Pure SU(3) lattice gauge theory in equilibrium

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1 Introduction

We study the 4-dimensional SU(3) lattice gauge theory with the compact formulation. In this formulation we consider compact link variables $U_\mu(x) \in \mathrm{SU}(3)$ rather than the Lie algebra-valued fields $A_\mu(x)$. The compact variables $U_\mu(x)$ become the fundamental fields to be integrated over the functional integral. One great advantage of the compact formulation is that we no longer require to fix the gauge.

Link variables are oriented, so they naturally live in the links between neighboring points on the lattice, with spacing a. Thus, the variable $U_\mu(x)$ is defined in the link between points x and $x + a\hat{\mu}$ in the positive μ direction, see Fig. 1a. We define the variable between points x and $x + a\hat{\mu}$ pointing in the negative direction via

$$U_{-\mu}(x + a\hat{\mu}) \equiv U_\mu^\dagger(x), \tag{1}$$

see Fig. 1b.

Under a gauge transformation the link variables transform as

$$U'_\mu(x) = \Omega(x) U_\mu(x) \Omega^\dagger(x + a\hat{\mu}), \tag{2}$$

where $\Omega(x) \in \mathrm{SU}(3)$. We refer to [1] for a complete discussion on lattice gauge theory.

(a) Link variable at point x in the μ direction. a is the lattice spacing.

(b) Link variable at point $x + a\hat{\mu}$ in the $-\mu$ direction.

For SU(N), the lattice gauge action reads

$$S[U] = \frac{\beta}{N} \sum_x \sum_{\mu<\nu} \text{Re Tr}\left[\mathbb{1} - U_{\mu\nu}(x)\right], \tag{3}$$

where $\beta = \frac{2N}{g^2}$. The plaquettes are defined by

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) \tag{4}$$

Sum of plaquette variables

$$S_{\text{P}} = \frac{1}{N} \sum_x \sum_{\mu<\nu} \text{Re Tr}[U_{\mu\nu}(x)] \tag{5}$$

we define

$$E_{\text{P}} = \frac{\langle S_{\text{P}} \rangle}{DV} \tag{6}$$

where $V = L^d$ is the volume of the lattice and $D = \frac{d(d-1)}{2}$ is the number of planes of rotation.

Staples

$$\Sigma_\mu(x) = \sum_{\nu \neq \mu} \left[U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu^\dagger(x+\hat{\mu}) + U_\nu^\dagger(x-\hat{\nu}) U_\mu(x-\hat{\nu}) U_\nu(x+\hat{\mu}-\hat{\nu}) \right] \tag{7}$$

The change in the action by a local update when changing $U_\mu(x) \to U'_\mu(x)$ is

$$\Delta S = -\frac{\beta}{N} \text{Re Tr}\left[\left(U'_\mu(x) - U_\mu(x) \right) \Sigma_\mu^\dagger \right] \tag{8}$$

In SU(3) the SU(2) elements are of the form

$$R = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; S = \begin{pmatrix} s_{11} & 0 & s_{12} \\ 0 & 1 & 0 \\ s_{21} & 0 & s_{22} \end{pmatrix}, \; T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & r_{21} & r_{22} \end{pmatrix}. \tag{9}$$

1.1 Polyakov Loop

The polyakov loop is defined as the product of the link variables in the Euclidean time direction over a closed loop

$$P(\vec{x}\,) = \text{Tr} \prod_{t_{\text{E}}=1}^{L_t} U_4(\vec{x}, t_{\text{E}}). \tag{10}$$

One important quantity involving Polyakov loops is their correlation. The correlation function of Polyakov loops is related to the static quark potential $V(r)$ as

$$\langle P(\vec{x})P(\vec{y})^\dagger\rangle \propto e^{-L_t a V(r)}\left(1+\mathcal{O}\left(e^{-L_t a \Delta E}\right)\right), \tag{11}$$

where $r = |\vec{x}-\vec{y}\,|$. Up to a constant the static quark potential is

$$aV(r) = -\log(\langle P(\vec{x})P(\vec{y})^\dagger\rangle)/L_t. \tag{12}$$

We use the Cornell potential to parametrize $V(r)$

$$V(r) = A + \frac{B}{r} + \sigma r, \tag{13}$$

here A is an irrelevant additive constant, B is the Coulomb part of the potential and σ is the so called *string tension*.

2 Setting physical scales

One method of defining a physical scale is related to the static potential. First the Sommer parameter is defined as $r_0 \approx 0.5$ fm, is a physical length where, roguhly speaking, the potential becomes linear.

First we compute the force

$$-F(r) = \frac{dV(r)}{dr} = \sigma - \frac{B}{r^2}. \tag{14}$$

Using experimental data for heavy quark-antiquark pairs ($\bar{b}b$ and $\bar{c}c$) it is found that

$$-F(r_0)r_0^2 = 1.65, \quad \text{where} \quad r_0 \simeq 0.5 \text{ fm}. \tag{15}$$

Using the form of the force in eq. (14)

$$-F(r)r_0^2 = \sigma r_0^2 - B = 1.65, \tag{16}$$

which implies

$$r_0 = \sqrt{\frac{1.65+B}{\sigma}}, \tag{17}$$

or in lattice units

$$\frac{r_0}{a} = \sqrt{\frac{1.65+B}{a^2\sigma}}. \tag{18}$$

The parameters B and $a^2\sigma$ can be obtained by a fit to the function

$$aV(an) = aA + \frac{B}{n} + a^2\sigma n, \tag{19}$$

where $r = an$. Once obtained the dimensionless quantity $X = r_0/a$, the lattice spacing in physical units is $a = 0.5/X$ fm.

3 Algorithms

3.1 Metropolis

1. Given some gauge field configuration, we go through all the lattice points in a lexicographic way. At each point x in the μ direction we generate a random SU(3) matrix $U'_\mu(x)$.

2. We compute the sum of the staples and compute the change of the action according to eq. (8). We generate a uniform random number $r \in [0, 1)$ and accept the change if $r \leq p$, where p is

$$p = \min(1, \exp(-\Delta S)). \tag{20}$$

3.2 Heatbath

The heatbath algorithm was first implemented by Creutz for the SU(2) theory [5] and generalized to SU(N) theories by Cabbibo and Marinari [6]. The idea for implementing the algorithm in SU(N) is to iterate over a set of SU(2) subgroups in SU(N) and apply the heatbath algorithm to these SU(2) elements.

1. Given some field configuration at a point x and direction μ, we find the sum of staples $\Sigma_\mu(x)$ in eq. (7).

2. We compute $W = U_\mu(x)\Sigma^\dagger_\mu(x)$ and define W_2 as the 2×2 submatrix of W that has the same block structure as R (or S or T).

3. W_2 is a complex 2×2 matrix which is generally not an element of SU(2). As described in Ref. [7], we generate a matrix proportional to an SU(2) element from W_2 as follows

$$V = \frac{1}{2}\left[W_2 - W_2^\dagger + \mathbb{1}\mathrm{Tr}\ W_2^\dagger\right] \tag{21}$$

4. We compute the determinant of V and take $\xi = \sqrt{\det V}$

5. We follow the original heatbath implementation of Creutz. We generate a uniform random number $r \in [\exp(-\frac{4}{3}\beta\xi), 1]$ and compute

$$x_0 = 1 + \frac{\log r}{\frac{2}{3}\beta\xi}. \tag{22}$$

   Now we generate a uniform random number $u \in [0, 1)$ and accept x_0 if $u > 1 - \sqrt{1 - x_0^2}$. Repeat this step until a x_0 is accepted.

6. We now generate a vector $\vec{x} = (x_1, x_2, x_3)$ in the unit 2-sphere uniformly distributed. This may be acomplished as follows. We generate three random numbers $\vec{r} = (r_1, r_2, r_3)$ in the interval $[-1, 1]$ and accept them if they are inside the unit 3-sphere. Otherwise we generate other three numbers until they are accepted. Once they are accepted we take

$$\vec{x} = \sqrt{1 - x_0^2} \frac{\vec{r}}{|\vec{r}|}. \tag{23}$$

7. We have generated the elements of a SU(2) matrix X

$$X = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \tag{24}$$

8. We take

$$R_2 = X \frac{V^\dagger}{\xi}. \tag{25}$$

   And transform this 2×2 matrix to the same block structure as the 3×3 matrix R (or S or T).

9. We go to step 2 and repeat the process to generate S (or T) but this time taking $W = RU_\mu(x)\Sigma_\mu(x)$ (or $W = SRU_\mu(x)\Sigma_\mu(x)$).

10. The new link element $U'_\mu(x)$ is

$$U'_\mu(x) = TSRU_\mu(x). \tag{26}$$

4 Results

We present simulation results for the 4-dimensional SU(3) lattice gague field theory. We implement two different local update algorithms, namely, Metropolis and Heatbath. Figure 2 shows the energy density E_{P} as a function of $\beta = \frac{6}{g^2}$. There is a maximum of the slope at $5 < \beta < 6$, however this is not an indication of a phase transition.

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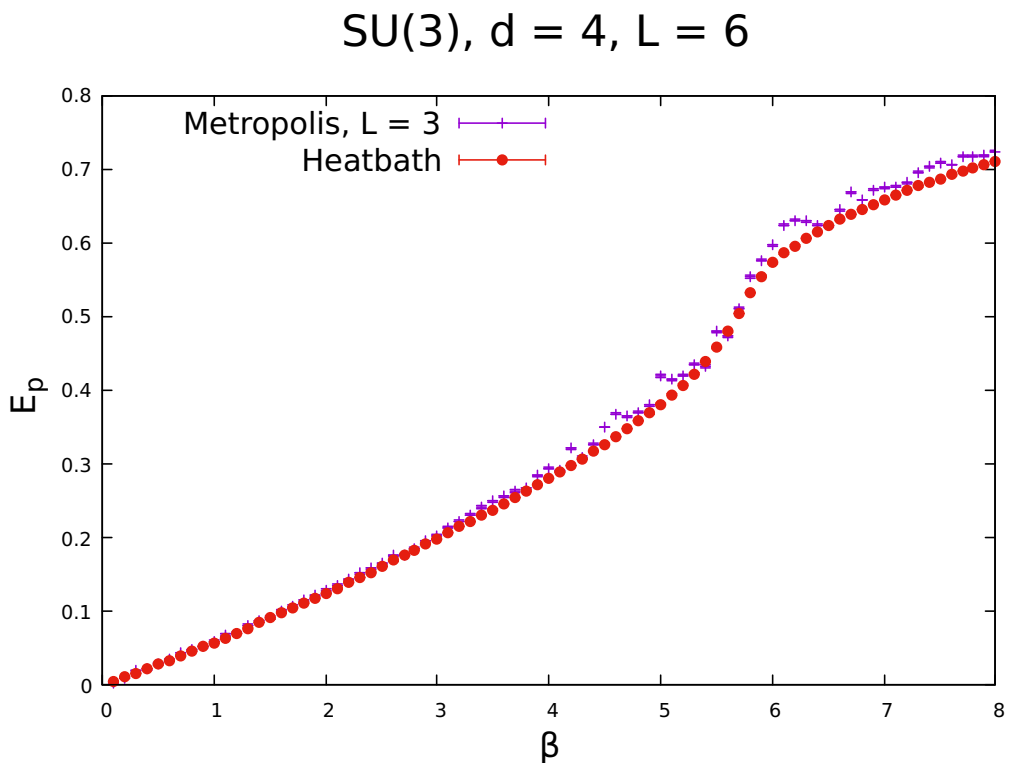

Figure 2: Energy density vs. $\beta = \frac{6}{g^2}$. We simulated the SU(3) pure gauge theory in four dimensions using the Metropolis and Heatbath algorithms.

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