}) = F'_{ret}(w_{ret}).$$

Since even after promotion, retirement is still possible, we impose the smooth-pasting conditions at w_{ret} again. Initially the agent is untrained, and thus the principal has to decide on the timing of promotion or retirement. In case the principal promotes the agent, her profit is $F_P(w) - K$. Putting everything together, the principal's value function must satisfy

$$F(w) \geq \max\{F_{ret}(w), F_P(w) - K\},$$

leading to the lower boundary condition $F(0) = 0$ and smooth-pasting conditions analogous to before (see Sannikov (2008))

$$F(w_P) = F_P(w_P) - K \quad \text{and} \quad F'(w_P) = F_P'(w_P),$$

where w_P is the promotion threshold. These smooth-pasting conditions are only relevant if promotion actually occurs before retirement. It may happen that for specific parameter constellations, immediate retirement is more beneficial than promotion. In that case, one would revert to the original smooth-pasting conditions (12) and simply compare which situation is more beneficial for the principal.

All scenarios can be summarized using a function $F_0 : [\widetilde{w}, \infty) \to \mathbb{R}$ (see Sannikov (2008)), with $\widetilde{w}$ denoting the numerical value of the agent's outside option in general. $F_0(w)$ represents the principal's option to receive the profit $F_0(w)$ by delivering to the agent a continuation value of w. The baseline scenario of retirement is captured by setting $\widetilde{w} = 0$ and $F_0(w) = F_{ret}(w)$. For the second scenario, we consider $\widetilde{w} > 0$ (the agent has an outside option) with again $F_0(w) = F_{ret}(w)$. The third scenario (replacement) is represented by $\widetilde{w} = 0$ and $F_0(w) = F_{ret}(w) + D$. For the last scenario (promotion), we again take $\widetilde{w} = 0$ and set $F_0(w) = \max F_{ret}(w),\, F_P(w) - K$. Using this representation, we can easily incorporate the different scenarios into the principal–agent problem by adjusting the expressions for the principal's and agent's continuation values. In addition to the consumption contract (c, a), the principal now also specifies a stopping time τ, at which the agent receives the continuation value W_τ^A while the principal receives the profit $F_0(W_\tau^A)$. In every scenario, there are two values of the agent's continuation utility at which the output process might be stopped: the lower point $w_L = \widetilde{w}$, and a higher point w_H (which might be the retirement or promotion threshold). The stopping time is defined by

$$\tau = \inf\{t : W_t^A = w_L \text{ or } W_t^A = w_H\}.$$

The adjusted continuation values for a given set of strategies (c, a) and stopping time τ then read (see Sannikov (2008))

$$\begin{aligned} W_t^P &= \mathbb{E}_t\left[\delta \int_t^\tau e^{-\delta(s-t)}(a_s - c_s)ds + e^{-\delta(\tau-t)} F_0(W_\tau^A)\right], \\ W_t^A &= \mathbb{E}_t\left[\delta \int_t^\tau e^{-\delta(s-t)}(u(c_s) - h(a_s))ds + e^{-\delta(\tau-t)} W_\tau^A\right]. \end{aligned}$$

This modification does not change the results concerning incentive compatibility, given that the problem effectively divides into two parts. For $t < \tau$, the results of the previous section still apply, while for $t \geq \tau$ the contract is in a different phase (with different payments) defined such that incentive compatibility is satisfied. The participation constraint remains the same for the first and third scenarios. For the second and last scenarios, we must additionally include the constraint that the agent's continuation value cannot drop below $\widetilde{W}_t^A$ at any time (due to the agent's outside option). This is automatically handled by the introduction of the stopping time.

2 Numerical Method

2.1 Finite Time Horizon Setting

This section explains how the HJB equation is solved using a implicit finite difference scheme. Sannikov (2008) onsiders an infinite-horizon setting, implying that the HJB equation reduces to a nonlinear ODE with the agent's continuation value as the only state variable. Using a finite-differences approach, we will consider Sannikov's model on a finite time horizon T and introduce a terminal condition along with adjusted boundary conditions. If the time horizon T is sufficiently large, we expect to reproduce the same results as Sannikov (2008). First we need to implement a consistent terminal condition. In line with Cvitanić and Zhang (2012) and Williams (2015), we assume that at the time horizon a final consumption payment C_T is made. We further assume that the principal has a risk-neutral utility function with respect to the final payment C_T, implying that the principal's continuation value at the horizon is given by

$$W_T^P(C_T) = -C_T.$$

Similarly, we assume the agent's utility from the final payment is

$$W_T^A = u(C_T).$$

Combining these two conditions, a consistent terminal condition for the principal's value function reads:

$$F(T, w) = -u^{-1}(w).$$

In a finite-horizon setting, the principal's value function also depends on time. The corresponding HJB equation is:

$$\max_{a,c} \left\{ F_t + \delta(a - c) + \delta F_w \left(w - u(c) + h(a)\right) + \frac{\delta^2\sigma^2}{2}(h'(a))^2 F_{ww} - \delta F \right\} = 0,$$

where we used equation (10) and added the time-derivative term F_t.we develop consistent boundary conditions for the finite-horizon case under the different scenarios. A consistent boundary condition at the lower boundary for the first, second, and last scenarios is given by

$$F(t, w_L) = 0.$$

In the case that the agent is retired at the value $w_{ret} > 0$, the final consumption payment $C_{ret,T} = u^{-1}(w_{ret})$ ensures that the agent remains at his continuation value w_{ret} until the time horizon T. Further, we assume that a constant retirement payment c_{ret} is paid if the agent is retired earlier at some $t < T$. Setting

$$c_{ret} = u^{-1}(w_{ret}),$$

we can verify this using the finite-horizon expression for the agent's continuation value under zero uncertainty and zero effort:

$$\begin{aligned} W_t^A = \delta \int_t^T e^{-\delta(s-t)} u(c_{ret}) + e^{-(T-t)} u(C_{ret,T}) &= w_{ret}\left(1 - e^{-\delta(T-t}\right) + w_{ret} e^{-(T-t)}, \\ &= w_{ret}. \end{aligned}$$

Using this, we can calculate the principal's profit $F_{ret}(t, w)$ from retiring the agent in the finite-horizon setting:

$$F_{ret}(t, w) = -\delta \int_t^T e^{-\delta(s-t)} c_{ret} - e^{-\delta(T-t)} C_{ret,T} = -u^{-1}(w).$$

The boundary condition for the first, second, and the promotion scenario (when considering a trained agent) is then:

$$F(t, w_H) = F_{ret}(t, w_H),$$

for some threshold $w_H > 0$ that must be determined. The boundary condition for the last scenario (promotion, initially untrained agent) is:

$$F(t, w_H) = \max\{F_{ret}(t, w_H), F_P(t, w_H) - K\},$$

where retirement or promotion happens at w_H.

The third scenario (replacement) is more complicated in the finite-horizon setting, due to the fact that D is determined endogenously as a function of time. The profit gained from replacing the agent is an unknown function $D(t)$, whose value at the horizon T can be calculated to be

$$D(T) = -u^{-1}(R_0) - C.$$

The profit at time horizon will always be negative and thus no replacement takes place. As we perform backward induction on a grid, we can calculate the value of $D(t)$ numerically by finding the maximum of the principal's profit at each time step. Without using a grid, one would have to guess the function $D(t)$ and then verify if the guess leads to a consistent solution. Assuming $D(t)$ is known, the boundary condition reads:

$$F(t, w_H) = F_{ret}(t, w_H) + D(t),$$

with w_H denoting the point of replacement in this scenario.

2.2 Finite Differences

efore we discretize the model, we calculate the optimal controls analytically. Considering the finite-horizon setting does not change the first-order conditions derived earlier. In order to calculate the optimal effort choice we have to specify the cost function. Here, we follow Sannikov (2008) and set the agent's utility and cost functions to:

$$u(c) = \sqrt{c} \quad \text{and} \quad h(a) = 0.5a^2 + 0.4a.$$

One can easily verify that this specification satisfies all our assumptions. The optimal consumption policy is then:

$$c^*(t,w) = \begin{cases} 0 & \text{for } F_w > 0 \\ \left(-\frac{F_w}{2}\right)^2 & \text{for } F_w \leq 0, \end{cases}$$

while the optimal effort strategy is given by

$$a^*(t,w) = -0.4 - \frac{1}{F_w + \delta\sigma^2 F_{ww}}. \tag{14}$$

in case (14) yields a negative value, will simply set $a = 0$. For a trained agent, the FOC for the optimal consumption policy does not change, while the optimal effort strategy a_P^* is given by a modified expression:

$$a_P^*(t,w) = -0.4 - \frac{\theta}{F_w + \frac{\delta\sigma^2}{\theta^2} F_{ww}}.$$

After the stopping time (i.e. once the agent is retired, quits, is replaced, or is promoted), the optimal controls are set to the values determined by the respective contractual terms.

Given that we can calculate explicit solutions for the optimal policies, we next discretize the reduced-form HJB equation

$$F_t + \delta(a^* - c^*) + \delta F_w \left(\omega - u(c^*) + h(a^*)\right) + \frac{\delta^2\sigma^2}{2}(h'(a^*))^2 F_{ww} - \delta F = 0$$

using an implicit finite-differences scheme. We choose an implicit scheme due to its superior stability and convergence properties (see Wilmott et al. (1998)). We set up an equally spaced lattice in the (t, w)-space using the grid points

$$\{(t_n, w_i) |\ n = 0, 1, \ldots, N,\ i = 0, 1, \ldots, I\},$$

where $t_n = n\,\Delta t$ and $w_i = w_L + i\,\Delta w$, with Δt and Δw being the grid spacings in time and state, respectively. Starting from the terminal condition, we can approximately calculate the value function at every grid point by iterating backward through the grid. At any time t_n with $n < N$, the value $F_{n,i} \approx F(t_n, w_i)$ for interior points $0 < i < I$ can be computed from the values at t_{n+1} using the discretized HJB equation:

$$
\begin{aligned}
0 =& \frac{F_{n+1,i} - F_{n,i}}{\Delta t} + \delta(a^*_{n+1,i} - c^*_{n+1,i}) \\
&+ \delta \frac{F_{n,i+1} - F_{n,i-1}}{2\Delta w}\big(w_i - u(c^*_{n+1,i}) + h(a^*_{n+1,i})\big) \\
&+ \frac{\delta^2\sigma^2}{2}\big(h'(a^*_{n+1,i})\big)^2 \frac{F_{n,i+1} - 2F_{n,i} + F_{n,i-1}}{(\Delta x)^2} - \delta F_{n,i}.
\end{aligned}
\tag{15}
$$

Here we have used central difference approximations for the first and second derivatives. Since optimal policies tend not to vary much with time, we evaluate the policy functions at t_{n+1}, $c_{n+1,i} \triangleq c(t_{n+1}, w_i)$ and $a_{n+1,i} \triangleq a(t_{n+1}, w_i)$, using the corresponding central differences expressions. Reordering the terms of (15), we obtain the following linear equation:

$$
\alpha_{n,i} F_{n,i-1} + \beta_{n,i} F_{n,i} + \gamma_{n,i} F_{n,i+1} = F_{n+1,i} + \delta_{n,i}, \tag{16}
$$

with the coefficients given by

$$
\begin{aligned}
\alpha_{n,i} &= \Delta t \left[\frac{\delta}{2\Delta w}\big(w_i - u(c^*_{n+1,i}) + h(a^*_{n+1,i})\big) - \frac{\delta^2\sigma^2}{2}\big(h'(a^*_{n+1,i})\big)^2 \frac{1}{(\Delta x)^2}\right], \\
\beta_{n,i} &= 1 + \Delta t \left[\delta + \delta^2\sigma^2\big(h'(a^*_{n+1,i})\big)^2 \frac{1}{(\Delta x)^2}\right], \\
\gamma_{n,i} &= \Delta t \left[-\frac{\delta}{2\Delta w}\big(w_i - u(c^*_{n+1,i}) + h(a^*_{n+1,i})\big) - \frac{\delta^2\sigma^2}{2}\big(h'(a^*_{n+1,i})\big)^2 \frac{1}{(\Delta x)^2}\right], \\
\delta_{n,i} &= \Delta t \delta(a^*_{n+1,i} - c^*_{n+1,i}).
\end{aligned}
$$

Equation (16) holds for all $0 < i < I$, leading to $I-1$ equations for $I+1$ unkowns. In order to calculate all unkowns, we have to add two more conditions. In principle one could add two linear interpolation equations for the boundary values $F_{n,0}$ and $F_{n,I}$ to complete the system, but this leads to an equation system which can not be solved in an efficient way. The main advantage of the implicit method lies in the use of boundary conditions at the lower and upper end of the grid in w-direction, leading to an equation system with tridiagonal coefficientmatrix, which can be solved in a very efficient way. Thus assuming that $F_{n,0}$ and $F_{n,I}$ are known, we can rewrite the discretized HJB equation for the point $i = 1$ and $i = I-1$ in the following way

$$
\begin{aligned}
\beta_{n,1} F_{n,1} + \gamma_{n,1} F_{n,2} &= F_{n+1,1} + \delta_{n,1} - \alpha_{n,1} F_{n,0}, \\
\alpha_{n,I-2} F_{n,I-1} + \beta_{n,I-1} F_{n,I-1} &= F_{n+1,I-1} + \delta_{n,I-1} - \alpha_{n,I-1} F_{n,I}.
\end{aligned}
$$

Putting the system of equations into matrix form, we obtain

$$
\begin{pmatrix}
\beta_{n,1} & \gamma_{n,1} & 0 & 0 & 0 & \dots & 0 \\
\alpha_{n,2} & \beta_{n,2} & \gamma_{n,2} & 0 & 0 & \dots & 0 \\
0 & \alpha_{n,3} & \beta_{n,3} & \gamma_{n,3} & 0 & \dots & 0 \\
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \alpha_{n,I-2} & \beta_{n,I-2} & \gamma_{n,I-2} \\
0 & \dots & 0 & 0 & 0 & \alpha_{n,I-1} & \beta_{n,I-1}
\end{pmatrix}
\begin{pmatrix}
F_{n,1} \\ F_{n,2} \\ F_{n,3} \\ \vdots \\ \vdots \\ F_{n,I-2} \\ F_{n,I-1}
\end{pmatrix}
=
\begin{pmatrix}
\tilde{\delta}_{n,1} \\ \tilde{\delta}_{n,2} \\ \tilde{\delta}_{n,3} \\ \vdots \\ \vdots \\ \tilde{\delta}_{n,I-2} \\ \tilde{\delta}_{n,I-1}
\end{pmatrix}
$$

with $\tilde{\delta}_{n,i}$ defined as

$$\tilde{\delta}_{n,i} \triangleq \begin{cases} F_{n+1,1} + \delta_{n,1} - \alpha_{n,1} F_{n,0} & \text{for} \quad i = 1 \\ F_{n+1,i} + \delta_{n,i} & \text{for} \quad 1 < i < I-1 \\ F_{n+1,I} + \delta_{n,I-1} - \gamma_{n,I-1} F_{n,I} & \text{for} \quad i = I-1. \end{cases}$$

This linear system can be solved using a tridiagonal solver algorithm (see Atkinson (1989)), which uses a forward loop to calculate

$$l_{n,i} = \begin{cases} \frac{\gamma_{n,1}}{\beta_{n,1}} & \text{for} \quad i = 1 \\ \frac{\gamma_{n,i}}{\beta_{n,i} - \alpha_{n,i} l_{n,i-1}} & \text{for} \quad i = 2, \dots, I-1 \end{cases}$$

and

$$y_{n,i} = \begin{cases} \frac{\tilde{\delta}_{n,1}}{\beta_{n,1}} & \text{for} \quad i = 1 \\ \frac{\tilde{\delta}_{n,i} - \alpha_{n,i} y_{n,i-1}}{\beta_{n,i} - \alpha_{n,i} l_{n,i-1}} & \text{for} \quad i = 2, \dots, I-1. \end{cases}$$

The solution is then obtained by performing a backward substitution starting at $i = I - 1$ and setting

$$F_{n,i} = \begin{cases} y_{n,i} & \text{for} \quad i = I-1 \\ y_{n,i} - l_{n,i} y_{n,i+1} & \text{for} \quad i = I-2, \dots, 1. \end{cases}$$

Depending on the scenario, we set the values of $F_{n,0}$ and $F_{n,I}$ accordingly. $F_{n,0}$ is determined by the lower boundary condition $F(t, w_L)$ as discussed in the previous section. The upper boundary condition is more complicated, since the point w_H at which retirement or promotion happens is unknown. Despite not knowing the exact value of w_H, we can ensure the grid is chosen sufficiently large such that at $i = I$ the agent has been retired, replaced, or promoted, providing an applicable boundary condition. Here we may have to try different grid sizes until a consistent solution is reached. Moreover, since we are dealing with an optimal stopping problem, we have to modify the backward loop of our tridiagonal solver algorithm slightly. Following Wilmott et al. (1998), we adjust the backward substitution as follows:

$$F_{n,i} = \begin{cases} \max\{y_{n,i}, F_0(t_n, w_i)\} & \text{for} \quad i = I-1 \\ \max\{y_{n,i} - l_{n,i} y_{n,i+1}, F_0(t_n, w_i)\} & \text{for} \quad i = I-2, \dots, 1. \end{cases}$$

As soon as we have calculated the value function at some time step t_n, we compute the optimal policies by replacing the analytical derivatives with the corresponding central difference formulas. We proceed in this manner until the stopping time is reached. At the boundaries, the optimal policies are then set according to the boundary conditions. We handle the replacement scenario's profit function $D(t)$ similarly. We assume that this function does not vary much over time, and thus we can use its value at t_{n+1} to determine the boundary condition at t_n. Then at t_n, we calculate $D(t_n)$ numerically by determining the maximum of the value function on the grid.

3 Results

In the attached code, we solve for the value function using the following parameter values:

$$\delta = 0.1, \quad \sigma = 1.0, \quad \tilde{w}_Q = 0.1, \quad C = 0.0341, \quad \theta = 1.5, \quad K = 0.1 \quad \text{and} \quad \tilde{w}_P = 0.2.$$

For first three scenario we use gird settings given by

$$T = 100, \quad N = 100.000, \quad w_{max} = 2, \quad \text{and} \quad I = 100.$$

which implies a grid spacing of $\Delta t = 0.001$ and $\Delta w = 0.02$. For the fourth scenario, we use $I = 200$, imypling a gird spacing of $\Delta w = 0.01$. Figure 1 depicts the value function, optimal consumption payments, and the optimal effort strategy for the retirement scenario. Figure 2 depicts the value function, optimal consumption payments, and the optimal effort strategy for the quitting scenario. Figure 3 depicts the value function, optimal consumption payments, and the optimal effort strategy for the replacement scenario. Figure **??** depicts the value function, optimal consumption payments, and the optimal effort strategy for the promotion scenario with an initially trained agent, while Figure 5 depicts the same with an initially untrained agent. Figure 2 depicts the value function, optimal consumption payments and the optimal effort strategy for the quitting scenario. Figure 3 depicts the value function, optimal consumption payments, the optimal effort strategy and the replacement profit for the replacment scenario. Figure **??** depicts the value function, optimal consumption payments and the optimal effort strategy in the promotion scenario with an intially trained agent, while **??** depicts the same with an initally untrained agent.

Figure 1: Retirement Scenario The picture depicts the value function, optimal consumption payments and the optimal effort strategy for the retirement scenario. We set the parameters to $\delta = 0.1$, $\sigma = 1.0$, and $T = 100$.

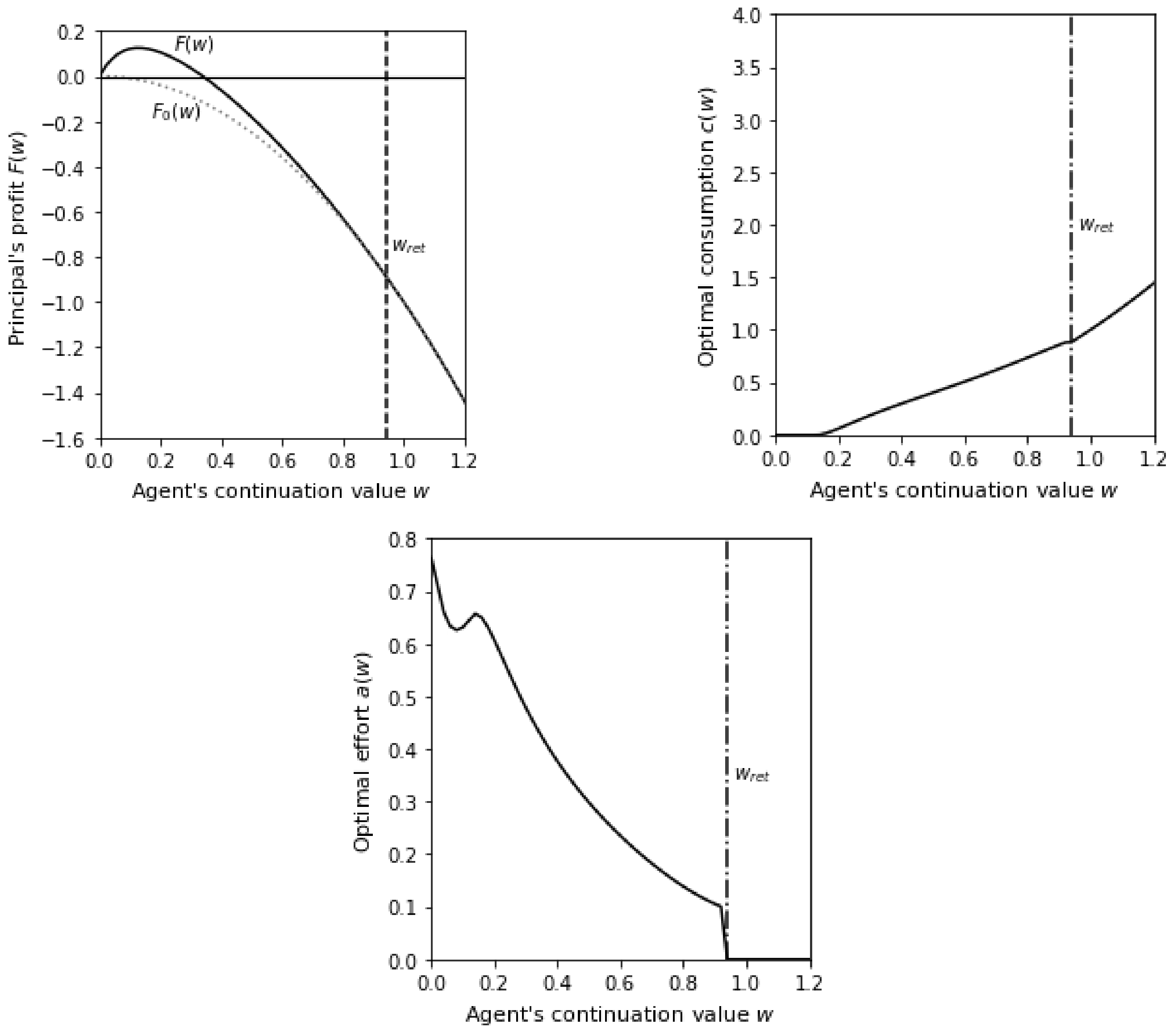


Figure 2: Quitting Scenario The picture depicts the value function, optimal consumption payments and the optimal effort strategy for the quitting scenario. We set the parameters to $\delta = 0.1$, $\sigma = 1.0$, $\tilde{w}_Q = 0.1$ and $T = 100$.

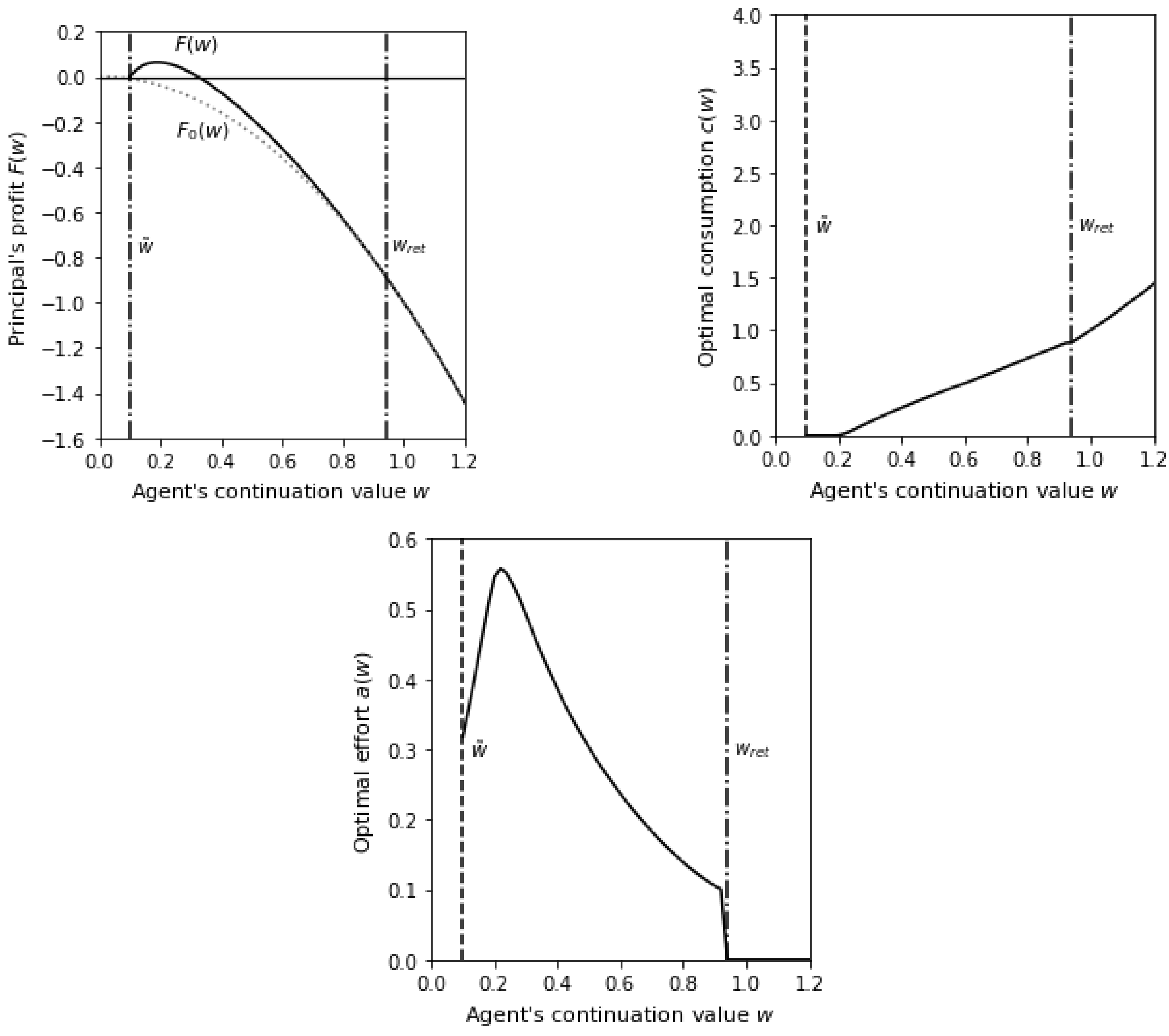


Figure 3: Replacement Scenario The picture depicts the value function, optimal consumption payments and the optimal effort strategy for the replacement scenario. We set the parameters to $\delta = 0.1$, $\sigma = 1.0$, $C = 0.0341$ and $T = 100$.

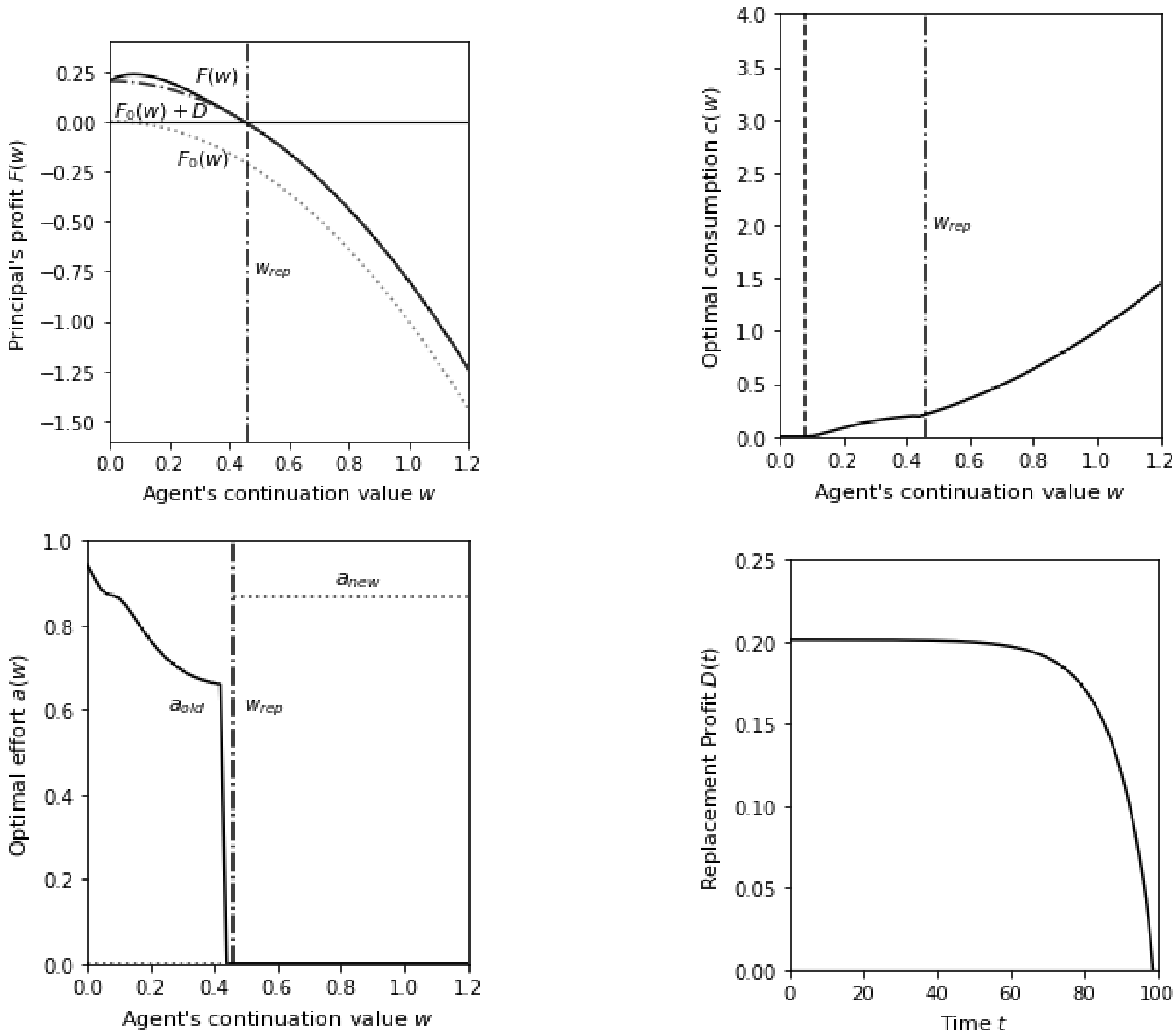


Figure 4: Promotion Scenario: Initially Trained Agent The picture depicts the value function, optimal consumption payments and the optimal effort strategy for the promotion scenario with an initially trained agent. We set the parameters to $\delta = 0.1$, $\sigma = 1.0$, $\theta = 1.5$, $K = 0.1$, $\tilde{w}_P = 0.2$ and $T = 100$.

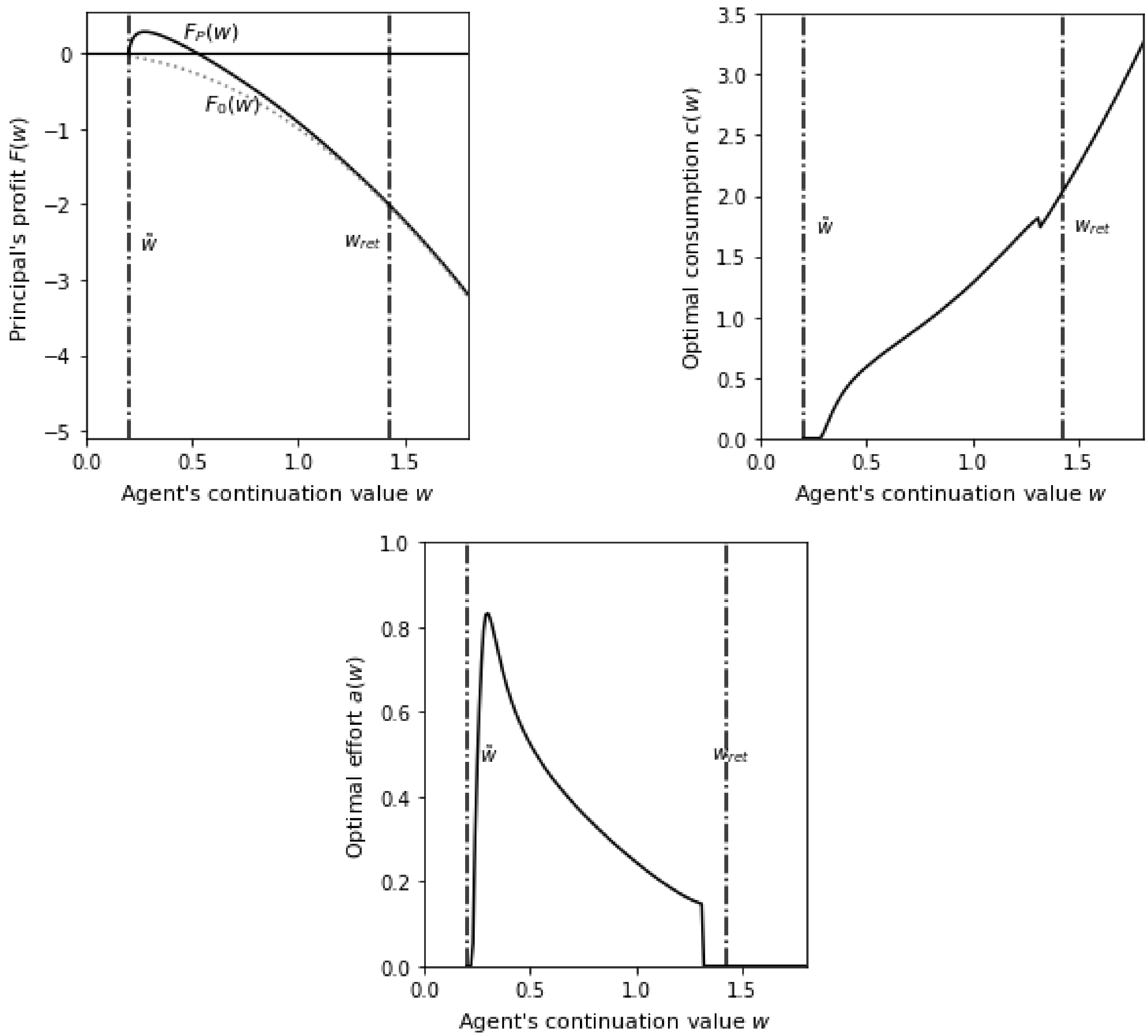
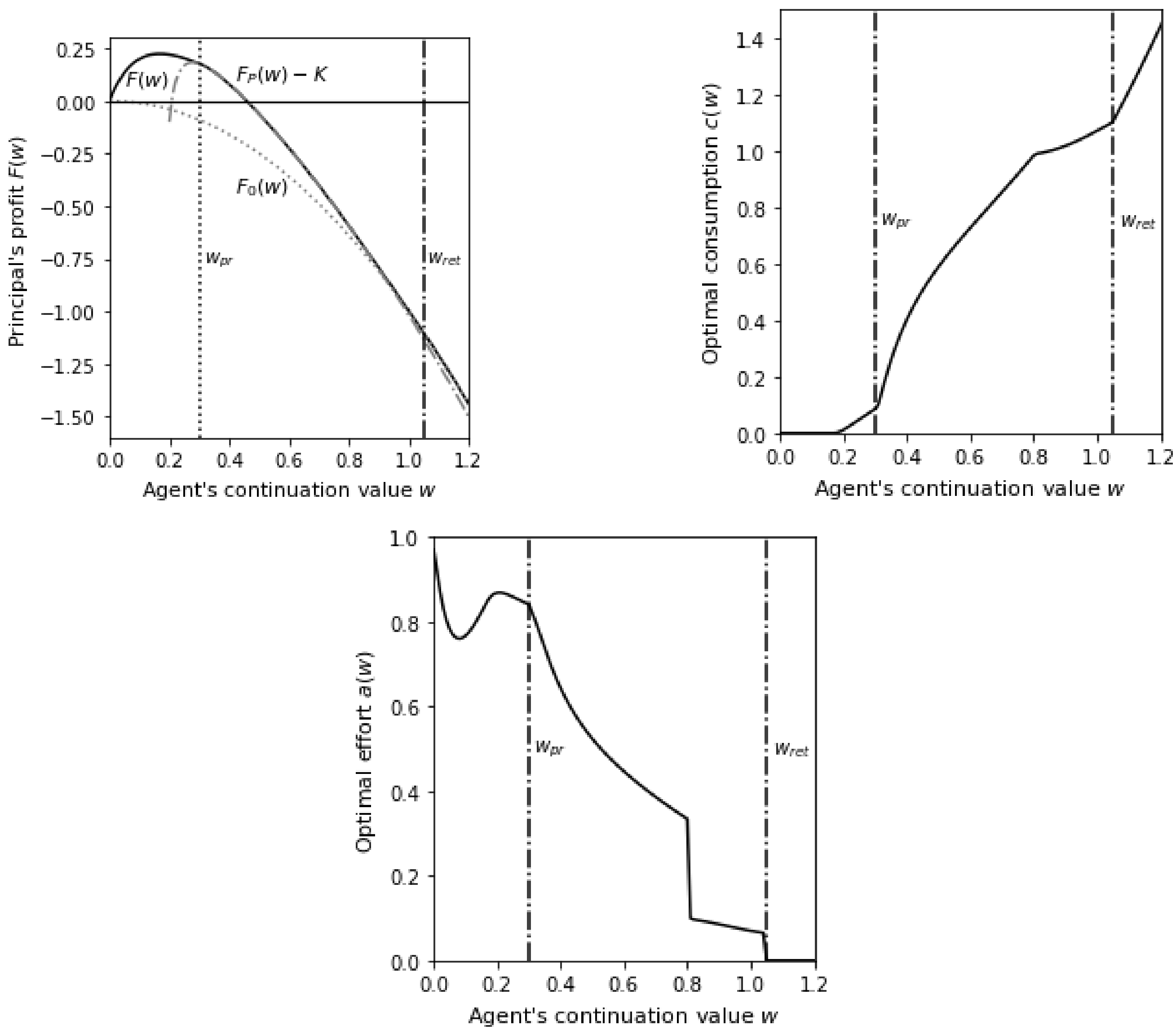


Figure 5: Promotion Scenario: Initially Untrained Agent The picture depicts the value function, optimal consumption payments and the optimal effort strategy for the promotion scenario with an initially untrained agent. We set the parameters to $\delta = 0.1$, $\sigma = 1.0$, $\theta = 1.5$, $K = 0.1$, $\tilde{w}_P = 0.2$ and $T = 100$.



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