

# Pure SU(3) lattice gauge theory in equilibrium

José Antonio García Hernández

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## 1 Introduction

We study the 4-dimensional SU(3) lattice gauge theory with the compact formulation. In this formulation we consider compact link variables  $U_\mu(x) \in \text{SU}(3)$  rather than the Lie algebra-valued fields  $A_\mu(x)$ . The compact variables  $U_\mu(x)$  become the fundamental fields to be integrated over the functional integral. One great advantage of the compact formulation is that we no longer require to fix the gauge.

Link variables are oriented, so they naturally live in the links between neighboring points on the lattice, with spacing  $a$ . Thus, the variable  $U_\mu(x)$  is defined in the link between points  $x$  and  $x + a\hat{\mu}$  in the positive  $\mu$  direction, see Fig. 1a. We define the variable between points  $x$  and  $x + a\hat{\mu}$  pointing in the negative direction via

$$U_{-\mu}(x + a\hat{\mu}) \equiv U_\mu^\dagger(x), \quad (1)$$

see Fig. 1b.

Under a gauge transformation the link variables transform as

$$U'_\mu(x) = \Omega(x)U_\mu(x)\Omega^\dagger(x + a\hat{\mu}), \quad (2)$$



(a) Link variable at point  $x$  in the  $\mu$  direction.  $a$  is the lattice spacing. (b) Link variable at point  $x + a\hat{\mu}$  in the  $-\mu$  direction.

Figure 1: Graphical representation of link variables.

where  $\Omega(x) \in \text{SU}(3)$ . We refer to [1] for a complete discussion on lattice gauge theory.

For  $\text{SU}(N)$ , the lattice gauge action reads

$$S[U] = \frac{\beta}{N} \sum_x \sum_{\mu < \nu} \text{Re Tr} [\mathbb{1} - U_{\mu\nu}(x)], \quad (3)$$

where  $\beta = \frac{2N}{g^2}$ . The plaquettes are defined by

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\mu} + \hat{\nu})U_\nu^\dagger(x) \quad (4)$$

Sum of plaquette variables

$$S_P = \frac{1}{N} \sum_x \sum_{\mu < \nu} \text{Re Tr}[U_{\mu\nu}(x)] \quad (5)$$

we define

$$E_P = \frac{\langle S_P \rangle}{DV} \quad (6)$$

where  $V = L^d$  is the volume of the lattice and  $D = \frac{d(d-1)}{2}$  is the number of planes of rotation.

Staples

$$\Sigma_\mu(x) = \sum_{\nu \neq \mu} [U_\nu(x)U_\mu(x + \hat{\nu})U_\nu^\dagger(x + \hat{\mu}) + U_\nu^\dagger(x - \hat{\nu})U_\mu(x - \hat{\nu})U_\nu(x + \hat{\mu} - \hat{\nu})] \quad (7)$$

The change in the action by a local update when changing  $U_\mu(x) \rightarrow U'_\mu(x)$  is

$$\Delta S = -\frac{\beta}{N} \text{Re Tr} [(U'_\mu(x) - U_\mu(x)) \Sigma_\mu^\dagger] \quad (8)$$

In  $\text{SU}(3)$  the  $\text{SU}(2)$  elements are of the form

$$R = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & 0 & s_{12} \\ 0 & 1 & 0 \\ s_{21} & 0 & s_{22} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & r_{21} & r_{22} \end{pmatrix}. \quad (9)$$

## 1.1 Polyakov Loop

The polyakov loop is defined as the product of the link variables in the Euclidean time direction over a closed loop

$$P(\vec{x}) = \text{Tr} \prod_{t_E=1}^{L_t} U_4(\vec{x}, t_E). \quad (10)$$

## 1.2 Static quark potential

One important quantity involving Polyakov loops is their correlation. The correlation function of Polyakov loops is related to the static quark potential  $V(r)$  as

$$\langle P(\vec{x})P(\vec{y})^\dagger \rangle \propto e^{-L_t a V(r)} (1 + \mathcal{O}(e^{-L_t a \Delta E})) , \quad (11)$$

where  $r = |\vec{x} - \vec{y}|$ . Numerically speaking, we compute

$$\langle P(\vec{x})P(\vec{y})^\dagger \rangle = \frac{1}{3V} \sum_{x,\mu=1,2,3} P(\vec{x})P(\vec{x} + r\hat{\mu})^\dagger . \quad (12)$$

Up to a constant the static quark potential is

$$aV(r) = -\log(\langle P(\vec{x})P(\vec{y})^\dagger \rangle)/L_t. \quad (13)$$

We use the Cornell potential to parametrize  $V(r)$

$$V(r) = A + \frac{B}{r} + \sigma r, \quad (14)$$

here  $A$  is an irrelevant additive constant,  $B$  is the Coulomb part of the potential and  $\sigma$  is the so called *string tension*.

## 2 Setting physical scales

One method of defining a physical scale is related to the static potential. First the Sommer parameter is defined as  $r_0 \approx 0.5$  fm, is a physical length where, roughly speaking, the potential becomes linear.

First we compute the force

$$-F(r) = \frac{dV(r)}{dr} = \sigma - \frac{B}{r^2}. \quad (15)$$

Using experimental data for heavy quark-antiquark pairs ( $\bar{b}b$  and  $\bar{c}c$ ) it is found that

$$-F(r_0)r_0^2 = 1.65, \quad \text{where } r_0 \simeq 0.5 \text{ fm}. \quad (16)$$

Using the form of the force in eq. (15)

$$-F(r)r_0^2 = \sigma r_0^2 - B = 1.65, \quad (17)$$

which implies

$$r_0 = \sqrt{\frac{1.65 + B}{\sigma}}, \quad (18)$$

or in lattice units

$$\frac{r_0}{a} = \sqrt{\frac{1.65 + B}{a^2 \sigma}}. \quad (19)$$

The parameters  $B$  and  $a^2 \sigma$  can be obtained by a fit to the function

$$aV(an) = aA + \frac{B}{n} + a^2 \sigma n, \quad (20)$$

where  $r = an$  and  $n \in \mathbb{N}$ . Once obtained the dimensionless quantity  $X = r_0/a$ , the lattice spacing in physical units is  $a = 0.5/X$  fm.

The temperature  $T$  can be extracted by

$$\frac{1}{T} = aL_t. \quad (21)$$

## 3 Algorithms

### 3.1 Metropolis

1. Given some gauge field configuration, we go through all the lattice points in a lexicographic way. At each point  $x$  in the  $\mu$  direction we generate a random  $SU(3)$  matrix  $U'_\mu(x)$ .
2. We compute the sum of the staples and compute the change of the action according to eq. (8). We generate a uniform random number  $r \in [0, 1)$  and accept the change if  $r \leq p$ , where  $p$  is

$$p = \min(1, \exp(-\Delta S)). \quad (22)$$

### 3.2 Heatbath

The heatbath algorithm was first implemented by Creutz for the  $SU(2)$  theory [5] and generalized to  $SU(N)$  theories by Cabbibo and Marinari [6]. The idea for implementing the algorithm in  $SU(N)$  is to iterate over a set of  $SU(2)$  subgroups in  $SU(N)$  and apply the heatbath algorithm to these  $SU(2)$  elements.

1. Given some field configuration at a point  $x$  and direction  $\mu$ , we find the sum of staples  $\Sigma_\mu(x)$  in eq. (7).
2. We compute  $W = U_\mu(x)\Sigma_\mu^\dagger(x)$  and define  $W_2$  as the  $2 \times 2$  submatrix of  $W$  that has the same block structure as  $R$  (or  $S$  or  $T$ ).

3.  $W_2$  is a complex  $2 \times 2$  matrix which is generally not an element of  $SU(2)$ . As described in Ref. [7], we generate a matrix proportional to an  $SU(2)$  element from  $W_2$  as follows

$$V = \frac{1}{2} \left[ W_2 - W_2^\dagger + \mathbf{1} \text{Tr } W_2^\dagger \right] \quad (23)$$

4. We compute the determinant of  $V$  and take  $\xi = \sqrt{\det V}$
5. We follow the original heatbath implementation of Creutz. We generate a uniform random number  $r \in [\exp(-\frac{4}{3}\beta\xi), 1]$  and compute

$$x_0 = 1 + \frac{\log r}{\frac{2}{3}\beta\xi}. \quad (24)$$

Now we generate a uniform random number  $u \in [0, 1)$  and accept  $x_0$  if  $u > 1 - \sqrt{1 - x_0^2}$ . Repeat this step until a  $x_0$  is accepted.

6. We now generate a vector  $\vec{x} = (x_1, x_2, x_3)$  in the unit 2-sphere uniformly distributed. This may be accomplished as follows. We generate three random numbers  $\vec{r} = (r_1, r_2, r_3)$  in the interval  $[-1, 1]$  and accept them if they are inside the unit 3-sphere. Otherwise we generate other three numbers until they are accepted. Once they are accepted we take

$$\vec{x} = \sqrt{1 - x_0^2} \frac{\vec{r}}{|\vec{r}|}. \quad (25)$$

7. We have generated the elements of a  $SU(2)$  matrix  $X$

$$X = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \quad (26)$$

8. We take

$$R_2 = X \frac{V^\dagger}{\xi}. \quad (27)$$

And transform this  $2 \times 2$  matrix to the same block structure as the  $3 \times 3$  matrix  $R$  (or  $S$  or  $T$ ).

9. We go to step 2 and repeat the process to generate  $S$  (or  $T$ ) but this time taking  $W = RU_\mu(x)\Sigma_\mu(x)$  (or  $W = SRU_\mu(x)\Sigma_\mu(x)$ ).
10. The new link element  $U'_\mu(x)$  is

$$U'_\mu(x) = T S R U_\mu(x). \quad (28)$$

## SU(3), d = 4, L = 6

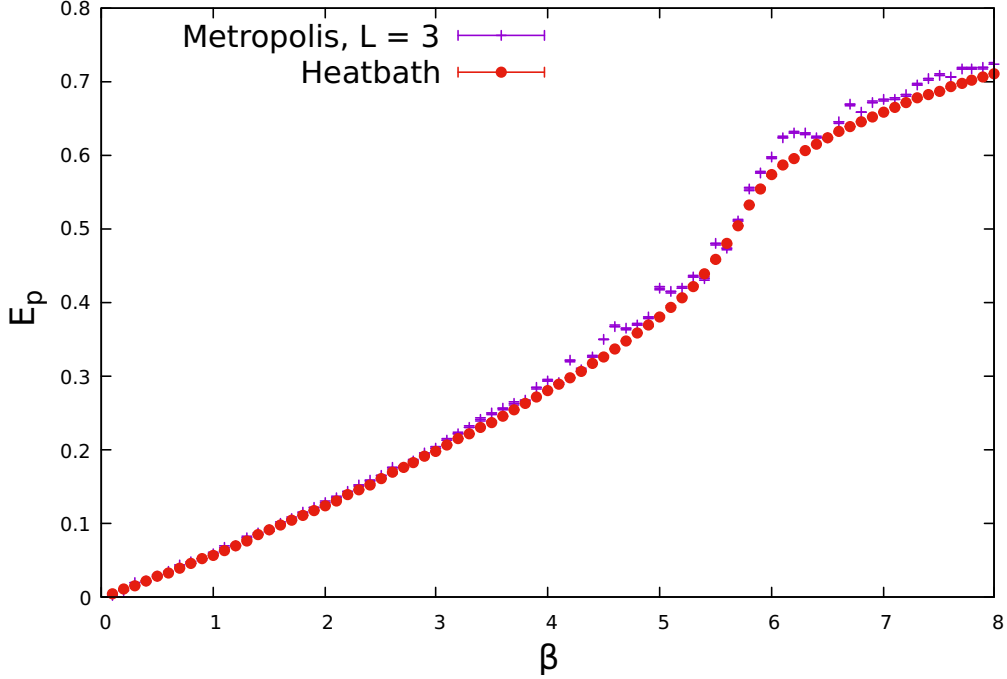


Figure 2: Energy density vs.  $\beta = \frac{6}{g^2}$ . We simulated the SU(3) pure gauge theory in four dimensions using the Metropolis and Heatbath algorithms.

## 4 Results

We present simulation results for the 4-dimensional SU(3) lattice gauge field theory. We implement two different local update algorithms, namely, Metropolis and Heatbath. Figure 2 shows the energy density  $E_P$  as a function of  $\beta = \frac{6}{g^2}$ . There is a maximum of the slope at  $5 < \beta < 6$ , however this is not an indication of a phase transition.

In figures 3 and 4 it is shown the expectation value of the Polyako loop averaged over the volume for two values of  $\beta$ .

$$V = 8^4, \beta = 6.063$$

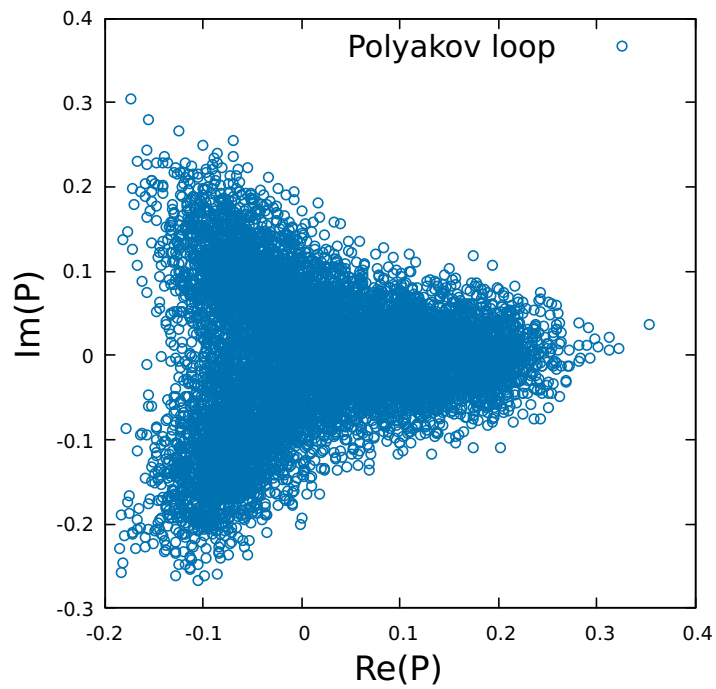


Figure 3: Scatter plot of the expectation value of the Polyakov loop at  $\beta = 6.063$  and volume  $V = 8^4$ .

$$V = 8^4, \beta = 5.7$$

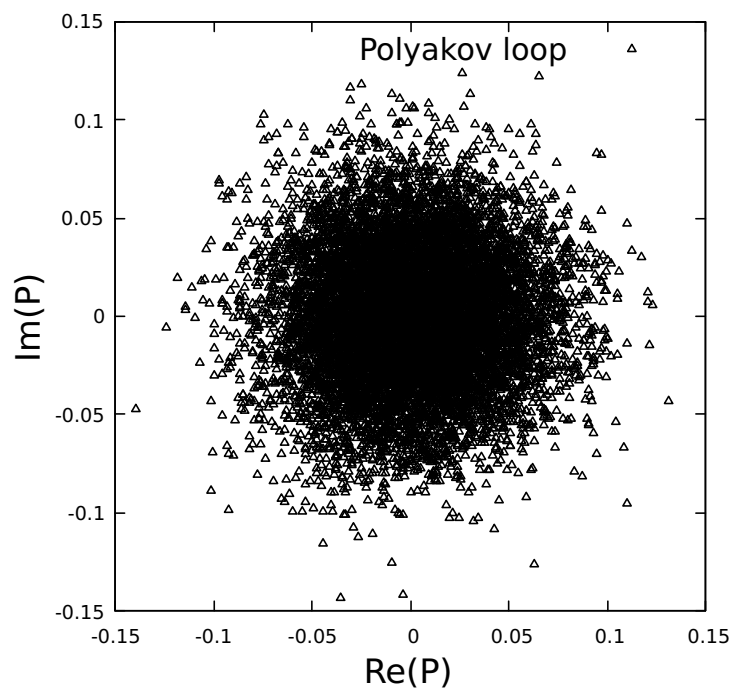


Figure 4: Scatter plot of the expectation value of the Polyakov loop at  $\beta = 5.7$  and volume  $V = 8^4$ .



$$V = 16^3 * 6, \beta = 5.7$$

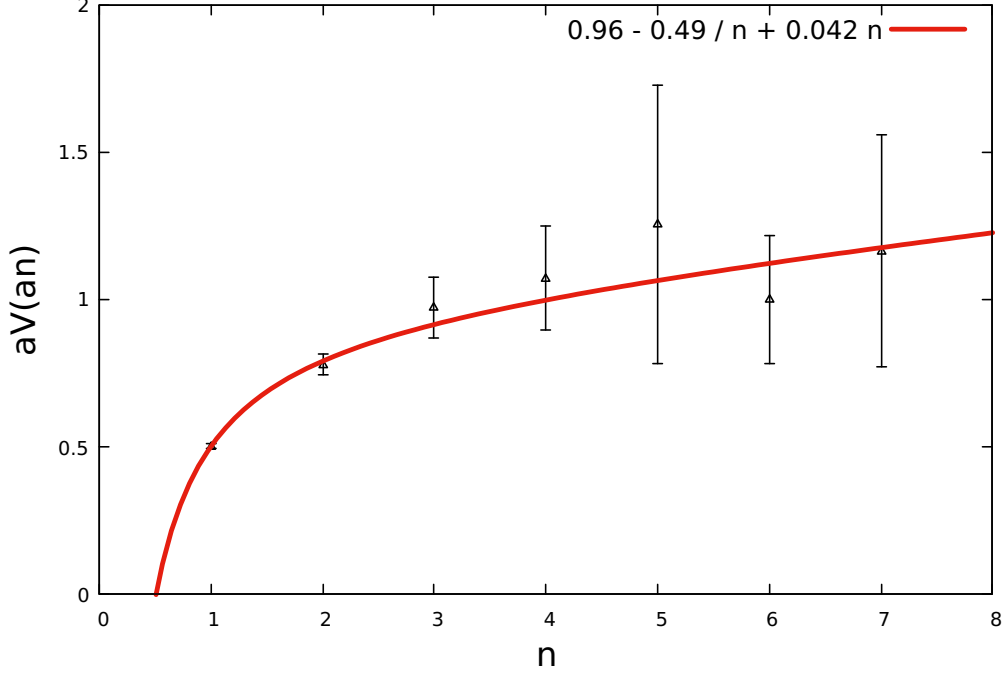


Figure 5: Static quark potential.

With a volume  $16^3 \times 6$  and at  $\beta = 5.7$  the static potential  $aV(an)$  using the correlation of the Polyakov loop is seen in figure 5. We obtained a string tension  $a^2\sigma = 0.042$  and  $B = -0.49$ , therefore

$$\frac{r_0}{a} = \sqrt{\frac{1.65 - 0.49}{0.042}} = 5.25, \quad (29)$$

and the lattice spacing in physical units corresponds to

$$a = 0.5/5.25 \text{ fm} = 0.095 \text{ fm}. \quad (30)$$

Finally the temperature corresponds to  $T = \frac{197 \text{ MeV}}{aL_t} = \frac{197 \text{ MeV}}{0.095 \cdot 6} = 345 \text{ MeV}$ .

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