Chapter 11

Aided Inertial Navigation

The topic of this chapter is aided inertial navigation. Inertial navigation itself is a large topic with a significant body of related literature. The presentation herein begins with a discussion of gravity and definition of the concept of specific force. Both of these concepts and their interrelation are critical to inertial navigation. Next, the chapter discusses the kinematics of inertial navigation in various reference frames. The input variables to the kinematic models are accelerations and angular rates. The acceleration vector is computed from a measured specific force vector by compensating for gravity and instrument errors. The dynamic evolution of the INS error state is a topic of detailed study. The chapter includes a detailed derivation of a state-space model, analytic analysis of simplified models, simulation analysis of the full linear state-space model, and derivation and discussion of instrument error models. Finally, the chapter concludes with sections that discuss initialization, INS aiding, and error state observability.

11.1 Gravitation and Specific Force

The purpose of this section is to distinguish between inertial and non-inertial forces and to define the concept of a specific force.

11.1.1 Gravitation

Newton's law of gravitation states that the force of gravitational attraction of mass m_1 on m_2 is defined by

$$\mathbf{F}_{12} = -\frac{G m_1 m_2}{\|\mathbf{p}_{12}\|^3}\mathbf{p}_{12} \tag{11.1}$$

where G is the universal gravitational constant, and $\mathbf{p}_{12} = \mathbf{p}_2 - \mathbf{p}_1$ is the position of center of mass $\mathbf{p}_2$ with respect to the center of mass $\mathbf{p}_1$. For the purposes of this book we consider inertial and gravitational masses to be identical. The negative sign indicates that the direction of the force is opposite that of $\mathbf{p}_{12}$ (i.e., m_2 is attracted towards m_1).

In particular, if $m_1 = M_e$ represents the mass of Earth and the subscript on m_2 is dropped, then the gravitational attraction of Earth on $m_2 = m$ is

$$\mathbf{F}_{e2} = -\frac{GM_e m}{\|\mathbf{p}_{e2}\|^3}\mathbf{p}_{e2} \tag{11.2}$$

and the gravitational attraction of m on Earth is

$$\mathbf{F}_{2e} = -\frac{GM_e m}{\|\mathbf{p}_{2e}\|^3}\mathbf{p}_{2e}. \tag{11.3}$$

Using Newton's second law, eqns. (11.2) and (11.3) yield the following two differential equations, respectively

$$\begin{aligned} \ddot{\mathbf{p}}_e^i &= -\frac{Gm}{\|\mathbf{p}_{2e}\|^3}\mathbf{p}_{2e}^i \\ \ddot{\mathbf{p}}_2^i &= -\frac{GM_e}{\|\mathbf{p}_{e2}\|^3}\mathbf{p}_{e2}^i; \end{aligned}$$

therefore, because $\mathbf{p}_{e2} = \mathbf{p}_2 - \mathbf{p}_e = -\mathbf{p}_{2e}$ we have

$$\begin{aligned} \ddot{\mathbf{p}}_{e2}^i &= -\frac{G(M_e + m)}{\|\mathbf{p}_{e2}\|^3}\mathbf{p}_{e2}^i \\ &\approx -\frac{GM_e}{\|\mathbf{p}_{e2}\|^3}\mathbf{p}_{e2}^i = \mathbf{G}^i \end{aligned}$$

where $\mathbf{G}$ is the gravitational acceleration defined as

$$\mathbf{G} = -\frac{GM_e}{\|\mathbf{p}_{e2}\|^3}\mathbf{p}_{e2}.$$

This relatively simple model of gravitation is derived for a central force field. It would be approximately valid for vehicles in space. For vehicles near the surface of Earth more detailed gravitational models are required, see Section 2.3.2.2.

11.1.2 Specific Force

Newtonian physics applies to inertial reference frames. An inertial reference frame is non-accelerating, non-rotating, and has no gravitational field. According to Newton's second law, in an inertial reference frame, the acceleration $\ddot{\mathbf{p}}^i$ of a mass m is proportional to the inertial (i.e., physically applied) forces $\mathbf{F}_I$

$$\mathbf{F}_I = m\ddot{\mathbf{p}}^i. \tag{11.4}$$

The quantity $\mathbf{f} = \frac{\mathbf{F}_I}{m}$ has units of acceleration and is referred to as the *specific force*. The specific force is the inertial force per unit mass required to produce the acceleration $\ddot{\mathbf{p}}^i$.

Examples of inertial forces include spring forces, friction, lift, thrust. An example that is important to the understanding of accelerometer operation is the support force applied by a mechanical structure to the case of an accelerometer. The force of gravity is not an inertial force.

When inertial forces are applied to a mass in the presence of the Earth gravitational field, the dynamic model for the position $\mathbf{p}$ of the mass m becomes

$$\begin{aligned} m\ddot{\mathbf{p}}^i &= \mathbf{F}_I - \frac{GM_e m}{\|\mathbf{p}\|^3}\mathbf{p}^i \\ \ddot{\mathbf{p}}^i &= \mathbf{f}^i + \mathbf{G}^i. \end{aligned} \tag{11.5}$$

Eqn. (11.5) is referred to as the fundamental equation of inertial navigation in the inertial reference frame. As will be discussed in Section 11.1.3, accelerometers measure the specific force vector

$$\mathbf{f}^i = \ddot{\mathbf{p}}^i - \mathbf{G}^i. \tag{11.6}$$

The accelerometer output is represented in the accelerometer frame of reference.

The accelerometer measurement of specific force, after transformation from accelerometer frame to inertial frame, can be used as an input to eqn. (11.5). Integration of eqn. (11.5) with $\mathbf{f}^i$ as an input would compute the velocity $\dot{\mathbf{p}}^i$ and the position $\mathbf{p}^i$. Sections 11.2 and 12.1.2 will derive equations similar to eqn. (11.5) that are applicable in alternative reference frames that are more convenient for navigation applications.

11.1.3 Accelerometers

The objective of this section is to describe the basic operation of an ideal accelerometer. Readers interested in more detailed discussion of real accelerometer designs should consult references such as [37, 73, 126].

Consider an accelerometer constructed via a spring-mass-damper system as depicted in Figure 11.1. The position vector of the mass m is $\mathbf{p}$. The position of the accelerometer case is $\mathbf{p}_c$. The case relative position of the mass is $\delta\mathbf{p}_c = \mathbf{p} - \mathbf{p}_c$. For this discussion, we assume that $\delta\mathbf{p}_c$ can be perfectly measured. The equilibrium position of mass m is $\delta\mathbf{p}_c = \mathbf{0}$.

The following discussion distinguishes between inertial (i.e., physically applied) and kinematic forces (e.g., gravity). In Figure 11.1, $\mathbf{F}_I$ represents a force physically applied to the accelerometer case.

In an inertial reference frames (i.e., the frame is not accelerating, not rotating, and has no gravitational field) by Newton's laws, the dynamic

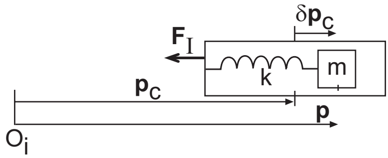

Figure 11.1: Basic accelerometer.

equation for the inertial acceleration of the mass m is

$$\ddot{\mathbf{p}} = -\frac{k}{m}\delta\mathbf{p}_c - \frac{b}{m}\delta\dot{\mathbf{p}}_c \tag{11.7}$$

where k is the spring constant and b is the viscous damping constant. Defining the accelerometer output as $\mathbf{f} = -\frac{k}{m}\delta\mathbf{p_c}$, then eqn. (11.7) can be rewritten as

$$\alpha\dot{\mathbf{f}} = -\mathbf{f} + \ddot{\mathbf{p}} \tag{11.8}$$

where the parameter $\alpha = \frac{b}{k}$ is the time constant of the sensor. The bandwidth of the accelerometer is determined by the parameter α. When the bandwidth of the acceleration signal $\ddot{\mathbf{p}}(t)$ is less that the sensor bandwidth, then $\alpha\dot{\mathbf{f}}(t)$ is small and the sensor maintains the condition

$$\mathbf{f}(t) = \ddot{\mathbf{p}}(t). \tag{11.9}$$

Note that $\mathbf{f}$ is a specific force with units of acceleration.

In the presence of a gravitational field, the accelerometer dynamic equation is

$$\ddot{\mathbf{p}} = -\frac{k}{m}\delta\mathbf{p}_c - \frac{b}{m}\delta\dot{\mathbf{p}}_c + \mathbf{G} \tag{11.10}$$

where $\mathbf{G}$ represents the position dependent gravitational acceleration. It can be shown by manipulations similar to those shown above that

$$\alpha\dot{\mathbf{f}} = -\mathbf{f} + \ddot{\mathbf{p}} - \mathbf{G}(\mathbf{p}). \tag{11.11}$$

When the acceleration signal $\ddot{\mathbf{p}}(t)$ is within the sensor bandwidth, the *specific force* output $\mathbf{f}$ is

$$\mathbf{f} = \ddot{\mathbf{p}} - \mathbf{G}(\mathbf{p}). \tag{11.12}$$

Eqn. (11.12) represents the accelerometer output equation (neglecting bandwidth effects). The equation does not make any assumptions about the accelerometer trajectory, but does recognize that the gravitational acceleration is location dependent.

The above discussion states that an accelerometer measures specific force or the relative acceleration between the case and the mass m. It does not detect accelerations that affect the case and mass m identically. It is useful to consider a few special cases.

- A (non-rotating) accelerometer with no applied forces is in free-fall with $\ddot{\mathbf{p}} = \mathbf{G}$; therefore, the accelerometer output is $\mathbf{f} = \mathbf{0}$.

- An accelerometer in stable orbit around Earth, is also in free-fall. It is constantly accelerating towards Earth with acceleration $\ddot{\mathbf{p}} = \mathbf{G}$; therefore, the accelerometer output is again $\mathbf{f} = \mathbf{0}$.

- Consider an accelerometer at rest on the Earth surface. In this case, the accelerometer is subject to the Earth's gravitational field and is caused to rotate about the Earth at the Earth rate ω_{ie}. Defining the origin of the inertial frame to be coincident with the Earth center of mass, we have that $\ddot{\mathbf{p}} = \mathbf{\Omega}_{ie}\mathbf{\Omega}_{ie}\mathbf{p}$ where $\mathbf{\Omega}_{ie} = [\boldsymbol{\omega}_{ie}\times]$; therefore, the accelerometer output is

$$\mathbf{f} = \mathbf{\Omega}_{ie}\mathbf{\Omega}_{ie}\mathbf{p} - \mathbf{G}, \tag{11.13}$$

  which is the inertial force applied by the supporting structure to the case to maintain the case in a stationary Earth relative position.

When using an accelerometer, the user must compensate the specific force output for the effects of gravity. To prepare for the subsequent analysis, define the local gravity vector as

$$\mathbf{g} = \mathbf{G}(\mathbf{p}) - \mathbf{\Omega}_{ie}\mathbf{\Omega}_{ie}\mathbf{p}. \tag{11.14}$$

The local gravity vector indicates the direction that a weight would hang at a location indicated by $\mathbf{p}$. Figure 11.2 shows the relation of these three accelerations. The vector magnitudes are not drawn to scale. In reality $\|\mathbf{\Omega}_{ie}\mathbf{\Omega}_{ie}\mathbf{p}\| \approx \frac{1}{300}\|\mathbf{G}\|$. In this figure, ϕ and ϕ_c denotes geodetic and geocentric latitude, respectively.

The specific force vector can be represented in any desired frame of reference. When this is done, the analyst should be careful with terminology and interpretation. For example, the specific force vector in the ECEF

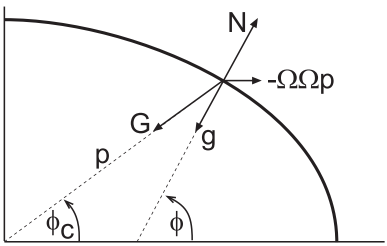

Figure 11.2: Effective gravity vector.

frame is

$$\begin{aligned}
\mathbf{f}^e &= \mathbf{R}_i^e \left(\ddot{\mathbf{p}}^i - \mathbf{g}^i\right) \\
&= \mathbf{R}_i^e \ddot{\mathbf{p}}^i - \mathbf{g}^e
\end{aligned}$$

which is the inertial acceleration minus gravity both represented in ECEF frame. If we let $\mathbf{p}^e$ represent the position vector in the ECEF frame, it is important to note that $\mathbf{R}_i^e \ddot{\mathbf{p}}^i \neq \frac{d^2}{dt^2}\mathbf{p}^e$, see Section 11.2.2.

11.1.4 Gravity Error

The purpose of this section is to derive an expression for the effect of position error on the computed navigation frame gravity vector. The second term in eqn. (11.14) is straightforward; therefore, this section focuses on the term $\mathbf{G}(\mathbf{p}) = -\frac{GM}{\|\mathbf{p}\|^3}\mathbf{p}$.

The analysis of this section will be used later in this chapter; however, on the first reading of the chapter, the majority of this section may be skipped. In that case, it is recommended that the reader proceed directly to the final result of eqn. (11.22) and the discussion that follows it.

Denote the actual and computed vehicle locations relative to the Earth center of mass by $\mathbf{p}$ and $\hat{\mathbf{p}}$, respectively. The error between these locations is $\delta\mathbf{p} = \mathbf{p} - \hat{\mathbf{p}}$. Using the first two terms in the Taylor's series expansion, we obtain the linear error model

$$\delta\mathbf{g} = \mathbf{G}(\mathbf{p}) - \mathbf{G}(\hat{\mathbf{p}}) = \left.\frac{\partial \mathbf{G}}{\partial \mathbf{p}}\right|_{\hat{\mathbf{p}}} \delta\mathbf{p}. \tag{11.15}$$

where

$$\begin{aligned}
\left.\frac{\partial \mathbf{G}}{\partial \mathbf{p}}\right|_{\hat{\mathbf{p}}} &= -GM\left[\frac{1}{(\hat{\mathbf{p}}^\top\hat{\mathbf{p}})^{\frac{3}{2}}}\mathbf{I} - \frac{3}{(\hat{\mathbf{p}}^\top\hat{\mathbf{p}})^{\frac{5}{2}}}\hat{\mathbf{p}}\hat{\mathbf{p}}^\top\right] && (11.16)\\
&= \frac{-GM}{(\hat{\mathbf{p}}^\top\hat{\mathbf{p}})^{\frac{5}{2}}}\left[\left(\hat{\mathbf{p}}^\top\hat{\mathbf{p}}\right)\mathbf{I} - 3\hat{\mathbf{p}}\hat{\mathbf{p}}^\top\right] && (11.17)\\
&= \frac{-GM}{(\hat{\mathbf{p}}^\top\hat{\mathbf{p}})^{\frac{5}{2}}}\begin{bmatrix} R^2 - 3\hat{x}_1^2 & -3\hat{x}_1\hat{x}_2 & -3\hat{x}_1\hat{x}_3 \\ -3\hat{x}_1\hat{x}_2 & R^2 - 3\hat{x}_2^2 & -3\hat{x}_2\hat{x}_3 \\ -3\hat{x}_1\hat{x}_3 & -3\hat{x}_2\hat{x}_3 & R^2 - 3\hat{x}_3^2 \end{bmatrix} && (11.18)
\end{aligned}$$

where $R = \|\hat{\mathbf{p}}\|$ and $\hat{\mathbf{p}} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]^\top$.

If $\mathbf{G}$, $\mathbf{p}$, $\hat{\mathbf{p}}$, and $\delta\mathbf{p}$ are each represented along the geographic frame north, east, and down axes, then

$$\hat{\mathbf{p}}^n = [0, 0, -R]^\top \tag{11.19}$$

and eqn. (11.18) reduces to

$$\left.\frac{\partial \mathbf{G}^n}{\partial \mathbf{p}^n}\right|_{\hat{\mathbf{p}}^n} = \frac{-GM}{R^3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{11.20}$$

Continuing to neglect the $\mathbf{\Omega}_{ie}\mathbf{\Omega}_{ie}\mathbf{p}$ term from eqn. (11.14), we have

$$\delta \mathbf{g}^n = \frac{-g}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\delta \mathbf{p}^n \tag{11.21}$$

where $g = \frac{GM}{R^2}$ and $\delta \mathbf{g}^n = [\delta g_n, \delta g_e, \delta g_d]^\top$.

The effect of the gravity error on the velocity error dynamics is considered in detail in Section 11.4.3. In that section, the vertical component of the position error is height above the ellipse δh, not error in the down component δd of the geodetic vector. The fact that $\delta h = -\delta d$ inserts one extra negative sign in the error analysis of that section

$$\delta \mathbf{g}^n = \frac{-g}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \delta n \\ \delta e \\ \delta h \end{bmatrix}. \tag{11.22}$$

The following two paragraphs briefly discuss simple models for the horizontal and vertical error dynamics. The purpose is to introduce the important effects that gravity has on the inertial system navigation systems.

In the navigation frame, the lateral position error differential equations can be represented as

$$\begin{aligned} \delta\ddot{n} &= \delta g_n + \alpha_n \\ \delta\ddot{e} &= \delta g_e + \alpha_e \end{aligned}$$

where all error sources other than those due to gravity are combined into the terms α_n and α_e. The full error model that defines α_n and α_e will be discussed in Section 11.4.1. From eqn. (11.22), $\delta g_n = \frac{-g}{R}\delta n$ and $\delta g_e = \frac{-g}{R}\delta e$; therefore, the dependence of the lateral position error on the gravity error has the form

$$\begin{aligned} \delta\ddot{n} + \frac{g}{R}\delta n &= \alpha_n \\ \delta\ddot{e} + \frac{g}{R}\delta e &= \alpha_e \end{aligned}$$

which are forced harmonic oscillators with natural frequency equal to the *Schuler frequency* $\omega_s = \sqrt{\frac{g}{R}}$. If the gravity error terms were absent, then the lateral position errors would be the pure integral of the errors α_n and

α_e. We see that the gravity error terms in the lateral dynamics have a stabilizing effect. Because they are small and stable, they are often dropped in the error analysis.

For the vertical error dynamics, the situation is different. The vertical position error differential equation can be written as

$$\delta\ddot{d} = \delta g_d + \alpha_d$$

where α_d represent all other error sources except for those due to gravity and from eqn. (11.21) we have $\delta g_d = 2\frac{g}{R}\delta d$. Therefore, the vertical position error model is

$$\delta\ddot{d} - 2\frac{g}{R}\delta d = \alpha_d.$$

This system has eigenvalues at $\pm\sqrt{2\frac{g}{R}}$ where the positive eigenvalue shows that the vertical error dynamics due to gravity error are unstable. Even though this eigenvalue is small, because the effect of the gravity error on the system is destabilizing, the effect of gravity error on the vertical dynamics cannot be neglected in the overall error analysis in Section 11.4.3.

11.2 INS Kinematic Equations

Let $\mathbf{p}$ represent the vector from the i-frame origin to the point $\mathbf{P}$. The a-frame is another arbitrary reference frame. The relative rate of rotation of frame a with respect to frame i represented in frame a is $\boldsymbol{\omega}^a_{ia}$. The notation $\boldsymbol{\Omega}^a_{ia} = [\boldsymbol{\omega}^a_{ia}\times]$ is defined in eqn. (B.15). It should be reviewed now as it will be used frequently throughout the chapter.

From eqn. (2.61) in Section 2.6.2, the second derivative of $\mathbf{p}$ in the i and a frames are related by

$$\ddot{\mathbf{p}}^i = \mathbf{R}^i_a \left[2\boldsymbol{\Omega}^a_{ia}\mathbf{v}^a + \boldsymbol{\Omega}^a_{ia}\boldsymbol{\Omega}^a_{ia}\mathbf{r}^a + \dot{\boldsymbol{\Omega}}^a_{ia}\mathbf{r}^a + \frac{d^2\mathbf{r}^a}{dt^2}\right] \tag{11.23}$$

where $\mathbf{r}$ represents the vector from the a-frame origin to the point $\mathbf{P}$. The $\ddot{\boldsymbol{\rho}}^i$ term from eqn. (2.61) has been dropped in eqn. (11.23). In eqn. (2.61) $\boldsymbol{\rho}$ represented the vector from the i-frame origin to the a-frame origin. In the instances in which this equation is used in this chapter, the origins of the a and i frames will be coincident, so that $\boldsymbol{\rho} = \mathbf{0}$. The symbol $\boldsymbol{\rho}$ will be reserved for another use later in the chapter.

From eqn. (11.5) we have that $\ddot{\mathbf{p}}^i = \mathbf{f}^i + \mathbf{G}^i$. Solving eqn. (11.23) for the second derivative of $\mathbf{r}^a$ yields

$$\frac{d^2\mathbf{r}^a}{dt^2} = \mathbf{f}^a + \mathbf{G}^a - 2\boldsymbol{\Omega}^a_{ia}\mathbf{v}^a - \left(\boldsymbol{\Omega}^a_{ia}\boldsymbol{\Omega}^a_{ia} + \dot{\boldsymbol{\Omega}}^a_{ia}\right)\mathbf{r}^a \tag{11.24}$$

$$= \mathbf{R}^a_b\mathbf{f}^b + \mathbf{G}^a - 2\boldsymbol{\Omega}^a_{ia}\mathbf{v}^a - \left(\boldsymbol{\Omega}^a_{ia}\boldsymbol{\Omega}^a_{ia} + \dot{\boldsymbol{\Omega}}^a_{ia}\right)\mathbf{r}^a. \tag{11.25}$$

These two equation allow a brief comparison between mechanized and strapdown inertial navigation systems.

In a *mechanized INS*, the accelerometers and gyros are attached to a platform that is designed to maintain its alignment with the a-frame as the vehicle maneuvers. Assuming that the platform is initially aligned with the a-frame, a main idea of the design is to apply torques to the platform so that the outputs of the gyros remain at zero (i.e., platform to a-frame alignment is maintained). This approach requires that the gyro biases be accurately calibrated. With the mechanized approach, the accelerometers measure the specific force in the a-frame. Given initial conditions $\mathbf{v}^a(0)$ and $\mathbf{p}^a(0)$ and alignment of the platform frame with the a-frame, eqn. (11.24) can be integrated to compute $\mathbf{v}^a(t)$ and $\mathbf{p}^a(t)$. The vehicle attitude with respect to the a-frame could be determined by measuring the angles between the platform and the vehicle.

In a *strapdown INS* the accelerometers and gyros are mounted on a platform that is rigidly attached to the vehicle. In this approach, the gyros experience the full rotational rate of the vehicle as it maneuvers; hence gyro scale factor accuracy (as well as bias accuracy) is important. In the strapdown implementation, the accelerometers measure the body frame (b-frame) specific force $\mathbf{f}^b$ which can be transformed into the a-frame using $\mathbf{f}^a = \mathbf{R}_b^a \mathbf{f}^b$. The strapdown INS approach requires initial conditions $\mathbf{v}^a(0)$ and $\mathbf{p}^a(0)$ and $\mathbf{R}_b^a(0)$. The gyro outputs (and navigation state) are used to compute $\boldsymbol{\omega}_{ab}^b(t)$ so that the INS can maintain $\mathbf{R}_b^a(t)$ computationally by integrating

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \boldsymbol{\Omega}_{ab}^b. \tag{11.26}$$

The strapdown INS integrates eqn. (11.25) to compute $\mathbf{v}^a(t)$ and $\mathbf{p}^a(t)$. The vehicle attitude is represented by $\mathbf{R}_b^a$.

This chapter will focus on strapdown inertial navigation. Throughout the presentation, the direction cosine representation of attitude will be assumed. Other representations exist (e.g., Euler angles, quaternions). Readers are encouraged to consider the quaternion representation (see Appendix D). It may have a steeper learning curve, but provides a computationally efficient and singularity-free attitude representation approach. The error modeling approach presented in this chapter is applicable to any of the attitude representations.

Equations for specific choices of the a-frame will be considered in the following subsections. For each frame, a differential equation will be derived for each of the position, velocity, and attitude.