Linear Algebra 1: Vectors

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Much of the material in these slides comes from Géron's notes:
https://github.com/ageron/handson-ml/blob/master/math_linear_algebra.ipynb

Learning outcomes

After this lecture you should be able to:

1. Define ‘vector’
2. Define vector operations (addition, multiplication by scalar, norm, dot product, etc.) and perform them by hand
3. Perform vector operations with Python

Vectors

A scalar is a single number

A vector is a list/array of scalars.

In machine learning, vectors are used for observations (“feature vectors”) and also for predictions.

Example:

$$x = \begin{pmatrix} 1800 \\ 3 \\ 2.5 \end{pmatrix}$$

Variable x is a vector of length 3, and $x_1 = 1800$

It represents a point in 3-dimensional space.

2D vectors

Vectors with 2 dimensions can be visualized as points on a plane.

$$u = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

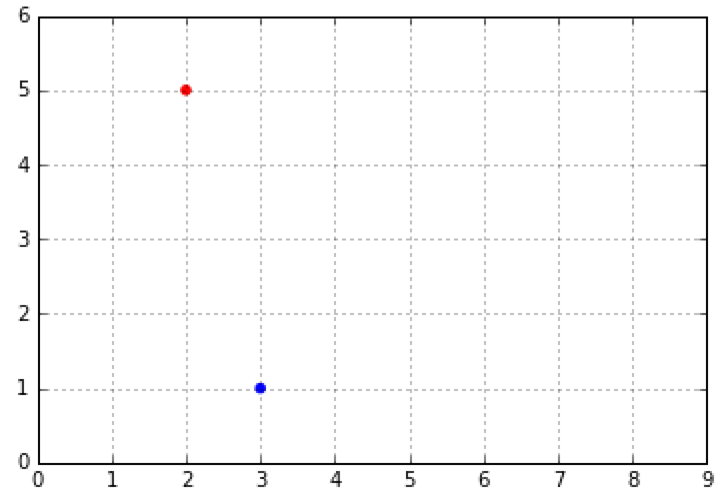
$$v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

These are column vectors.

There are also row vectors:

$$u = (6 \ \ 2.1 \ \ 0)$$

$$v = (7 \ \ 1.2 \ \ 4.9)$$

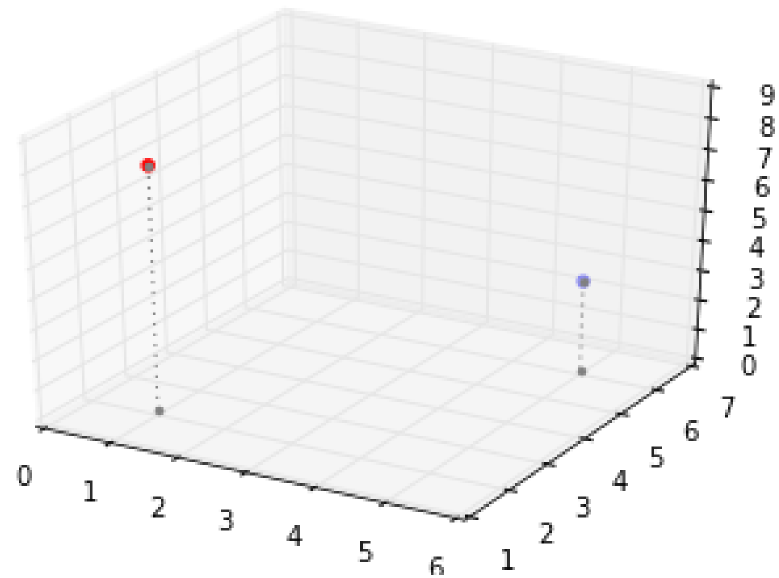


3D vectors

Vectors with 3 dimensions can be visualized as points in 3-dimensional space.

$$u = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$$

$$v = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix}$$



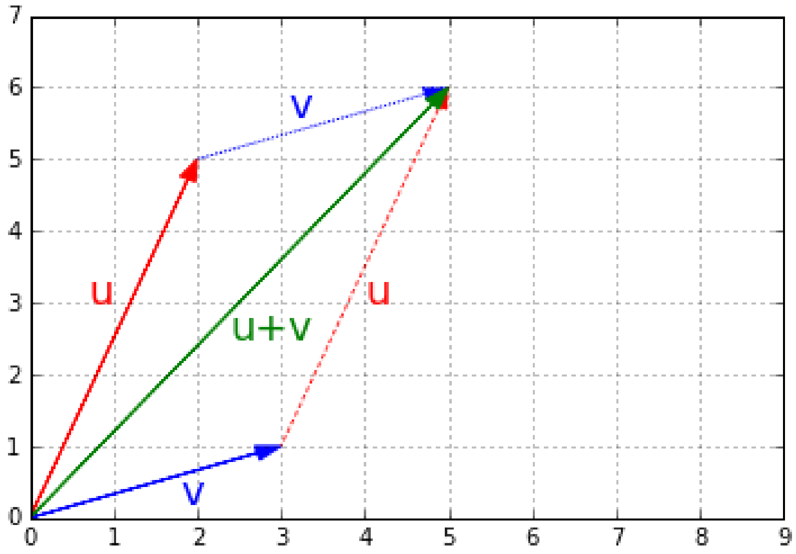
Vectors in Python

```python
# you can use a regular Python list
x = [1.2, -5.3]
# but we will use NumPy arrays
x = np.array([1.2, -5.3])
# getting the size of a vector with NumPy
x.size
# getting an element of a vector
x[0]
```

Vector addition

Easy – just "vectorized" addition

$$\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}5\\6\end{pmatrix}$$



Some properties of vector addition:

vector addition commutes:

$$u+v=v+u$$

vector addition is associative:

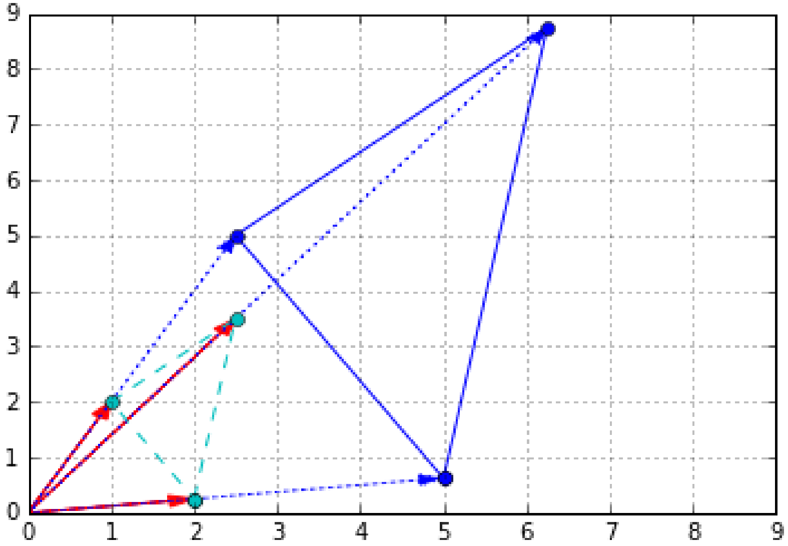
$$u+(v+w)=(v+u)+w$$

In Numpy, just add Numpy arrays using '+'.

Multiplication by a scalar

This is also very simple:

$$5\begin{pmatrix}-2\\5\end{pmatrix} = \begin{pmatrix}-10\\30\end{pmatrix}$$



Associative:

$$a(bu) = (ab)u$$

Distributes over vector addition:

$$k(u+v) = ku + kv$$

In Numpy, just use * operator on scalar and array.

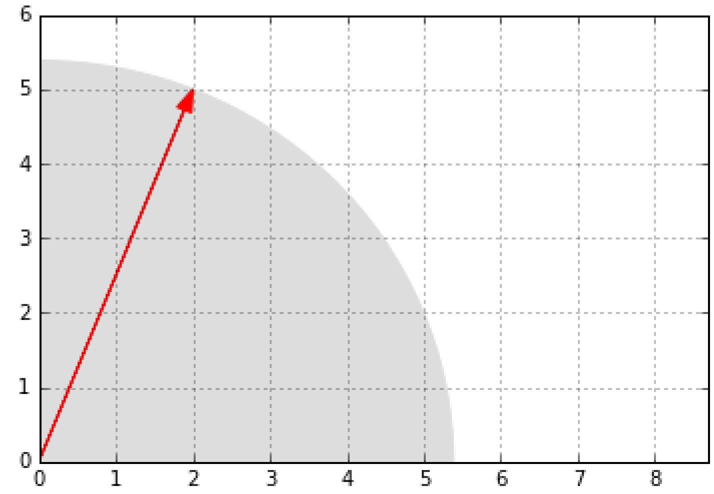
Norms

The norm of a vector u is a measure of the length of u. It's written $\|u\|$.

It's an operator that <u>takes a vector and gives a scalar</u>.

We'll use "Euclidean" norm -- also called L^2 norm.

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$



The norm of $\binom{2}{5}$ is about 5.4

$$\sqrt{2^2 + 5^2} \cong 5.38$$

```
import numpy.linalg as LA
u = np.array([2, 5])
LA.norm(u)
```

Zero, unit, and normalized vectors

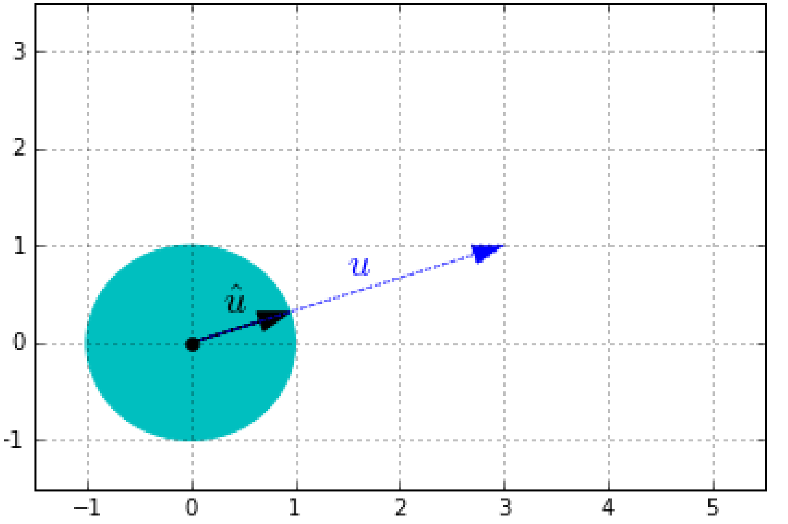
A zero vector is... a vector of zeros

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A unit vector is a vector of ones.

A normalized vector $\hat{u}$ is a unit vector in the same direction as u.

$$\hat{u} = \frac{u}{\|u\|}$$



normalized vector

Dot product of two vectors

Definition: (dot product also known as "inner product")

$$u \cdot v = \sum_i u_i \times v_i$$

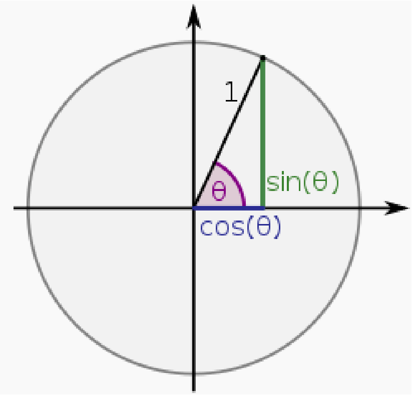
Example:

$$\binom{3}{1} \cdot \binom{2}{5} = 3(2) + 1(5) = 11$$

An alternative definition is used in physics:

$$u \cdot v = \|u\| \times \|v\| \times \cos(\theta)$$

where θ is the angle between u and v




```
# dot product with NumPy
np.dot(u,v)
# alternative
u.dot(v)
```

Commutative: $u \cdot v = v \cdot u$

Associative: ?

Associates with scalar multiplication:

$$k(u \cdot v) = ku \cdot kv$$

Distributes over vector addition:

$$u \cdot (v + w) = (u \cdot v) + u \cdot w)$$

Using dot product to compute angles

We know

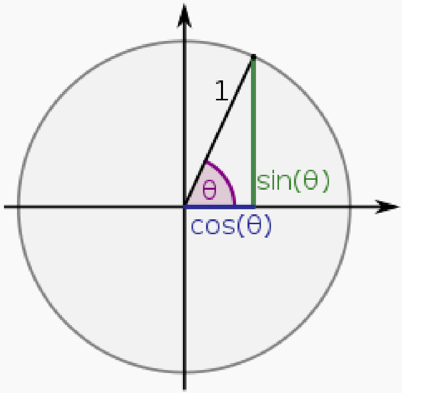
$$u \cdot v = \|u\| \times \|v\| \times \cos(\theta)$$

or equivalently

$$\cos(\theta) = \frac{u \cdot v}{\|u\| \times \|v\|}$$

or equivalently

$$\theta = \arccos\left(\frac{u \cdot v}{\|u\| \times \|v\|}\right)$$



If $u \cdot v$ is 0, then how are u and v related?

Example:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2 - 2 = 0$$

Summary

1. Vectors of length n represent points in n-dimensional space
2. Operations on vectors:
   - addition
   - multiplication by scalar
   - norm
   - dot product
3. Special vectors: zero, unit, normalized
4. Vectors and vector operations in NumPy