CHAPTER 13

Nonlinear Kalman filtering

> It appears that no particular approximate [nonlinear] filter is consistently better than any other, though ... any nonlinear filter is better than a strictly linear one.
> —Lawrence Schwartz and Edwin Stear [Sch68]

All of our discussion to this point has considered linear filters for linear systems. Unfortunately, linear systems do not exist. All systems are ultimately nonlinear. Even the simple $I = V/R$ relationship of Ohm's Law is only an approximation over a limited range. If the voltage across a resistor exceeds a certain threshold, then the linear approximation breaks down. Figure 13.1 shows a typical relationship between the current through a resistor and the voltage across the resistor. At small input voltages the relationship is approximately linear, but if the power dissipated by the resistor exceeds some threshold then the relationship becomes highly nonlinear. Even a device as simple as a resistor is only approximately linear, and even then only in a limited range of operation.

So we see that linear systems do not really exist. However, many systems are close enough to linear that linear estimation approaches give satisfactory results. But "close enough" can only be carried so far. Eventually, we run across a system that does not behave linearly even over a small range of operation, and our linear approaches for estimation no longer give good results. In this case, we need to explore nonlinear estimators.

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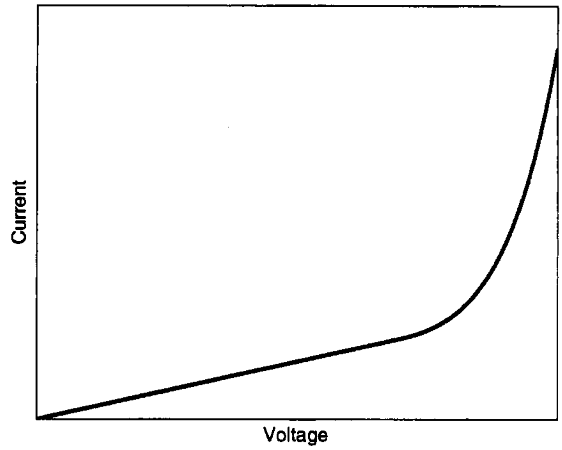

Figure 13.1 Typical current/voltage relationship for a resistor. The relationship is linear for a limited range of operation, but becomes highly nonlinear beyond that range.

Nonlinear filtering can be a difficult and complex subject. It is certainly not as mature, cohesive, or well understood as linear filtering. There is still a lot of room for advances and improvement in nonlinear estimation techniques. However, some nonlinear estimation methods have become (or are becoming) widespread. These techniques include nonlinear extensions of the Kalman filter, unscented filtering, and particle filtering.

In this chapter, we will discuss some nonlinear extensions of the Kalman filter. The Kalman filter that we discussed earlier in this book directly applies only to linear systems. However, a nonlinear system can be linearized as discussed in Section 1.3, and then linear estimation techniques (such as the Kalman or H_∞ filter) can be applied. This chapter discusses those types of approaches to nonlinear Kalman filtering.

In Section 13.1, we will discuss the linearized Kalman filter. This will involve finding a linear system whose states represent the deviations from a nominal trajectory of a nonlinear system. We can then use the Kalman filter to estimate the deviations from the nominal trajectory, and hence obtain an estimate of the states of the nonlinear system. In Section 13.2, we will extend the linearized Kalman filter to directly estimate the states of a nonlinear system. This filter, called the extended Kalman filter (EKF), is undoubtedly the most widely used nonlinear state estimation technique that has been applied in the past few decades. In Section 13.3, we will discuss "higher-order" approaches to nonlinear Kalman filtering. These approaches involve more than a direct linearization of the nonlinear system, hence the expression "higher order." Such methods include second-order Kalman filtering, iterated Kalman filtering, sum-based Kalman filtering, and grid-based Kalman filtering. These filters provide ways to reduce the linearization errors that are inherent in the EKF. They typically provide estimation performance that is better than the EKF, but they do so at the price of higher complexity and computational expense.

Section 13.4 covers parameter estimation using Kalman filtering. Sometimes, an engineer wants to estimate the parameters of a system but does not care about estimating the states. This becomes a system identification problem. The system equations are generally nonlinear functions of the system parameters. System parameters are usually considered to be constant, or slowly time-varying, and a nonlinear Kalman filter (or any other nonlinear state estimator) can be adapted to estimate system parameters.

13.1 THE LINEARIZED KALMAN FILTER

In this section, we will show how to linearize a nonlinear system, and then use Kalman filtering theory to estimate the deviations of the state from a nominal state value. This will then give us an estimate of the state of the nonlinear system. We will derive the linearized Kalman filter from the continuous-time viewpoint, but the analogous derivation for discrete-time or hybrid systems are straightforward.

Consider the following general nonlinear system model:

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y &= h(x, v, t) \\ w &\sim (0, Q) \\ v &\sim (0, R) \end{aligned} \tag{13.1}$$

The system equation $f(\cdot)$ and the measurement equation $h(\cdot)$ are nonlinear functions. We will use Taylor series to expand these equations around a nominal control u_0, nominal state x_0, nominal output y_0, and nominal noise values w_0 and v_0. These nominal values (all of which are functions of time) are based on *a priori* guesses of what the system trajectory might look like. For example, if the system equations represent the dynamics of an airplane, then the nominal control, state, and output might be the planned flight trajectory. The *actual* flight trajectory will differ from this nominal trajectory due to mismodeling, disturbances, and other unforeseen effects. But the actual trajectory should be close to the nominal trajectory, in which case the Taylor series linearization should be approximately correct. The Taylor series linearization of Equation (13.1) gives

$$\begin{aligned} \dot{x} &\approx f(x_0, u_0, w_0, t) + \left.\frac{\partial f}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial f}{\partial u}\right|_0 (u - u_0) + \\ &\qquad \left.\frac{\partial f}{\partial w}\right|_0 (w - w_0) \\ &= f(x_0, u_0, w_0, t) + A\Delta x + B\Delta u + L\Delta w \\ y &\approx h(x_0, v_0, t) + \left.\frac{\partial h}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial h}{\partial v}\right|_0 (v - v_0) \\ &= h(x_0, v_0, t) + C\Delta x + M\Delta v \end{aligned} \tag{13.2}$$

The definitions of the partial derivative matrices A, B, C, L, and M are apparent from the above equations. The 0 subscript on the partial derivatives means that they are evaluated at the nominal control, state, output, and noise values. The definitions of the deviations Δx, Δu, Δw, and Δv are also apparent from the above equations.

Let us assume that the nominal noise values $w_0(t)$ and $v_0(t)$ are both equal to 0 for all time. [If they are not equal to 0 then we should be able to write them as the sum of a known deterministic part and a zero-mean part, redefine the noise quantities, and rewrite Equation (13.1) so that the nominal noise values are equal to 0. See Problem 13.1]. Since $w_0(t)$ and $v_0(t)$ are both equal to 0, we see that $\Delta w(t) = w(t)$ and $\Delta v(t) = v(t)$. Further assume that the control $u(t)$ is perfectly known. In general, this is a reasonable assumption. After all, the control input $u(t)$ is determined by our control system, so there should not be any uncertainty in its value. This means that $u_0(t) = u(t)$ and $\Delta u(t) = 0$. However, in reality there may be uncertainties in the outputs of our control system because they are connected to actuators that have biases and noise. If this is the case then we can express the control as $u_0(t) + \Delta u(t)$, where $u_0(t)$ is known and $\Delta u(t)$ is a zero-mean random variable, rewrite the system equations with a perfectly known control signal, and include $\Delta u(t)$ as part of the process noise (see Problem 13.2). Now we define the nominal system trajectory as

$$\begin{aligned} \dot{x}_0 &= f(x_0, u_0, w_0, t) \\ y_0 &= h(x_0, v_0, t) \end{aligned} \tag{13.3}$$

We define the deviation of the true state derivative from the nominal state derivative, and the deviation of the true measurement from the nominal measurement, as follows:

$$\begin{aligned} \Delta\dot{x} &= \dot{x} - \dot{x}_0 \\ \Delta y &= y - y_0 \end{aligned} \tag{13.4}$$

With these definitions Equation (13.2) becomes

$$\begin{aligned} \Delta\dot{x} &= A\Delta x + Lw \\ &= A\Delta x + \tilde{w} \\ \tilde{w} &\sim (0, \tilde{Q}), \qquad \tilde{Q} = LQL^T \\ \Delta y &= C\Delta x + Mv \\ &= C\Delta x + \tilde{v} \\ \tilde{v} &\sim (0, \tilde{R}), \qquad \tilde{R} = MRM^T \end{aligned} \tag{13.5}$$

The above equation is a linear system with state Δx and measurement Δy, so we can use a Kalman filter to estimate Δx. The inputs to the filter consist of Δy, which is the difference between the actual measurement y and the nominal measurement y_0. The Δx that is output from the Kalman filter is an estimate of the difference between the actual state x and the nominal state x_0. The Kalman filter equations for the linearized Kalman filter are

$$\begin{aligned} \Delta\hat{x}(0) &= 0 \\ P(0) &= E\left[(\Delta x(0) - \Delta\hat{x}(0))(\Delta x(0) - \Delta\hat{x}(0))^T\right] \\ \Delta\dot{\hat{x}} &= A\Delta\hat{x} + K(\Delta y - C\Delta\hat{x}) \\ K &= PC^T\tilde{R}^{-1} \\ \dot{P} &= AP + PA^T + \tilde{Q} - PC^T\tilde{R}^{-1}CP \\ \hat{x} &= x_0 + \Delta\hat{x} \end{aligned} \tag{13.6}$$

For the Kalman filter, P is equal to the covariance of the estimation error. In the linearized Kalman filter this is no longer true because of errors that creep into the linearization of Equation (13.2). However, if the linearization errors are small then P should be approximately equal to the covariance of the estimation error. The linearized Kalman filter can be summarized as follows.

The continuous-time linearized Kalman filter

1. The system equations are given as

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y &= h(x, v, t) \\ w &\sim (0, Q) \\ v &\sim (0, R) \end{aligned} \tag{13.7}$$

The nominal trajectory is known ahead of time:

$$\begin{aligned} \dot{x}_0 &= f(x_0, u_0, 0, t) \\ y_0 &= h(x_0, 0, t) \end{aligned} \tag{13.8}$$

2. Compute the following partial derivative matrices evaluated at the nominal trajectory values:

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_0 \\ L &= \left.\frac{\partial f}{\partial w}\right|_0 \\ C &= \left.\frac{\partial h}{\partial x}\right|_0 \\ M &= \left.\frac{\partial h}{\partial v}\right|_0 \end{aligned} \tag{13.9}$$

3. Compute the following matrices:

$$\begin{aligned} \tilde{Q} &= LQL^T \\ \tilde{R} &= MRM^T \end{aligned} \tag{13.10}$$

4. Define Δy as the difference between the actual measurement y and the nominal measurement y_0:

$$\Delta y = y - y_0 \tag{13.11}$$

5. Execute the following Kalman filter equations:

$$\begin{aligned} \Delta\hat{x}(0) &= 0 \\ P(0) &= E\left[(\Delta x(0) - \Delta\hat{x}(0))(\Delta x(0) - \Delta\hat{x}(0))^T\right] \\ \Delta\dot{\hat{x}} &= A\Delta\hat{x} + K(\Delta y - C\Delta\hat{x}) \\ K &= PC^T\tilde{R}^{-1} \\ \dot{P} &= AP + PA^T + \tilde{Q} - PC^T\tilde{R}^{-1}CP \end{aligned} \tag{13.12}$$

6. Estimate the state as follows:

$$\hat{x} = x_0 + \Delta\hat{x} \tag{13.13}$$

The hybrid linearized Kalman filter and the discrete-time linearized Kalman filter are not presented here, but if the development above is understood then their derivations should be straightforward.

13.2 THE EXTENDED KALMAN FILTER

The previous section obtained a linearized Kalman filter for estimating the states of a nonlinear system. The derivation was based on linearizing the nonlinear system around a nominal state trajectory. The question that arises is, How do we know the nominal state trajectory? In some cases it may not be straightforward to find the nominal trajectory. However, since the Kalman filter estimates the state of the system, we can use the Kalman filter estimate as the nominal state trajectory. This is sort of a bootstrap method. We linearize the nonlinear system around the Kalman filter estimate, and the Kalman filter estimate is based on the linearized system. This is the idea of the extended Kalman filter (EKF), which was originally proposed by Stanley Schmidt so that the Kalman filter could be applied to nonlinear spacecraft navigation problems [Bel67].

In Section 13.2.1, we will present the EKF for continuous-time systems with continuous-time measurements. In Section 13.2.2, we will present the hybrid EKF, which is the EKF for continuous-time systems with discrete-time measurements. In Section 13.2.3, we will present the EKF for discrete-time systems with discrete-time measurements.

13.2.1 The continuous-time extended Kalman filter

Combine the $\dot{x}_0$ expression in Equation (13.3) with the $\Delta\dot{\hat{x}}$ expression in Equation (13.6) to obtain

$$\dot{x}_0 + \Delta\dot{\hat{x}} = f(x_0, u_0, w_0, t) + A\Delta\hat{x} + K[y - y_0 - C(\hat{x} - x_0)] \tag{13.14}$$

Now choose $x_0(t) = \hat{x}(t)$ so that $\Delta\hat{x}(t) = 0$ and $\Delta\dot{\hat{x}}(t) = 0$. In other words, our linearization trajectory $x_0(t)$ is equal to our linearized Kalman filter estimate $\hat{x}(t)$. Then the nominal measurement expression in Equation (13.3) becomes

$$\begin{aligned} y_0 &= h(x_0, v_0, t) \\ &= h(\hat{x}, v_0, t) \end{aligned} \tag{13.15}$$

and Equation (13.14) becomes

$$\dot{\hat{x}} = f(\hat{x}, u, w_0, t) + K\left[y - h(\hat{x}, v_0, t)\right] \tag{13.16}$$

This is equivalent to the linearized Kalman filter except that we have chosen $x_0 = \hat{x}$, and we have rearranged the equations to obtain $\hat{x}$ directly. The Kalman gain K is the same as that presented in Equation (13.6). But this formulation inputs the measurement y directly, and outputs the state estimate $\hat{x}$ directly. This is often referred to as the extended Kalman–Bucy filter because Richard Bucy collaborated with Rudolph Kalman in the first publication of the continuous-time Kalman filter [Kal61]. The continuous-time EKF can be summarized as follows.

The continuous-time extended Kalman filter

1. The system equations are given as

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y &= h(x, v, t) \\ w &\sim (0, Q) \\ v &\sim (0, R) \end{aligned} \tag{13.17}$$

2. Compute the following partial derivative matrices evaluated at the current state estimate:

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} \\ L &= \left.\frac{\partial f}{\partial w}\right|_{\hat{x}} \\ C &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}} \\ M &= \left.\frac{\partial h}{\partial v}\right|_{\hat{x}} \end{aligned} \tag{13.18}$$

3. Compute the following matrices:

$$\begin{aligned} \tilde{Q} &= LQL^T \\ \tilde{R} &= MRM^T \end{aligned} \tag{13.19}$$

4. Execute the following Kalman filter equations:

$$\begin{aligned} \hat{x}(0) &= E[x(0)] \\ P(0) &= E\left[(x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^T\right] \\ \dot{\hat{x}} &= f(\hat{x}, u, w_0, t) + K\left[y - h(\hat{x}, v_0, t)\right] \\ K &= PC^T\tilde{R}^{-1} \\ \dot{P} &= AP + PA^T + \tilde{Q} - PC^T\tilde{R}^{-1}CP \end{aligned} \tag{13.20}$$

where the nominal noise values are given as $w_0 = 0$ and $v_0 = 0$.

■ EXAMPLE 13.1

In this example, we will use the continuous-time EKF to estimate the state of a two-phase permanent magnet synchronous motor. The system equations are given in Example 1.4 and are repeated here:

$$\begin{aligned} \dot{i}_a &= \frac{-R}{L}i_a + \frac{\omega\lambda}{L}\sin\theta + \frac{u_a + q_1}{L} \\ \dot{i}_b &= \frac{-R}{L}i_b - \frac{\omega\lambda}{L}\cos\theta + \frac{u_b + q_2}{L} \\ \dot{\omega} &= \frac{-3\lambda}{2J}i_a\sin\theta + \frac{3\lambda}{2J}i_b\cos\theta - \frac{F\omega}{J} + q_3 \\ \dot{\theta} &= \omega \end{aligned} \tag{13.21}$$

where i_a and i_b are the currents in the two windings, θ and ω are the angular position and velocity of the rotor, R and L are the winding resistance and inductance, λ is the flux constant, and F is the coefficient of viscous friction. The control inputs u_a and u_b consist of the applied voltages across the two windings, and J is the moment of inertia of the motor shaft and load. The state is defined as

$$x = \begin{bmatrix} i_a & i_b & \omega & \theta \end{bmatrix}^T \tag{13.22}$$

The q_i terms are process noise due to uncertainty in the control inputs (q_1 and q_2) and the load torque (q_3). The partial derivative A matrix is obtained as

$$\begin{aligned} A &= \frac{\partial f}{\partial x} \\ &= \begin{bmatrix} -R/L & 0 & \lambda s/L & x_3 \lambda c/L \\ 0 & -R/L & -\lambda c/L & x_3 \lambda s/L \\ -3\lambda s/2/J & 3\lambda c/2/J & -F/J & -3\lambda(x_1 c + x_2 s)/2/J \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{13.23}$$

where we have used the notation $s = \sin x_4$ and $c = \cos x_4$. Suppose that we can measure the winding currents with sense resistors so our measurement equations are

$$\begin{aligned} y(1) &= i_a + v(1) \\ y(2) &= i_b + v(2) \end{aligned} \tag{13.24}$$

where $v(1)$ and $v(2)$ are independent zero-mean white noise processes with standard deviations equal to 0.1 amps. The nominal control inputs are set to

$$\begin{aligned} u_a(t) &= \sin(2\pi t) \\ u_b(t) &= \cos(2\pi t) \end{aligned} \tag{13.25}$$

The actual control inputs are equal to the nominal values plus q_1 and q_2 (electrical noise terms), which are independent zero-mean white noise processes with standard deviations equal to 0.01 amps. The noise due to load torque disturbances (q_3) has a standard deviation of 0.5 rad/sec^2. Measurements are obtained continuously. Even though our measurements consist only of the winding currents and the system is nonlinear, we can use a continuous-time EKF (implemented in analog circuitry or very fast digital logic) to estimate the rotor position and velocity. The simulation results are shown in Figure 13.2. The four states are estimated quite well. In particular, the rotor position estimate is so good that the true and estimated rotor position traces are not distinguishable in Figure 13.2.

The P matrix quantifies the uncertainty in the state estimates. If the nonlinearities in the system and measurement are not too severe, then the P matrix should give us an idea of how accurate our estimates are. In this example, the standard deviations of the state estimation errors were obtained from the simulation and then compared with the diagonal elements of the steady-state P matrix that came out of the Kalman filter. Table 13.1 shows a comparison of the estimation errors that were determined by simulation and

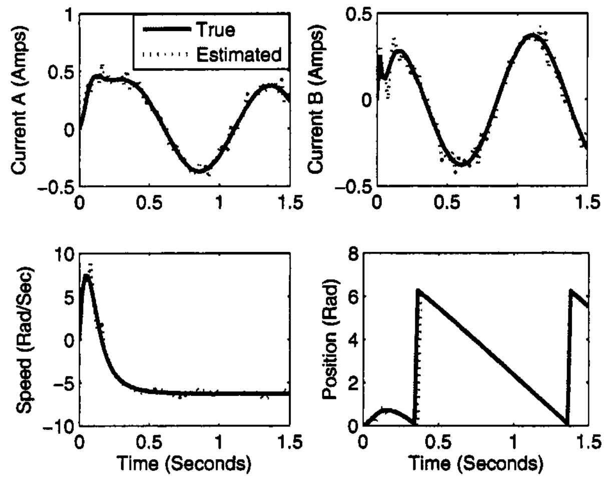

Figure 13.2 Continuous extended Kalman filter simulation results for the two-phase permanent magnet synchronous motor of Example 13.1.

Table 13.1 Example 13.1 results showing one standard deviation state estimation errors determined from simulation results and determined from the P matrix of the EKF. These results are for the two-phase permanent magnet motor simulation. This table shows that the P matrix gives a good indication of the magnitude of the EKF state estimation errors.

	Simulation	P Matrix
Winding A Current	0.054 amps	0.094 Amps
Winding B Current	0.052 amps	0.094 Amps
Speed	0.26 rad/sec	0.44 rad/sec
Position	0.013 rad	0.025 rad

the theoretical estimation errors based on the P matrix. We see that the P matrix gives a good indication of the magnitude of the estimation errors.

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13.2.2 The hybrid extended Kalman filter

Many real engineering systems are governed by continuous-time dynamics whereas the measurements are obtained at discrete instants of time. In this section, we will derive the hybrid EKF, which considers systems with continuous-time dynamics and discrete-time measurements. This is the most common situation encountered in practice.

Suppose we have a continuous-time system with discrete-time measurements as follows:

$$
\begin{aligned}
\dot{x} &= f(x, u, w, t) \\
y_k &= h_k(x_k, v_k) \\
w(t) &\sim (0, Q) \\
v_k &\sim (0, R_k)
\end{aligned}
\tag{13.26}
$$

The process noise $w(t)$ is continuous-time white noise with covariance Q, and the measurement noise v_k is discrete-time white noise with covariance R_k. Between measurements we propagate the state estimate according to the known nonlinear dynamics, and we propagate the covariance as derived in the continuous-time EKF of Section 13.2.1 using Equation (13.20). Recall that the $\dot{P}$ expression from Equation (13.20) is given as

$$
\dot{P} = AP + PA^T + LQL^T - PC^T(MRM^T)^{-1}CP \tag{13.27}
$$

In the hybrid EKF, we should not include the R term in the $\dot{P}$ equation because we are integrating P between measurement times, during which we do not have any measurements. Another way of looking at it is that in between measurement times we have measurements with infinite covariance ($R = \infty$), so the last term on the right side of the $\dot{P}$ equation goes to zero. This gives us the following for the time-update equations of the hybrid EKF:

$$
\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}, u, w_0, t) \\
\dot{P} &= AP + PA^T + LQL^T
\end{aligned}
\tag{13.28}
$$

where A and L are given in Equation (13.18). The above equations propagate $\hat{x}$ from $\hat{x}_{k-1}^+$ to $\hat{x}_k^-$, and P from P_{k-1}^+ to P_k^-. Note that w_0 is the nominal process noise in the above equation; that is, $w_0(t) = 0$.

At each measurement time, we update the state estimate and the covariance as derived in the discrete-time Kalman filter (Chapter 5):

$$
\begin{aligned}
K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\
\hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, v_0, t_k)] \\
P_k^+ &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k M_k R_k M_k^T K_k^T
\end{aligned}
\tag{13.29}
$$

where v_0 is the nominal measurement noise; that is, $v_0 = 0$. H_k is the partial derivative of $h_k(x_k, v_k)$ with respect to x_k, and M_k is the partial derivative of $h_k(x_k, v_k)$ with respect to v_k. H_k and M_k are evaluated at $\hat{x}_k^-$.

Note that P_k and K_k cannot be computed offline because they depend on H_k and M_k, which depend on $\hat{x}_k^-$, which in turn depends on the noisy measurements. Therefore, a steady-state solution does not (in general) exist to the extended Kalman filter. However, some efforts at obtaining steady-state approximations to the extended Kalman filter have been reported in [Saf78].

The hybrid EKF can be summarized as follows.

The hybrid extended Kalman filter

1. The system equations with continuous-time dynamics and discrete-time measurements are given as follows:

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y_k &= h_k(x_k, v_k) \\ w(t) &\sim (0, Q) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.30}$$

2. Initialize the filter as follows:

$$\begin{aligned} \hat{x}_0^+ &= E[x_0] \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{13.31}$$

3. For $k = 1, 2, \cdots$, perform the following.

   (a) Integrate the state estimate and its covariance from time $(k-1)^+$ to time k^- as follows:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, 0, t) \\ \dot{P} &= AP + PA^T + LQL^T \end{aligned} \tag{13.32}$$

   where F and L are given in Equation (13.18). We begin this integration process with $\hat{x} = \hat{x}_{k-1}^+$ and $P = P_{k-1}^+$. At the end of this integration we have $\hat{x} = \hat{x}_k^-$ and $P = P_k^-$.

   (b) At time k, incorporate the measurement y_k into the state estimate and estimation covariance as follows:

$$\begin{aligned} K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k(y_k - h_k(\hat{x}_k^-, 0, t_k)) \\ P_k^+ &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k M_k R_k M_k^T K_k^T \end{aligned} \tag{13.33}$$

   H_k and M_k are the partial derivatives of $h_k(x_k, v_k)$ with respect to x_k and v_k, and are both evaluated at $\hat{x}_k^-$. Note that other equivalent expressions can be used for K_k and P_k^+, as is apparent from Equation (5.19).

■ **EXAMPLE 13.2**

In this example, we will use the continuous-time EKF and the hybrid EKF to estimate the altitude x_1, velocity x_2, and constant ballistic coefficient $1/x_3$ of a body as it falls toward earth. A range-measuring device measures the altitude of the falling body. This example (or a variant thereof) is given in several places, for example [Ath68, Ste94, Jul00]. The equations for this system are

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= \rho_0 \exp(-x_1/k) x_2^2 / 2x_3 - g + w_2 \\ \dot{x}_3 &= w_3 \\ y &= x_1 + v \end{aligned} \tag{13.34}$$

As usual, w_i is the noise that affects the ith process equation, and v is the measurement noise. ρ_0 is the air density at sea level, k is a constant that defines the relationship between air density and altitude, and g is the acceleration due to gravity. The partial derivative matrices for this system are given as follows:

$$\begin{aligned}
A &= \frac{\partial f}{\partial x} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \\
A_{21} &= -\rho_0 \exp(-x_1/k) x_2^2/2kx_3 \\
A_{22} &= \rho_0 \exp(-x_1/k) x_2/x_3 \\
A_{23} &= -\rho_0 \exp(-x_1/k) x_2^2/2x_3^2 \\
C = H &= \frac{\partial h}{\partial x} \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{13.35}$$

We will use the continuous-time system equations to simulate the system. For the hybrid system we suppose that we obtain range measurements every 0.5 seconds. The constants that we will use are given as

$$\begin{aligned}
\rho_0 &= 0.0034 \text{ lb-sec}^2/\text{ft}^4 \\
g &= 32.2 \text{ ft/sec}^2 \\
k &= 22000 \text{ ft} \\
E[v^2(t)] &= 100 \text{ ft}^2 \\
E[w_i^2(t)] &= 0 \quad (i = 1, 2, 3)
\end{aligned} \tag{13.36}$$

The initial conditions of the system and the estimator are given as

$$\begin{aligned}
x_0 &= \begin{bmatrix} 100,000 & -6,000 & 1/2,000 \end{bmatrix}^T \\
\hat{x}_0^+ &= \begin{bmatrix} 100,010 & -6,100 & 1/2,500 \end{bmatrix}^T \\
P_0^+ &= \begin{bmatrix} 500 & 0 & 0 \\ 0 & 20,000 & 0 \\ 0 & 0 & 1/250,000 \end{bmatrix}
\end{aligned} \tag{13.37}$$

We use rectangular integration with a step size of 0.4 msec to simulate the system, the continuous-time EKF, and the hybrid EKF (with a measurement time of 0.5 sec). Figure 13.3 shows estimation-error magnitudes averaged over 100 simulations for the altitude, velocity, and ballistic coefficient reciprocal of the falling body. We see that the continuous-time EKF appears to perform better in general than the hybrid EKF. This is to be expected since more measurements are incorporated in the continuous-time EKF. The RMS estimation errors averaged over 100 simulations was 2.8 feet for the continuous-time EKF and 5.1 feet for the hybrid EKF for altitude estimation, 1.2 feet/s for the continuous-time EKF and 2.0 feet/s for the hybrid EKF for velocity estimation, and 213 for the continuous-time EKF and 246 for the hybrid EKF for the reciprocal of ballistic coefficient estimation. Of course, a continuous-time EKF (in analog hardware) would be more difficult to implement, tune, and modify than a hybrid EKF (in digital hardware).

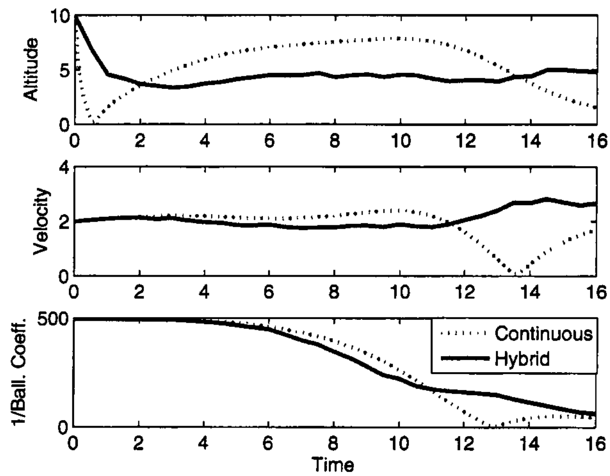

Figure 13.3 Example 13.2 altitude, velocity, and ballistic coefficient reciprocal estimation-error magnitudes of a falling body averaged over 100 simulations. The continuous-time EKF generally performs better than the hybrid EKF.

$\nabla\nabla\nabla$

13.2.3 The discrete-time extended Kalman filter

In this section, we will derive the discrete-time EKF, which considers discrete-time dynamics and discrete-time measurements. This situation is often encountered in practice. Even if the underlying system dynamics are continuous time, the EKF usually needs to be implemented in a digital computer. This means that there might not be enough computational power to integrate the system dynamics as required in a continuous-time EKF or a hybrid EKF. So the dynamics are often discretized (see Section 1.4) and then a discrete-time EKF can be used.

Suppose we have the system model

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.38}$$

We perform a Taylor series expansion of the state equation around $x_{k-1} = \hat{x}_{k-1}^{+}$ and $w_{k-1} = 0$ to obtain the following:

$$\begin{aligned} x_k &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) + \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} (x_{k-1} - \hat{x}_{k-1}^+) + \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+} w_{k-1} \\ &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}^+) + L_{k-1}w_{k-1} \\ &= F_{k-1}x_{k-1} + \left[f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) - F_{k-1}\hat{x}_{k-1}^+\right] + L_{k-1}w_{k-1} \\ &= F_{k-1}x_{k-1} + \tilde{u}_{k-1} + \tilde{w}_{k-1} \end{aligned} \tag{13.39}$$

F_{k-1} and L_{k-1} are defined by the above equation. The known signal $\tilde{u}_k$ and the noise signal $\tilde{w}_k$ are defined as follows:

$$\begin{aligned} \tilde{u}_k &= f_k(\hat{x}_k^+, u_k, 0) - F_k\hat{x}_k^+ \\ \tilde{w}_k &\sim (0, L_kQ_kL_k^T) \end{aligned} \tag{13.40}$$

We linearize the measurement equation around $x_k = \hat{x}_k^-$ and $v_k = 0$ to obtain

$$\begin{aligned} y_k &= h_k(\hat{x}_k^-, 0) + \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_k^-} (x_k - \hat{x}_k^-) + \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_k^-} v_k \\ &= h_k(\hat{x}_k^-, 0) + H_k(x_k - \hat{x}_k^-) + M_kv_k \\ &= H_kx_k + \left[h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^-\right] + M_kv_k \\ &= H_kx_k + z_k + \tilde{v}_k \end{aligned} \tag{13.41}$$

H_k and M_k are defined by the above equation. The known signal z_k and the noise signal $\tilde{v}_k$ are defined as

$$\begin{aligned} z_k &= h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^- \\ \tilde{v}_k &\sim (0, M_kR_kM_k^T) \end{aligned} \tag{13.42}$$

We have a linear state-space system in Equation (13.39) and a linear measurement in Equation (13.41). That means we can use the standard Kalman filter equations to estimate the state. This results in the following equations for the discrete-time extended Kalman filter.

$$\begin{aligned} P_k^- &= F_{k-1}P_{k-1}^+F_{k-1}^T + L_{k-1}Q_{k-1}L_{k-1}^T \\ K_k &= P_k^-H_k^T(H_kP_k^-H_k^T + M_kR_kM_k^T)^{-1} \\ \hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) \\ z_k &= h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^- \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k(y_k - H_k\hat{x}_k^- - z_k) \\ &= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, 0)] \\ P_k^+ &= (I - K_kH_k)P_k^- \end{aligned} \tag{13.43}$$

The discrete-time EKF can be summarized as follows.

The discrete-time extended Kalman filter

1. The system and measurement equations are given as follows:

$$
\begin{aligned}
x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\
y_k &= h_k(x_k, v_k) \\
w_k &\sim (0, Q_k) \\
v_k &\sim (0, R_k)
\end{aligned} \tag{13.44}
$$

2. Initialize the filter as follows:

$$
\begin{aligned}
\hat{x}_0^+ &= E(x_0) \\
P_0^+ &= E[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T]
\end{aligned} \tag{13.45}
$$

3. For $k = 1, 2, \cdots$, perform the following.

(a) Compute the following partial derivative matrices:

$$
\begin{aligned}
F_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} \\
L_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+}
\end{aligned} \tag{13.46}
$$

(b) Perform the time update of the state estimate and estimation-error covariance as follows:

$$
\begin{aligned}
P_k^- &= F_{k-1} P_{k-1}^+ F_{k-1}^T + L_{k-1} Q_{k-1} L_{k-1}^T \\
\hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0)
\end{aligned} \tag{13.47}
$$

(c) Compute the following partial derivative matrices:

$$
\begin{aligned}
H_k &= \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_k^-} \\
M_k &= \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_k^-}
\end{aligned} \tag{13.48}
$$

(d) Perform the measurement update of the state estimate and estimation-error covariance as follows:

$$
\begin{aligned}
K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\
\hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, 0)] \\
P_k^+ &= (I - K_k H_k) P_k^-
\end{aligned} \tag{13.49}
$$

Note that other equivalent expressions can be used for K_k and P_k^+, as is apparent from Equation (5.19).