

Case of varying rotation axis Clearly, the second term of the sum in (227) captures through $\boldsymbol{\omega}_n \times \boldsymbol{\omega}_{n+1} \neq 0$ the effect that a varying rotation axis has on the integrated orientation. For its practical usage, we notice that given usual IMU sampling times $\Delta t \leq 0.01s$, and the usual near-collinearity of $\boldsymbol{\omega}_n$ and $\boldsymbol{\omega}_{n+1}$ due to inertia, this second-order term takes values of the order of $10^{-6}\|\boldsymbol{\omega}\|^2$, or easily smaller. Terms with higher multiplicities of $\boldsymbol{\omega}\Delta t$ are even smaller and have been neglected.

Please note also that, while all zeroth-order integrators result in unit quaternions by construction (because they are computed as the product of two unit quaternions), this is not the case for the first-order integrator due to the sum in (227). Hence, when using the first-order integrator, and even if the summed term is small as stated, users should take care to check the evolution of the quaternion norm over time, and eventually re-normalize the quaternion if needed, using quaternion updates of the form $\mathbf{q} \leftarrow \mathbf{q}/\|\mathbf{q}\|$. Only if the constant axis assumption holds, then (228) holds too and this normalization is no longer necessary.

5 Error-state kinematics for IMU-driven systems

5.1 Motivation

We wish to write the error-estate equations of the kinematics of an inertial system integrating accelerometer and gyrometer readings with bias and noise, using the Hamilton quaternion to represent the orientation in space or *attitude*.

Accelerometer and gyrometer readings come typically from an Inertial Measurement Unit (IMU). Integrating IMU readings leads to dead-reckoning positioning systems, which drift with time. Avoiding drift is a matter of fusing this information with absolute position readings such as GPS or vision.

The error-state Kalman filter (ESKF) is one of the tools we may use for this purpose. Within the Kalman filtering paradigm, these are the most remarkable assets of the ESKF (Madyastha et al., 2011):

- The orientation error-state is minimal (*i.e.*, it has the same number of parameters as degrees of freedom), avoiding issues related to over-parametrization (or redundancy) and the consequent risk of singularity of the involved covariances matrices, resulting typically from enforcing constraints.
- The error-state system is always operating close to the origin, and therefore far from possible parameter singularities, gimbal lock issues, or the like, providing a guarantee that the linearization validity holds at all times.
- The error-state is always small, meaning that all second-order products are negligible. This makes the computation of Jacobians very easy and fast. Some Jacobians may even be constant or equal to available state magnitudes.

- The error dynamics are slow because all the large-signal dynamics have been integrated in the nominal-state. This means that we can apply KF corrections (which are the only means to observe the errors) at a lower rate than the predictions.

5.2 The error-state Kalman filter explained

In error-state filter formulations, we speak of true-, nominal- and error-state values, the true-state being expressed as a suitable composition (linear sum, quaternion product or matrix product) of the nominal- and the error- states. The idea is to consider the nominal-state as large-signal (integrable in non-linear fashion) and the error-state as small signal (thus linearly integrable and suitable for linear-Gaussian filtering).

The error-state filter can be explained as follows. On one side, high-frequency IMU data \mathbf{u}_m is integrated into a nominal-state \mathbf{x} . This nominal state does not take into account the noise terms \mathbf{w} and other possible model imperfections. As a consequence, it will accumulate errors. These errors are collected in the error-state $\delta\mathbf{x}$ and estimated with the Error-State Kalman Filter (ESKF), this time incorporating all the noise and perturbations. The error-state consists of small-signal magnitudes, and its evolution function is correctly defined by a (time-variant) linear dynamic system, with its dynamic, control and measurement matrices computed from the values of the nominal-state. In parallel with integration of the nominal-state, the ESKF predicts a Gaussian estimate of the error-state. It only predicts, because by now no other measurement is available to correct these estimates. The filter correction is performed at the arrival of information other than IMU (*e.g.* GPS, vision, etc.), which is able to render the errors observable and which happens generally at a much lower rate than the integration phase. This correction provides a posterior Gaussian estimate of the error-state. After this, the error-state's mean is injected into the nominal-state, then reset to zero. The error-state's covariances matrix is conveniently updated to reflect this reset. The system goes on like this forever.

5.3 System kinematics in continuous time

The definition of all the involved variables is summarized in Table 3. Two important decisions regarding conventions are worth mentioning:

- The angular rates $\boldsymbol{\omega}$ are defined *locally* with respect to the nominal quaternion. This allows us to use the gyrometer measurements $\boldsymbol{\omega}_m$ directly, as they provide body-referenced angular rates.
- The angular error $\delta\boldsymbol{\theta}$ is also defined *locally* with respect to the nominal orientation. This is not necessarily the optimal way to proceed, but it corresponds to the choice in most IMU-integration works —what we could call the *classical approach*. There exists evidence (Li and Mourikis, 2012) that a globally-defined angular error has better properties. This will be explored too in the present document, Section 7, but most of the developments, examples and algorithms here are based in this locally-defined angular error.

Table 3: All variables in the error-state Kalman filter.

Magnitude	True	Nominal	Error	Composition	Measured	Noise
Full state ⁽¹⁾	\mathbf{x}_t	\mathbf{x}	$\delta\mathbf{x}$	$\mathbf{x}_t = \mathbf{x} \oplus \delta\mathbf{x}$		
Position	\mathbf{p}_t	\mathbf{p}	$\delta\mathbf{p}$	$\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}$		
Velocity	\mathbf{v}_t	\mathbf{v}	$\delta\mathbf{v}$	$\mathbf{v}_t = \mathbf{v} + \delta\mathbf{v}$		
Quaternion ^(2,3)	\mathbf{q}_t	\mathbf{q}	$\delta\mathbf{q}$	$\mathbf{q}_t = \mathbf{q} \otimes \delta\mathbf{q}$		
Rotation matrix ^(2,3)	\mathbf{R}_t	\mathbf{R}	$\delta\mathbf{R}$	$\mathbf{R}_t = \mathbf{R} \delta\mathbf{R}$		
Angles vector ⁽⁴⁾			$\delta\boldsymbol{\theta}$	$\delta\mathbf{q} = e^{\delta\boldsymbol{\theta}/2}$ $\delta\mathbf{R} = e^{[\delta\boldsymbol{\theta}]_{\times}}$		
Accelerometer bias	\mathbf{a}_{bt}	\mathbf{a}_b	$\delta\mathbf{a}_b$	$\mathbf{a}_{bt} = \mathbf{a}_b + \delta\mathbf{a}_b$		\mathbf{a}_w
Gyrometer bias	$\boldsymbol{\omega}_{bt}$	$\boldsymbol{\omega}_b$	$\delta\boldsymbol{\omega}_b$	$\boldsymbol{\omega}_{bt} = \boldsymbol{\omega}_b + \delta\boldsymbol{\omega}_b$		$\boldsymbol{\omega}_w$
Gravity vector	\mathbf{g}_t	\mathbf{g}	$\delta\mathbf{g}$	$\mathbf{g}_t = \mathbf{g} + \delta\mathbf{g}$		
Acceleration	\mathbf{a}_t				\mathbf{a}_m	\mathbf{a}_n
Angular rate	$\boldsymbol{\omega}_t$				$\boldsymbol{\omega}_m$	$\boldsymbol{\omega}_n$

⁽¹⁾ the symbol \oplus indicates a generic composition

⁽²⁾ indicates non-minimal representations

⁽³⁾ see Table 4 for the composition formula in case of globally-defined angular errors

⁽⁴⁾ exponentials defined as in (101) and (69, 77)

5.3.1 The true-state kinematics

The true kinematic equations are

$$\dot{\mathbf{p}}_t = \mathbf{v}_t \quad (230a)$$

$$\dot{\mathbf{v}}_t = \mathbf{a}_t \quad (230b)$$

$$\dot{\mathbf{q}}_t = \frac{1}{2}\mathbf{q}_t \otimes \boldsymbol{\omega}_t \quad (230c)$$

$$\dot{\mathbf{a}}_{bt} = \mathbf{a}_w \quad (230d)$$

$$\dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w \quad (230e)$$

$$\dot{\mathbf{g}}_t = 0 \quad (230f)$$

Here, the true acceleration \mathbf{a}_t and angular rate $\boldsymbol{\omega}_t$ are obtained from an IMU in the form of noisy sensor readings \mathbf{a}_m and $\boldsymbol{\omega}_m$ in body frame, namely²³

$$\mathbf{a}_m = \mathbf{R}_t^\top (\mathbf{a}_t - \mathbf{g}_t) + \mathbf{a}_{bt} + \mathbf{a}_n \quad (231)$$

$$\boldsymbol{\omega}_m = \boldsymbol{\omega}_t + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n \quad (232)$$

with $\mathbf{R}_t \triangleq \mathbf{R}\{\mathbf{q}_t\}$. With this, the true values can be isolated (this means that we have inverted the measurement equations),

$$\mathbf{a}_t = \mathbf{R}_t (\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \quad (233)$$

$$\boldsymbol{\omega}_t = \boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n. \quad (234)$$

Substituting above yields the kinematic system

$$\dot{\mathbf{p}}_t = \mathbf{v}_t \quad (235a)$$

$$\dot{\mathbf{v}}_t = \mathbf{R}_t (\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \quad (235b)$$

$$\dot{\mathbf{q}}_t = \frac{1}{2} \mathbf{q}_t \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n) \quad (235c)$$

$$\dot{\mathbf{a}}_{bt} = \mathbf{a}_w \quad (235d)$$

$$\dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w \quad (235e)$$

$$\dot{\mathbf{g}}_t = 0 \quad (235f)$$

which we may name $\dot{\mathbf{x}}_t = f_t(\mathbf{x}_t, \mathbf{u}, \mathbf{w})$. This system has state \mathbf{x}_t , is governed by IMU noisy readings \mathbf{u}_m , and is perturbed by white Gaussian noise \mathbf{w} , all defined by

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{p}_t \\ \mathbf{v}_t \\ \mathbf{q}_t \\ \mathbf{a}_{bt} \\ \boldsymbol{\omega}_{bt} \\ \mathbf{g}_t \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} \mathbf{a}_m - \mathbf{a}_n \\ \boldsymbol{\omega}_m - \boldsymbol{\omega}_n \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} \mathbf{a}_w \\ \boldsymbol{\omega}_w \end{bmatrix}. \quad (236)$$

It is to note in the above formulation that the gravity vector \mathbf{g}_t is going to be estimated by the filter. It has a constant evolution equation, (235f), as corresponds to a magnitude that is known to be constant. The system starts at a fixed and arbitrarily known initial orientation $\mathbf{q}_t(t=0) = \mathbf{q}_0$, which, being generally not in the horizontal plane, makes the initial gravity vector generally unknown. For simplicity it is usually taken $\mathbf{q}_0 = (1, 0, 0, 0)$

²³It is common practice to neglect the Earth's rotation rate $\boldsymbol{\omega}_\mathcal{E}$ in the rotational kinematics described in (232), which would otherwise be $\boldsymbol{\omega}_m = \boldsymbol{\omega}_t + \mathbf{R}_t^\top \boldsymbol{\omega}_\mathcal{E} + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n$. Considering a non-null Earth rotation rate is, in the vast majority of practical cases, unjustifiably complicated. However, we notice that when employing high-end IMU sensors with very small noises and biases, a value of $\omega_\mathcal{E} = 15^\circ/\text{h} \approx 7.3 \cdot 10^{-5} \text{ rad/s}$ might become directly measurable; in such cases, in order to keep the IMU error model valid, the rate $\boldsymbol{\omega}_\mathcal{E}$ should not be neglected in the formulation.

and thus $\mathbf{R}_0 = \mathbf{R}\{\mathbf{q}_0\} = \mathbf{I}$. We estimate \mathbf{g}_t expressed in frame \mathbf{q}_0 , and not \mathbf{q}_t expressed in a horizontal frame, so that the initial uncertainty in orientation is transferred to an initial uncertainty on the gravity direction. We do so to improve linearity: indeed, equation (235b) is now linear in \mathbf{g} , which carries all the uncertainty, and the initial orientation \mathbf{q}_0 is known without uncertainty, so that \mathbf{q} starts with no uncertainty. Once the gravity vector is estimated the horizontal plane can be recovered and, if desired, the whole state and recovered motion trajectories can be re-oriented to reflect the estimated horizontal. See (Lupton and Sukkarieh, 2009) for further justification. This is of course optional, and the reader is free to remove all equations related to gravity from the system and adopt a more classical approach of considering $\mathbf{g} \triangleq (0, 0, -9.8xx)$, with xx the appropriate decimal digits of the gravity vector on the site of the experiment, and an uncertain initial orientation \mathbf{q}_0 .

5.3.2 The nominal-state kinematics

The nominal-state kinematics corresponds to the modeled system without noises or perturbations,

$$\dot{\mathbf{p}} = \mathbf{v} \quad (237a)$$

$$\dot{\mathbf{v}} = \mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g} \quad (237b)$$

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \quad (237c)$$

$$\dot{\mathbf{a}}_b = 0 \quad (237d)$$

$$\dot{\boldsymbol{\omega}}_b = 0 \quad (237e)$$

$$\dot{\mathbf{g}} = 0. \quad (237f)$$

5.3.3 The error-state kinematics

The goal is to determine the linearized dynamics of the error-state. For each state equation, we write its composition (in Table 3), solving for the error state and simplifying all second-order infinitesimals. We give here the full error-state dynamic system and proceed afterwards with comments and proofs.

$$\dot{\delta \mathbf{p}} = \delta \mathbf{v} \quad (238a)$$

$$\dot{\delta \mathbf{v}} = -\mathbf{R}[\mathbf{a}_m - \mathbf{a}_b]_{\times} \delta \boldsymbol{\theta} - \mathbf{R} \delta \mathbf{a}_b + \delta \mathbf{g} - \mathbf{R} \mathbf{a}_n \quad (238b)$$

$$\dot{\delta \boldsymbol{\theta}} = -[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b]_{\times} \delta \boldsymbol{\theta} - \delta \boldsymbol{\omega}_b - \boldsymbol{\omega}_n \quad (238c)$$

$$\dot{\delta \mathbf{a}}_b = \mathbf{a}_w \quad (238d)$$

$$\dot{\delta \boldsymbol{\omega}}_b = \boldsymbol{\omega}_w \quad (238e)$$

$$\dot{\delta \mathbf{g}} = 0. \quad (238f)$$

Equations (238a), (238d), (238e) and (238f), respectively of position, both biases, and gravity errors, are derived from linear equations and their error-state dynamics is trivial.

As an example, consider the true and nominal position equations (235a) and (237a), their composition $\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}$ from Table 3, and solve for $\dot{\delta\mathbf{p}}$ to obtain (238a).

Equations (238b) and (238c), of velocity and orientation errors, require some non-trivial manipulations of the non-linear equations (235b) and (235c) to obtain the linearized dynamics. Their proofs are developed in the following two sections.

Equation (238b): The linear velocity error. We wish to determine $\dot{\delta\mathbf{v}}$, the dynamics of the velocity errors. We start with the following relations

$$\mathbf{R}_t = \mathbf{R}(\mathbf{I} + [\delta\boldsymbol{\theta}]_{\times}) + O(\|\delta\boldsymbol{\theta}\|^2) \quad (239)$$

$$\dot{\mathbf{v}} = \mathbf{R}\mathbf{a}_{\mathcal{B}} + \mathbf{g}, \quad (240)$$

where (239) is the small-signal approximation of \mathbf{R}_t , and in (240) we rewrote (237b) but introducing $\mathbf{a}_{\mathcal{B}}$ and $\delta\mathbf{a}_{\mathcal{B}}$, defined as the large- and small-signal accelerations in body frame,

$$\mathbf{a}_{\mathcal{B}} \triangleq \mathbf{a}_m - \mathbf{a}_b \quad (241)$$

$$\delta\mathbf{a}_{\mathcal{B}} \triangleq -\delta\mathbf{a}_b - \mathbf{a}_n \quad (242)$$

so that we can write the true acceleration in inertial frame as a composition of large- and small-signal terms,

$$\mathbf{a}_t = \mathbf{R}_t(\mathbf{a}_{\mathcal{B}} + \delta\mathbf{a}_{\mathcal{B}}) + \mathbf{g} + \delta\mathbf{g}. \quad (243)$$

We proceed by writing the expression (235b) of $\dot{\mathbf{v}}_t$ in two different forms (left and right developments), where the terms $O(\|\delta\boldsymbol{\theta}\|^2)$ have been ignored,

$$\begin{aligned} \dot{\mathbf{v}} + \dot{\delta\mathbf{v}} &= \boxed{\dot{\mathbf{v}}_t} = \mathbf{R}(\mathbf{I} + [\delta\boldsymbol{\theta}]_{\times})(\mathbf{a}_{\mathcal{B}} + \delta\mathbf{a}_{\mathcal{B}}) + \mathbf{g} + \delta\mathbf{g} \\ \mathbf{R}\mathbf{a}_{\mathcal{B}} + \mathbf{g} + \dot{\delta\mathbf{v}} &= \mathbf{R}\mathbf{a}_{\mathcal{B}} + \mathbf{R}\delta\mathbf{a}_{\mathcal{B}} + \mathbf{R}[\delta\boldsymbol{\theta}]_{\times}\mathbf{a}_{\mathcal{B}} + \mathbf{R}[\delta\boldsymbol{\theta}]_{\times}\delta\mathbf{a}_{\mathcal{B}} + \mathbf{g} + \delta\mathbf{g} \end{aligned}$$

This leads after removing $\mathbf{R}\mathbf{a}_{\mathcal{B}} + \mathbf{g}$ from left and right to

$$\dot{\delta\mathbf{v}} = \mathbf{R}(\delta\mathbf{a}_{\mathcal{B}} + [\delta\boldsymbol{\theta}]_{\times}\mathbf{a}_{\mathcal{B}}) + \mathbf{R}[\delta\boldsymbol{\theta}]_{\times}\delta\mathbf{a}_{\mathcal{B}} + \delta\mathbf{g} \quad (244)$$

Eliminating the second order terms and reorganizing some cross-products (with $[\mathbf{a}]_{\times}\mathbf{b} = -[\mathbf{b}]_{\times}\mathbf{a}$), we get

$$\dot{\delta\mathbf{v}} = \mathbf{R}(\delta\mathbf{a}_{\mathcal{B}} - [\mathbf{a}_{\mathcal{B}}]_{\times}\delta\boldsymbol{\theta}) + \delta\mathbf{g}, \quad (245)$$

then, recalling (241) and (242),

$$\dot{\delta\mathbf{v}} = \mathbf{R}(-[\mathbf{a}_m - \mathbf{a}_b]_{\times}\delta\boldsymbol{\theta} - \delta\mathbf{a}_b - \mathbf{a}_n) + \delta\mathbf{g} \quad (246)$$

which after proper rearranging leads to the dynamics of the linear velocity error,

$$\boxed{\dot{\delta\mathbf{v}} = -\mathbf{R}[\mathbf{a}_m - \mathbf{a}_b]_{\times}\delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g} - \mathbf{R}\mathbf{a}_n}. \quad (247)$$

To further clean up this expression, we can often times assume that the accelerometer noise is white, uncorrelated and isotropic²⁴,

$$\mathbb{E}[\mathbf{a}_n] = 0 \quad \mathbb{E}[\mathbf{a}_n \mathbf{a}_n^\top] = \sigma_a^2 \mathbf{I}, \quad (248)$$

that is, the covariance ellipsoid is a sphere centered at the origin, which means that its mean and covariances matrix are invariant upon rotations (*Proof*: $\mathbb{E}[\mathbf{R}\mathbf{a}_n] = \mathbf{R}\mathbb{E}[\mathbf{a}_n] = 0$ and $\mathbf{E}[(\mathbf{R}\mathbf{a}_n)(\mathbf{R}\mathbf{a}_n)^\top] = \mathbf{R}\mathbb{E}[\mathbf{a}_n \mathbf{a}_n^\top] \mathbf{R}^\top = \mathbf{R}\sigma_a^2 \mathbf{I} \mathbf{R}^\top = \sigma_a^2 \mathbf{I}$). Then we can redefine the accelerometer noise vector, with absolutely no consequences, according to

$$\mathbf{a}_n \leftarrow \mathbf{R}\mathbf{a}_n \quad (249)$$

which gives

$$\boxed{\dot{\delta \mathbf{v}} = -\mathbf{R} [\mathbf{a}_m - \mathbf{a}_b]_\times \delta \boldsymbol{\theta} - \mathbf{R} \delta \mathbf{a}_b + \delta \mathbf{g} - \mathbf{a}_n}. \quad (250)$$

Equation (238c): The orientation error. We wish to determine $\dot{\delta \boldsymbol{\theta}}$, the dynamics of the angular errors. We start with the following relations

$$\dot{\mathbf{q}}_t = \frac{1}{2} \mathbf{q}_t \otimes \boldsymbol{\omega}_t \quad (251)$$

$$\dot{\mathbf{q}} = \frac{1}{2} \mathbf{q} \otimes \boldsymbol{\omega}, \quad (252)$$

which are the true- and nominal- definitions of the quaternion derivatives.

As we did with the acceleration, we group large- and small-signal terms in the angular rate for clarity,

$$\boldsymbol{\omega} \triangleq \boldsymbol{\omega}_m - \boldsymbol{\omega}_b \quad (253)$$

$$\delta \boldsymbol{\omega} \triangleq -\delta \boldsymbol{\omega}_b - \boldsymbol{\omega}_n, \quad (254)$$

so that $\boldsymbol{\omega}_t$ can be written with a nominal part and an error part,

$$\boldsymbol{\omega}_t = \boldsymbol{\omega} + \delta \boldsymbol{\omega}. \quad (255)$$

We proceed by computing $\dot{\mathbf{q}}_t$ by two different means (left and right developments)

$$\begin{aligned} (\mathbf{q} \otimes \dot{\delta \mathbf{q}}) &= \boxed{\dot{\mathbf{q}}_t} = \frac{1}{2} \mathbf{q}_t \otimes \boldsymbol{\omega}_t \\ \dot{\mathbf{q}} \otimes \delta \mathbf{q} + \mathbf{q} \otimes \dot{\delta \mathbf{q}} &= \frac{1}{2} \mathbf{q} \otimes \delta \mathbf{q} \otimes \boldsymbol{\omega}_t \\ \frac{1}{2} \mathbf{q} \otimes \boldsymbol{\omega} \otimes \delta \mathbf{q} + \mathbf{q} \otimes \dot{\delta \mathbf{q}} &= \end{aligned}$$

²⁴This assumption cannot be made in cases where the three *XYZ* accelerometers are not identical.

simplifying the leading \mathbf{q} and isolating $\delta\dot{\mathbf{q}}$ we obtain

$$\begin{aligned}
\begin{bmatrix} 0 \\ \delta\dot{\boldsymbol{\theta}} \end{bmatrix} &= \boxed{2\delta\dot{\mathbf{q}}} = \delta\mathbf{q} \otimes \boldsymbol{\omega}_t - \boldsymbol{\omega} \otimes \delta\mathbf{q} \\
&= [\mathbf{q}]_R(\boldsymbol{\omega}_t)\delta\mathbf{q} - [\mathbf{q}]_L(\boldsymbol{\omega})\delta\mathbf{q} \\
&= \begin{bmatrix} 0 & -(\boldsymbol{\omega}_t - \boldsymbol{\omega})^\top \\ (\boldsymbol{\omega}_t - \boldsymbol{\omega}) & -[\boldsymbol{\omega}_t + \boldsymbol{\omega}]_\times \end{bmatrix} \begin{bmatrix} 1 \\ \delta\boldsymbol{\theta}/2 \end{bmatrix} + O(\|\delta\boldsymbol{\theta}\|^2) \\
&= \begin{bmatrix} 0 & -\delta\boldsymbol{\omega}^\top \\ \delta\boldsymbol{\omega} & -[2\boldsymbol{\omega} + \delta\boldsymbol{\omega}]_\times \end{bmatrix} \begin{bmatrix} 1 \\ \delta\boldsymbol{\theta}/2 \end{bmatrix} + O(\|\delta\boldsymbol{\theta}\|^2)
\end{aligned} \tag{256}$$

which results in one scalar- and one vector- equalities

$$0 = \delta\boldsymbol{\omega}^\top \delta\boldsymbol{\theta} + O(\|\delta\boldsymbol{\theta}\|^2) \tag{257a}$$

$$\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{\omega} - [\boldsymbol{\omega}]_\times \delta\boldsymbol{\theta} - \frac{1}{2} [\delta\boldsymbol{\omega}]_\times \delta\boldsymbol{\theta} + O(\|\delta\boldsymbol{\theta}\|^2). \tag{257b}$$

The first equation leads to $\delta\boldsymbol{\omega}^\top \delta\boldsymbol{\theta} = O(\|\delta\boldsymbol{\theta}\|^2)$, which is formed by second-order infinitesimals, not very useful. The second equation yields, after neglecting all second-order terms,

$$\delta\dot{\boldsymbol{\theta}} = -[\boldsymbol{\omega}]_\times \delta\boldsymbol{\theta} + \delta\boldsymbol{\omega} \tag{258}$$

and finally, recalling (253) and (254), we get the linearized dynamics of the angular error,

$$\boxed{\delta\dot{\boldsymbol{\theta}} = -[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b]_\times \delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b - \boldsymbol{\omega}_n} . \tag{259}$$

5.4 System kinematics in discrete time

The differential equations above need to be integrated into differences equations to account for discrete time intervals $\Delta t > 0$. The integration methods may vary. In some cases, one will be able to use exact closed-form solutions. In other cases, numerical integration of varying degree of accuracy may be employed. Please refer to the Appendices for pertinent details on integration methods.

Integration needs to be done for the following sub-systems:

1. The nominal state.
2. The error-state.
 - (a) The deterministic part: state dynamics and control.
 - (b) The stochastic part: noise and perturbations.

5.4.1 The nominal state kinematics

We can write the differences equations of the nominal-state as

$$\mathbf{p} \leftarrow \mathbf{p} + \mathbf{v} \Delta t + \frac{1}{2}(\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g}) \Delta t^2 \quad (260a)$$

$$\mathbf{v} \leftarrow \mathbf{v} + (\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g}) \Delta t \quad (260b)$$

$$\mathbf{q} \leftarrow \mathbf{q} \otimes \mathbf{q}\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \Delta t\} \quad (260c)$$

$$\mathbf{a}_b \leftarrow \mathbf{a}_b \quad (260d)$$

$$\boldsymbol{\omega}_b \leftarrow \boldsymbol{\omega}_b \quad (260e)$$

$$\mathbf{g} \leftarrow \mathbf{g} , \quad (260f)$$

where $x \leftarrow f(x, \bullet)$ stands for a time update of the type $x_{k+1} = f(x_k, \bullet_k)$, $\mathbf{R} \triangleq \mathbf{R}\{\mathbf{q}\}$ is the rotation matrix associated to the current nominal orientation \mathbf{q} , and $\mathbf{q}\{v\}$ is the quaternion associated to the rotation v , according to (101).

We can also use more precise integration, please see the Appendices for more information.

5.4.2 The error-state kinematics

The deterministic part is integrated normally (in this case we follow the methods in App. C.2), and the integration of the stochastic part results in random impulses (see App. E), thus,

$$\delta \mathbf{p} \leftarrow \delta \mathbf{p} + \delta \mathbf{v} \Delta t \quad (261a)$$

$$\delta \mathbf{v} \leftarrow \delta \mathbf{v} + (-\mathbf{R}[\mathbf{a}_m - \mathbf{a}_b]_{\times} \delta \boldsymbol{\theta} - \mathbf{R} \delta \mathbf{a}_b + \delta \mathbf{g}) \Delta t + \mathbf{v}_i \quad (261b)$$

$$\delta \boldsymbol{\theta} \leftarrow \mathbf{R}^{\top} \{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \Delta t\} \delta \boldsymbol{\theta} - \delta \boldsymbol{\omega}_b \Delta t + \boldsymbol{\theta}_i \quad (261c)$$

$$\delta \mathbf{a}_b \leftarrow \delta \mathbf{a}_b + \mathbf{a}_i \quad (261d)$$

$$\delta \boldsymbol{\omega}_b \leftarrow \delta \boldsymbol{\omega}_b + \boldsymbol{\omega}_i \quad (261e)$$

$$\delta \mathbf{g} \leftarrow \delta \mathbf{g} . \quad (261f)$$

Here, \mathbf{v}_i , $\boldsymbol{\theta}_i$, \mathbf{a}_i and $\boldsymbol{\omega}_i$ are the random impulses applied to the velocity, orientation and bias estimates, modeled by white Gaussian processes. Their mean is zero, and their covariances matrices are obtained by integrating the covariances of \mathbf{a}_n , $\boldsymbol{\omega}_n$, \mathbf{a}_w and $\boldsymbol{\omega}_w$ over the step time Δt (see App. E),

$$\mathbf{V}_i = \sigma_{\mathbf{a}_n}^2 \Delta t^2 \mathbf{I} \quad [m^2/s^2] \quad (262)$$

$$\boldsymbol{\Theta}_i = \sigma_{\boldsymbol{\omega}_n}^2 \Delta t^2 \mathbf{I} \quad [rad^2] \quad (263)$$

$$\mathbf{A}_i = \sigma_{\mathbf{a}_w}^2 \Delta t \mathbf{I} \quad [m^2/s^4] \quad (264)$$

$$\boldsymbol{\Omega}_i = \sigma_{\boldsymbol{\omega}_w}^2 \Delta t \mathbf{I} \quad [rad^2/s^2] \quad (265)$$

where $\sigma_{\mathbf{a}_n}[m/s^2]$, $\sigma_{\boldsymbol{\omega}_n}[rad/s]$, $\sigma_{\mathbf{a}_w}[m/s^2\sqrt{s}]$ and $\sigma_{\boldsymbol{\omega}_w}[rad/s\sqrt{s}]$ are to be determined from the information in the IMU datasheet, or from experimental measurements.

5.4.3 The error-state Jacobian and perturbation matrices

The Jacobians are obtained by simple inspection of the error-state differences equations in the previous section.

To write these equations in compact form, we consider the nominal state vector \mathbf{x} , the error state vector $\delta\mathbf{x}$, the input vector \mathbf{u}_m , and the perturbation impulses vector \mathbf{i} , as follows (see App. E.1 for details and justifications),

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \mathbf{q} \\ \mathbf{a}_b \\ \boldsymbol{\omega}_b \\ \mathbf{g} \end{bmatrix}, \quad \delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{p} \\ \delta\mathbf{v} \\ \delta\boldsymbol{\theta} \\ \delta\mathbf{a}_b \\ \delta\boldsymbol{\omega}_b \\ \delta\mathbf{g} \end{bmatrix}, \quad \mathbf{u}_m = \begin{bmatrix} \mathbf{a}_m \\ \boldsymbol{\omega}_m \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} \mathbf{v}_i \\ \boldsymbol{\theta}_i \\ \mathbf{a}_i \\ \boldsymbol{\omega}_i \end{bmatrix} \quad (266)$$

The error-state system is now

$$\delta\mathbf{x} \leftarrow f(\mathbf{x}, \delta\mathbf{x}, \mathbf{u}_m, \mathbf{i}) = \mathbf{F}_x(\mathbf{x}, \mathbf{u}_m) \cdot \delta\mathbf{x} + \mathbf{F}_i \cdot \mathbf{i}, \quad (267)$$

The ESKF prediction equations are written:

$$\hat{\delta\mathbf{x}} \leftarrow \mathbf{F}_x(\mathbf{x}, \mathbf{u}_m) \cdot \hat{\delta\mathbf{x}} \quad (268)$$

$$\mathbf{P} \leftarrow \mathbf{F}_x \mathbf{P} \mathbf{F}_x^\top + \mathbf{F}_i \mathbf{Q}_i \mathbf{F}_i^\top, \quad (269)$$

where $\delta\mathbf{x} \sim \mathcal{N}\{\hat{\delta\mathbf{x}}, \mathbf{P}\}$ ²⁵; \mathbf{F}_x and \mathbf{F}_i are the Jacobians of $f()$ with respect to the error and perturbation vectors; and \mathbf{Q}_i is the covariances matrix of the perturbation impulses.

The expressions of the Jacobian and covariances matrices above are detailed below. All state-related values appearing herein are extracted directly from the nominal state.

$$\mathbf{F}_x = \left. \frac{\partial f}{\partial \delta\mathbf{x}} \right|_{\mathbf{x}, \mathbf{u}_m} = \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I} & -\mathbf{R}[\mathbf{a}_m - \mathbf{a}_b]_\times \Delta t & -\mathbf{R}\Delta t & 0 & \mathbf{I}\Delta t \\ 0 & 0 & \mathbf{R}^\top\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b)\Delta t\} & 0 & -\mathbf{I}\Delta t & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \quad (270)$$

$$\mathbf{F}_i = \left. \frac{\partial f}{\partial \mathbf{i}} \right|_{\mathbf{x}, \mathbf{u}_m} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q}_i = \begin{bmatrix} \mathbf{V}_i & 0 & 0 & 0 \\ 0 & \boldsymbol{\Theta}_i & 0 & 0 \\ 0 & 0 & \mathbf{A}_i & 0 \\ 0 & 0 & 0 & \boldsymbol{\Omega}_i \end{bmatrix}. \quad (271)$$

²⁵ $x \sim \mathcal{N}\{\mu, \Sigma\}$ means that x is a Gaussian random variable with mean and covariances matrix specified by μ and Σ .

Please note particularly that \mathbf{F}_x is the system's transition matrix, which can be approximated to different levels of precision in a number of ways. We showed here one of its simplest forms (the Euler form). See Appendices B to D for further reference.

Please note also that, being the mean of the error $\delta\mathbf{x}$ initialized to zero, the linear equation (268) always returns zero. You should of course skip line (268) in your code. I recommend that you write it, though, but that you comment it out so that you are sure you did not forget anything.

And please note, finally, that you should NOT skip the covariance prediction (269)!! In effect, the term $\mathbf{F}_i \mathbf{Q}_i \mathbf{F}_i^\top$ is not null and therefore this covariance grows continuously – as it must be in any prediction step.

6 Fusing IMU with complementary sensory data

At the arrival of other kind of information than IMU, such as GPS or vision, we proceed to correct the ESKF. In a well-designed system, this should render the IMU biases observable and allow the ESKF to correctly estimate them. There are a myriad of possibilities, the most popular ones being GPS + IMU, monocular vision + IMU, and stereo vision + IMU. In recent years, the combination of visual sensors with IMU has attracted a lot of attention, and thus generated a lot of scientific activity. These vision + IMU setups are very interesting for use in GPS-denied environments, and can be implemented on mobile devices (typically smart phones), but also on UAVs and other small, agile platforms.

While the IMU information has served so far to make predictions to the ESKF, this other information is used to correct the filter, and thus observe the IMU bias errors. The correction consists of three steps:

1. observation of the error-state via filter correction,
2. injection of the observed errors into the nominal state, and
3. reset of the error-state.

These steps are developed in the following sections.

6.1 Observation of the error state via filter correction

Suppose as usual that we have a sensor that delivers information that depends on the state, such as

$$\mathbf{y} = h(\mathbf{x}_t) + v, \quad (272)$$

where $h()$ is a general nonlinear function of the system state (the true state), and v is a white Gaussian noise with covariance \mathbf{V} ,

$$v \sim \mathcal{N}\{0, \mathbf{V}\}. \quad (273)$$