14.4 Monte Carlo simulation of 2-d U(1) lattice gauge theory

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We call $U_\mu(x)$ to a U(1) variable at point x in the μ direction. For U(1), the lattice gauge action reads

$$S[U] = \beta \sum_x \sum_{\mu<\nu} \text{Re } [1 - U_{\mu\nu}(x)], \tag{1}$$

where $\beta = \frac{2}{e^2}$ and e being the gauge coupling. The plaquettes are defined by

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^*(x+\hat{\nu}) U_\nu^*(x) \tag{2}$$

The sum of plaquette variables is computed using

$$S_{\text{P}} = \sum_x \sum_{\mu<\nu} \text{Re } U_{\mu\nu}(x), \tag{3}$$

so the expectation value of an elementary Wilson loop or plaquette reads

$$E_{\text{P}} = \frac{\langle S_{\text{P}} \rangle}{DV}, \tag{4}$$

where $V = L^d$ is the volume of the lattice and $D = \frac{d(d-1)}{2}$ is the number of planes of rotation in d dimensions.

We define the sum of staples as

$$\Sigma_\mu(x) = \sum_{\nu \neq \mu} [U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu^*(x+\hat{\mu}) + U_\nu^*(x-\hat{\nu}) U_\mu(x-\hat{\nu}) U_\nu(x+\hat{\mu}-\hat{\nu})]. \tag{5}$$

The change in the action by a local update when changing $U_\mu(x) \to U'_\mu(x)$ is

$$\Delta S = -\beta \text{ Re } \left[\left(U'_\mu(x) - U_\mu(x) \right) \Sigma_\mu^* \right]. \tag{6}$$

We use the Metropolis algorithm for the simulation as decribed in the following steps:

1. Given some gauge field configuration, we go through all the lattice points in a lexicographic way. At each point x in the μ direction we generate a random U(1) variable $U'_\mu(x)$.

2. We compute the sum of the staples and compute the change of the action according to eq. (6). We generate a uniform random number $r \in [0, 1)$ and accept the change if $r \leq p$, where p is

$$p = \min(1, \exp(-\Delta S)). \tag{7}$$

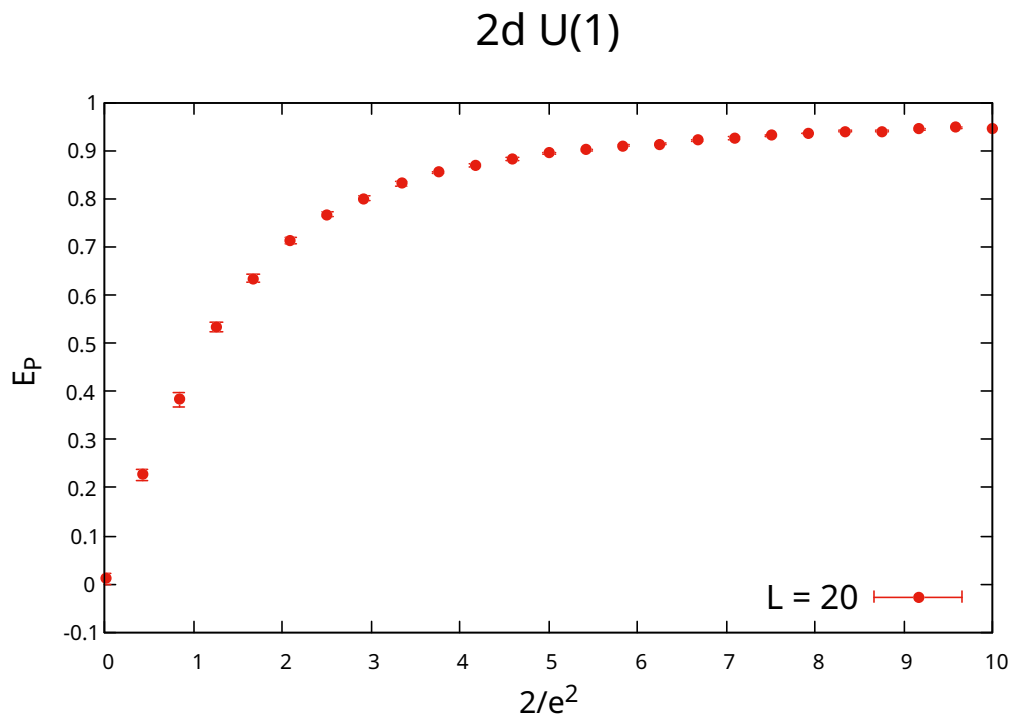

Figure 1: Expectation value of the plaquette variable vs. $\beta = 2/e^2$ with a volume $V = 20^2$. The errors were computed using the Jackknife error. The parameters of the simulation were: at each value of β, 5000 measurements were made separated by 20 sweeps and a thermalization of 1000 sweeps.