Differential Calculus Review

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Limits and Continuity

1. Give the $\varepsilon-\delta$ definition for the limit statement; $\lim\limits_{x\to c} f(x)=L$.
2. Find a number δ such that if $|x-2|<\delta$, then $|4x-8|<\varepsilon$, where
   (a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.005$
3. Use the definition of limit to find a suitable δ that shows
   (a) $\lim\limits_{x\to -2} 3x-x^2=-10$
   (b) $\lim\limits_{x\to 1} 3x^2-4x=-1$
4. Find the limit, if it exists
   (a) $\lim\limits_{x\to 0}\dfrac{(2+x)^{-1}-2^{-1}}{x}$ (b) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-9}}{2x-6}$
   (c) $\lim\limits_{x\to \pi/3^-}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$ (d) $\lim\limits_{x\to 0^+}\left(\arctan {}^1\!/_x\right)$
   (e) $\lim\limits_{x\to 0^+}(e^{2x}+x)^{1/(3x)}$ (f) $\lim\limits_{x\to 0}\left(\dfrac{x-\tan 3x}{x+\tan 2x}\right)$
   (g) $\lim\limits_{x\to\infty}\left[\sqrt{x^2+7x}-x\right]$ (h) $\lim\limits_{x\to\infty} x\ln\left(\dfrac{x^2}{x^2-1}\right)$
5. (a) For what value(s) of the constant c is the function f everywhere continuous?
   $$f(x)=\begin{cases} cx+7 & if \quad x\le 2 \\ (cx)^2+1 & if \quad x>2 \end{cases}$$
   (b) Determine where f is discontinuous and state which of the three conditions of continuity fail there.
   $$f(x)=\begin{cases} \sqrt{1-x} & if \quad x\le 1 \\ 2+\ln e^{x-4} & if \quad 1<x<5 \\ (3-x)^2 & if \quad x>5 \end{cases}$$

Differentiation

1. State both forms of the definition of the derivative of a function at a, then use one or the other to find the derivative of the functions at the indicated point.
   (a) $f(x)=4x^2-x$ at $a=2$
   (b) $f(x)=\sqrt{2x+1}$ at $a=4$. Identify the domain of f and the domain of f'.
2. Find an equation of the tangent line to the curve $y=\sqrt{x}-\dfrac{1}{\sqrt{x}}$ for $x=4$.

3. An arrow is shot upward on mars with a velocity of 60 meters per second. Its height (in meters) after t seconds is given by $s(t) = 60t - 1.9t^2$. (a) what is the maximum height attained by the arrow? (b) With what velocity will the arrow hit mars?

4. Find the indicated derivative and simplify.

(a) $y = x^3(4 - 5x^2)$, $\dfrac{d^2y}{dx^2}$

(b) $y = \dfrac{(x^2-1)^{3/2}}{3x^3}$, $\dfrac{dy}{dx}$

(c) $f(x) = \cos^3\left(\dfrac{2}{x^2+3}\right)$, $f'(1)$ to 4 decimal places

(d) $y = \sqrt[3]{x + \sqrt[4]{x}}$, $\dfrac{dy}{dx}$

(e) $y = e^{2x}\sec 3x$, $\dfrac{d^2y}{dx^2}$

(f) $y \ln x = x \arctan y$, $\dfrac{dy}{dx}$

(g) $y = x^{\sin x^2}$, $\dfrac{dy}{dx}$

(h) $F(x) = \displaystyle\int_1^{\tan x} \sqrt{t^3 - 1}\ dt$, $F'(x)$

5. Related rates

(a) Assume *R* and *S* are differentiable functions of *t*. If they are related by the equation; $3R^2 + 2S^3 = 57$, find $\dfrac{dR}{dt}$ at the instant when $R = 2$, $S = 1$, and $\dfrac{dS}{dt} = 0.1$.

(b) A 5-meter-long ladder is leaning against the side of a house. The foot of the ladder is pulled away from the house at a rate of 0.4 m/sec. Determine how fast the top of the ladder is descending when the foot of the ladder is 3 meters from the house.

(c) The height of a cylinder is increasing at a rate of 2 centimeters per minute. Find the rate of change of the volume of the cylinder with respect to time when the height is 10 centimeters if the radius is constant at 4 cm.

6 Determine the absolute extrema of the function on the indicated interval.

(a) $f(x) = x\sqrt{2x+6}$; $[-5/2, -1/2]$

(b) $f(x) = 5x^{2/3} + 2x^{5/3}$; $[-8, 8]$

7. State the Mean Value Theorem.

Determine whether the Mean Value Theorem applies to the function on the indicated interval and, if so, find a value '*c*' guaranteed by the theorem.

(a) $f(x) = \dfrac{7x-6}{x}$; $[1,6]$

(b) $f(x) = \sin 2x - 2\sin x$; $[\pi, 2\pi]$

8. For each function below,

(a) determine the domain (unless otherwise specified)

(b) find any intercepts and determine any asymptotes

(c) compute $f'(x)$ and $f''(x)$

(d) Determine the critical numbers, identify the intervals on which *f* is increasing and those on which *f* is decreasing, and find any relative extrema.

(e) identify the intervals on which f is concave up and those on which f is concave down, then determine the coordinates of any points of inflection.
(f) sketch the graph of the function

$$f(x) = (x-2)^3(3x+14)$$
$$f(x) = \frac{27(x-1)^2}{x^3}, \; x > 0$$

9. (a) A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 28 ft/sec. At what rate is his distance from second base decreasing when he is halfway to first base? Round the result to the nearest hundredth if necessary.

   (b) Find the point on the graph of $y = \sqrt{x+1}$ closest to the point (3, 0).

10. (a) Find the differential of the function $y = x^2 \cos 2x$.

    (b) Find values of dy and Δy for $y = x^3 - 2x$ when $x = 2$ and $\Delta x = 0.1$.

    (c) The measurement of the edge of a piece of square floor tile is found to be 12 inches with a possible error of 0.02 inches. Use differentials to approximate the maximum possible error in the calculated area and the relative error in the calculation.

Integration

1. Estimate the area under the graph of $y = \sqrt{4 - x^3}$ over the interval [0,1.5] using three approximating rectangles and right endpoints.
2. Express the limit as a definite integral on the given interval.

   $$\lim_{n\to\infty} \sum_{i=1}^{n} 3e^{r_i} \sin r_i \Delta r \; ; \; [-1,1]$$

3. Evaluate the integral by interpreting it in terms of area (note: you cannot find an antiderivative for each term of the integrand)

   $$\int_{-3}^{0} \left[x + 3 + \sqrt{9 - x^2} \right] dx$$

4. Evaluate the integral

(a) $\displaystyle\int_{6}^{8} \frac{x}{\sqrt{100 - x^2}} \, dx$ (b) $\displaystyle\int_{-5}^{2} \frac{x}{(x+6)^{2/3}} \, dx$

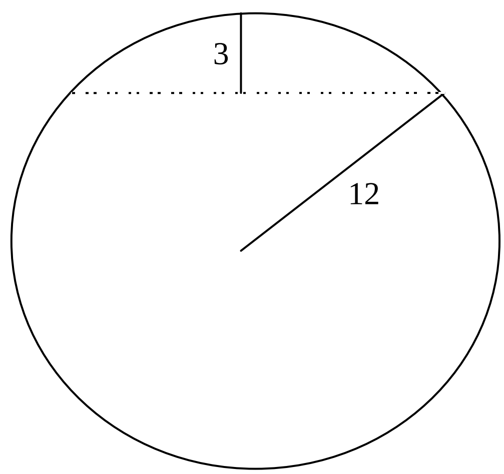
(c) $\displaystyle\int_{0}^{3} \frac{2x+3}{1+x^2} \, dx$ (d) $\displaystyle\int_{1}^{e^3} \frac{1 - \ln x}{x} \, dx$

(e) $\displaystyle\int_0^{\pi/12} (\sin 3x + \cos 3x)^2 \; dx$

(f) $\displaystyle\int \tan^2 x \sec^2 x \; dx$

Applications of Integration

1. Let R be the region bounded by the graphs of $y = \dfrac{x}{2}$ and $y = \sqrt{x}$.
   (a) Find the area of *R.*
   (b) Find the volume of the solid obtained by rotating *R* about the *x*-axis.
   (c) Set up an integral that represents the volume of the solid obtained by rotating *R* about the line $y = 4$.
   (d) Find the volume of the solid obtained by rotating *R* about the *y*-axis.
2. Set up a definite integral(s) that represents the area of the triangle with vertices (0,0), (-2,3), and (2,1).
3. The cap of a sphere of radius 12 has been removed. Find the volume of the remainder of the sphere if the height of the cap is 3 as shown.



4. A force of 10 lb is required to hold a spring stretched 4 in. beyond its natural length. How much work is done (in pound-inches) in stretching it from its natural length to 9 in. beyond its natural length?

5. The linear density of a 48 meter long rod is $72/\sqrt{x+1}$ kg/m, where *x*, is measured in meters from one end of the rod. Find the average density of the rod.