Econometrics

Introduction to Time Series

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Outline

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Presentation

- Time series: observations over time of a variable (unemployment rate, growth, ...) labeled Y_t
- Index: date (day, hour, year, quarter, ...)
- Y_t is called a *stochastic process*
- Data should be *stationary* (see below)
- Stationarity is needed because we will work with long series
- In past chapters, we worked with either cross sections or short periods (in the panel chapter) so it was not an issue
- Time series methods are very specific and few things refer to past chapters

Main source: course by A-C Disdier (University Paris 1)
Secondary source: *Basic Econometrics* by Gujarati

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Data

Source: Gujarati

- US time series 1947-2007, by quarter (1947-I to 2007-IV), seasonally adjusted, in billion year 2000 USD
- Data were collected on the website of the *Federal Reserve Bank* of St Louis
- Variables: `year`, `quarter`, `gdp`, `dpi` *(real disposable personal income)*, `pce` *(real personal consumption expenditures)*

Stationarity: Definition

- A process is **second order stationary** if its expected value and variance are constant over time
- And if the covariance between 2 periods depends only on the gap between these 2 periods and not on the date of observation

Formally:

- $\forall t,\ E(Y_t) = m$
- $\forall t,\ V(Y_t) = \sigma^2 < \infty$
- $\forall t, t+h,\ cov(Y_t, Y_{t+h}) = \gamma_h$

Basic Features

- A series is stationary if it results from a stationary process
- It should have no trend, no seasonality, or any element that changes with time
- If upward trend: not stationary because the expected value increases with time
- If seasonality: not stationary because strong correlation between Y_t and Y_{t-12}
- For instance, a white noise ε_t is stationary because by definition $E(\varepsilon_t) = 0$, $V(\varepsilon_t) = \sigma^2$ and $cov(u_t, u_{t+h}) = 0$

Spurious Regressions

- Assume we have a *random walk*: $y_t = y_{t-1} + u_t$ with u_t a white noise
- Regressing a random walk on another can give significant results even if they have nothing to do with each other
- Why: they are not *stationary* and so the usual OLS methods we are used to do not work
- Indeed, $E(y_t) = 0$ but $V(y_t) = \infty$
- However, their *first differences* are stationary and regressing one on another gives a non-significant result, as expected
- Indeed, $y_t - y_{t-1} = u_t$ is a white noise, which is stationary

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How to Check Stationarity

- Through the *autocorrelation functions* and *correlograms*
- If no autocorrelation: no memory, so stationary
- If autocorrelation of order 1: short memory, so can be fixed easily
- We will focus on easy examples in this introductory chapter

Simple Autocorrelation Function

Correlation between Y_t and Y_{t+h} (2 observations h periods apart):

$$\rho_h = \frac{cov(Y_t, Y_{t+h})}{\sigma_{Y_t}\sigma_{Y_{t+h}}} = \frac{\sum_{t=1}^{T-h}(Y_t - \bar{Y})(Y_{t+h} - \bar{Y})}{\sqrt{\sum_{t=1}^{T-h}(Y_t - \bar{Y})^2}\sqrt{\sum_{t=1}^{T-h}(Y_{t+h} - \bar{Y})^2}}$$

Where:

- T: number of observations
- $\bar{Y}$: mean of Y on the $T - h$ periods
- Caveat: mean and variance have to be recomputed for each h
- This function usually decreases quickly with h

Sample Autocorrelation Function

If T is sufficiently large, then the following function is a good approximation of the simple autocorrelation function:

$$\hat{\rho}_h = \frac{\sum_{t=1}^{T-h}(Y_t - \bar{Y})(Y_{t+h} - \bar{Y})}{\sum_{t=1}^{T}(Y_t - \bar{Y})^2}$$

Where T is the number of observations and $\bar{Y}$ is the mean of Y on the T periods.

- $\hat{\rho}_h \in [-1; 1]$
- The graph $\hat{\rho}_h$ as a function of h is a summary of the characteristics of the autocorrelation of the series and is called a *correlogram*
- For a white noise, this function should always be around zero
- Stata: `ac` and `corrgram`

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Partial Autocorrelation Function

- Partial correlation: correlation between 2 variables once correlation with other variables has been removed
- Example: partial correlation between x_1 and x_2, removing the effect of x_3:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

- Partial autocorrelation of lag h: correlation between Y_t and Y_{t-h}, while the influence of other intermediary lags $h + i$ $(i < h)$ has been removed
- Lags $h + i$ are of the type Y_{t-1}, Y_{t-2}, ..., Y_{t-h+1}; they are all the observations that took place between Y_t and Y_{t-h}

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Interpretation of Stata Output

- Graph of the autocorrelation function (ACF) of Lgdp up to *lag* 36

Interpretation of columns:

- AC: *autocorrelation function*
- PAC: *partial autocorrelation* (correlation between y_t and y_{t-h} after having removed intermediary y's)
- Q: statistic Q, also called the *Portmanteau statistic*, is used to test autocorrelation
- Prob>Q: the p-value of statistic Q
- Plots: graphs of AC and PAC

Statistic Q

Statistic Q (here, Ljung-Box version): under the null hypothesis H_0 (all the $\rho_h = 0$), with m *lags* tested, we get:

$$Q = N(N+2)\sum_{h=1}^{m} \frac{\rho_h^2}{N-h} \to \chi^2_m$$

- If Q is large (p-value small), we reject the null of no autocorrelation
- This indicates the series is not stationary

TS Process (Trend Stationary)

- *Trend stationary*: deterministic non-stationary process
- General form: $Y_t = f_t + u_t$ where f_t is a polynomial function of time and u_t a stationary process
- Polynomial of degree 1: $Y_t = b_0 + b_1 t + u_t$
- Property: the effect of a shock at time t is **transitory**
- The model is deterministic because the series always gets back to its long-term aspect, which is its trend line

DS Process (Difference Stationary) - Part 1

- *Difference stationary*: random non-stationary process (stochastic)
- Of order 1: Y_t function of Y_{t-1}
- Can be made stationary by using a *difference filter* (differentiate the series): $\Delta = (1 - L)^d$
- With L the lag operator and $d \in \mathbb{N}^*$ the differentiating parameter (also called the integration parameter)
- $(1 - L)^d Y_t = b + u_t$
- Example: with $d = 1$ if $Y_t = Y_{t-1} + b + u_t$ then $\Delta Y_t = Y_t - Y_{t-1} = b + u_t$
- That makes ΔY_t stationary

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DS Process (Difference Stationary) - Part 2

- Property: the effect of a shock at time t lasts **forever** on future values
- It is thus a permanent effect, however decreasing

Derivation:

- $Y_1 = Y_0 + b + u_1$
- $Y_2 = Y_1 + b + u_2 = Y_0 + b + u_1 + b + u_2 = Y_0 + 2b + u_1 + u_2$
- $Y_3 = Y_2 + b + u_3 = Y_0 + 2b + u_1 + u_2 + b + u_3 = Y_0 + 3b + u_1 + u_2 + u_3$
- And so on, so that for any t:

$$Y_t = Y_0 + t \cdot b + \sum_{i=1}^{t} u_i$$

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Unit Root Test (1)

- A variable has a *unit root* if its first difference is stationary
- For example, a random walk where $y_t = y_{t-1} + u_t$ has a unit root
- We need to test if a variable has a unit root: if yes, then we will use its first difference instead, and if not we will leave it as is
- Since variables are not *a priori* stationary, we cannot estimate the following equation: $y_t = \rho y_{t-1} + u_t$ and use a Student test to test if $\rho = 1$, because unfortunately all usual tests are wrong in case of non-stationarity
- The *Dickey-Fuller* (DF) test was specifically designed to test stationarity while avoiding this issue

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Unit Root Test (2)

- Assume that $y_t = \rho y_{t-1} + u_t$. We subtract y_{t-1} on both sides, and we get:
- $y_t - y_{t-1} = (\rho - 1)y_{t-1} + u_t$. We call $\delta = (\rho - 1)$, so:
- $\Delta y_t = \delta y_{t-1} + u_t$
- Now, thanks to the first difference on the left handside, we can estimate this equation and test the null hypothesis $H_0 : \delta = 0$
- Dickey & Fuller showed that the Student statistic t of δ follows a *Tau* distribution, and not the usual Student T distribution
- The quantiles of the Tau distribution were computed by the authors, who published the statistical tables
- The DF test can be run in 3 different ways: we need to try all of them because we don't know which one is the good one (the Tau quantiles are different according to the form tested)

The 3 Forms of the Test

- (1) *Random walk*: $\Delta y_t = \delta y_{t-1} + u_t$
- (2) *Random walk* with *drift*: $\Delta y_t = \beta_1 + \delta y_{t-1} + u_t$
- (3) *Random walk* with *drift* and deterministic *trend*: $\Delta y_t = \beta_1 + \beta_1 t + \delta y_{t-1} + u_t$

Interpretation:

- If the null $(H_0 : \delta = 0)$ is rejected, it means that y_t is **stationary**
- If the null is not rejected, it means that y_t is non-stationary *or* that it has a stochastic *trend* $\Rightarrow$ we must use its first difference
- Stata: `dfuller`
- Caveat: this procedure assumes that the error term is a white noise, which is not true in case of autocorrelation $\Rightarrow$ use instead the *Augmented Dickey Fuller* (ADF) test in that case

The Augmented Dickey Fuller (ADF) Test

- The classical DF test assumes that the u_t's are uncorrelated
- If they are correlated, we need to take this into account and "augment" the estimated model using the lagged values of Δy_t

The three cases become:

- Case (1): $\Delta y_t = \delta y_{t-1} + \sum_{i=1}^{m} \alpha_i \Delta y_{t-i} + u_t$
- Case (2): $\Delta y_t = \beta_0 + \delta y_{t-1} + \sum_{i=1}^{m} \alpha_i \Delta y_{t-i} + u_t$
- Case (3): $\Delta y_t = \beta_0 + \beta_1 t + \delta y_{t-1} + \sum_{i=1}^{m} \alpha_i \Delta y_{t-i} + u_t$
- Where $\Delta y_{t-1} = y_{t-1} - y_{t-2}$ etc.
- In our case, since we have quarterly values, we can choose 4 *lags* and so $m = 4$

Transforming Non-Stationary Series (1)

- We know it is impossible to use non-stationary series in regular regressions, since we are at risk of ending with a spurious regression
- If a series has a unit root, then its first difference ($\Delta y_t = y_t - y_{t-1}$) is stationary and we can use it in a regression
- We need to check that Δy_t is really stationary
- If using the first difference solves the issue, then the variable is a *difference-stationary process* (DSP)

Transforming Non-Stationary Series (2)

- But the variable could also be a *trend-stationary process* (TSP), i.e. a stationary process around the *trend line*
- To make the series stationary, we need to regress it on time, and the residual of this regression should be stationary
- That is: estimate $y_t = a + b \cdot t + u_t$, and compute $\hat{u}_t = y_t - \hat{a} - \hat{b} \cdot t$
- Value $\hat{u}_t$ should be a *detrended* series
- Remark: the *trend* can be non-linear: we may have $y_t = a + b \cdot t + c \cdot t^2 + u_t$

Integration

- A non-stationary series is said to be *integrated*
- If y_t is non-stationary but if $\Delta y_t = y_t - y_{t-1}$ is stationary, y_t is integrated of order 1 because we need to differentiate it 1 time to make it stationary
- If we need to differentiate a series d times to make it stationary, then it is integrated of order d
- Notation: $y_t \sim I(d)$, with d the *integration parameter*
- A series $I(0)$ is stationary

To make a series stationary, we often differentiate it or its log.

Definition

- If 2 series are non-stationary but a linear combination of them is stationary, then they are said to be *cointegrated*
- Assume the following model: $Lpce_t = b_1 + b_2 Ldpi_t + u_t \quad (1)$
- u_t is a linear combination of $Lpce_t$ and $Ldpi_t$: $u_t = Lpce_t - b_1 - b_2 Ldpi_t$
- If u_t is stationary, then these 2 variables are cointegrated and even if they are individually non-stationary, they can still be used in a regression as is, without having to differentiate them
- This can be generalized to any number of regressors
- Remark: an economic interpretation would be that 2 variables are cointegrated if they have a **long-term equilibrium relationship**

Testing for Cointegration: The Engel-Granger Test

- Assume the following model: $Lpce_t = b_1 + b_2 Ldpi_t + u_t \quad (1)$
- To test if *Lpce* and *Ldpi* are cointegrated, we just need to test the residuals of regression (1)
- We could use the Dickey-Fuller test
- But since the residuals ($\hat{u}$) are computed with the estimated parameter ($\hat{b}_2$) and not the true one, the critical values are a bit different
- So we can use the `dfuller` command and use the critical values from MacKinnon
- Cointegrated variables can be used in so-called *Error-correction models* (ECM)

- Now we know that *Lpce* and *Ldpi* are cointegrated: we could use them "as is" in a regression:
  $Lpce_t = b_1 + b_2 Ldpi_t + b_3 t + u_t$ (we add the time trend since we just showed that this made the residual stationary)
- What's more interesting is that the fact that they are cointegrated indicates that there is a **long-term relationship** between the two
- u_t can be thought of as the equilibrium error
- The *Granger representation theorem* states that if 2 variables are cointegrated, the relationship between the two can be expressed as an Error Correction Mechanism
- Such an ECM is: $\Delta Lpce_t = a_0 + a_1 \Delta Ldpi_t + a_2 u_{t-1} + \varepsilon_t$

Error-Correction Models (ECM) - Part 2

- The original model is: $Lpce_t = b_1 + b_2 Ldpi_t + b_3 t + u_t \quad (1)$
- Consider the following corresponding ECM:
  $\Delta Lpce_t = a_0 + a_1 \Delta Ldpi_t + a_2 u_{t-1} + \varepsilon_t \quad (2)$
- Also written: $\Delta Lpce_t = a_0 + a_1 \Delta Ldpi_t + a_2 [Lpce_{t-1} - b_1 - b_2 Ldpi_{t-1} - b_3(t-1)] + \varepsilon_t$
- a_2 is expected to be **negative**: if $u_{t-1} > 0$, i.e. if $Lpce_{t-1} > b_1 - b_2 Ldpi_{t-1} - b_3(t-1)$, then $Lpce_{t-1}$ is too high at period $t-1$, so that in the next period $Lpce_t$ should be lower to restore the equilibrium
- If a_2 is significant, it means that *pce* adjusts to *dpi* with a lag
- $|\hat{a}_2|$ gives the proportion of the discrepancy between long-term and short-term *pce* that is corrected within a quarter (since the time unit is a quarter)
- $\hat{a}_1$ from (2) is the *short-term* consumption elasticity
- $\hat{b}_2$ from (1) is the *long-term* consumption elasticity

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