2d $\lambda\phi^4$ theory

This code simulates the 1-component $\lambda\phi^4$ theory.

Requisites

- A Fortran compiler: gfortran
- Makefile installed
- Use Linux

Theory

Let $\phi(x) \in \mathbb{R}$ be a field. The Lagrangian density in Euclidian time reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x) + \frac{1}{2}m^2\phi(x)^2 + \frac{1}{4}\lambda\phi(x)^4,$$

where $\lambda \geq 0$, for the potential to be bounded from below. The continuum action reads

$$S[\phi] = \int dx^d \; \mathcal{L}(\phi, \partial_\mu\phi).$$

With the lattice regularization

$$\begin{aligned}
\phi(x) &\to \phi_x \\
\partial_\mu\phi(x) &\to \frac{\phi_{x+a\hat{\mu}} - \phi_x}{a} \\
\int dx^d &\to \sum_x a^d
\end{aligned}$$

The unit vectors are defined as

$$\hat{1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \hat{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the lattice the action takes the form

$$S[\phi] = \sum_x a^d \left\{ \frac{1}{2}\sum_\mu \left(\frac{\phi_{x+a\hat{\mu}} - \phi_x}{a} \right)^2 + \frac{1}{2}m^2\phi_x^2 + \frac{1}{4}\lambda\phi_x^4 \right\}.$$

In lattice units we set $a = 1$.

Expanding the action through the lattice

$$
\begin{aligned}
S[\phi] &= \frac{1}{2}\sum_{\mu=1}^{2}\left(\phi_{(1,1)+\hat{\mu}} - \phi_{(1,1)}\right)^2 + \frac{1}{2}m^2\phi_{(1,1)}^2 + \frac{1}{4}\lambda\phi_{(1,1)}^4 \\
&+ \frac{1}{2}\sum_{\mu=1}^{2}\left(\phi_{(1,2)+\hat{\mu}} - \phi_{(1,2)}\right)^2 + \frac{1}{2}m^2\phi_{(1,2)}^2 + \frac{1}{4}\lambda\phi_{(1,2)}^4 \\
&\vdots \\
&+ \frac{1}{2}\sum_{\mu=1}^{2}\left(\phi_{(i,j-1)+\hat{\mu}} - \phi_{(i,j-1)}\right)^2 + \frac{1}{2}m^2\phi_{(i,j-1)}^2 + \frac{1}{4}\lambda\phi_{(i,j-1)}^4 \\
&\vdots \\
&+ \frac{1}{2}\sum_{\mu=1}^{2}\left(\phi_{(i-1,j)+\hat{\mu}} - \phi_{(i-1,j)}\right)^2 + \frac{1}{2}m^2\phi_{(i-1,j)}^2 + \frac{1}{4}\lambda\phi_{(i-1,j)}^4 \\
&+ \frac{1}{2}\sum_{\mu=1}^{2}\left(\phi_{(i,j)+\hat{\mu}} - \phi_{(i,j)}\right)^2 + \frac{1}{2}m^2\phi_{(i,j)}^2 + \frac{1}{4}\lambda\phi_{(i,j)}^4 \\
&\vdots \\
&= \cdots \\
&+ \frac{1}{2}\left[\left(\phi_{(i+1,j-1)} - \phi_{(i,j-1)}\right)^2 + \left(\phi_{(i,j)} - \phi_{(i,j-1)}\right)^2\right] + \frac{1}{2}m^2\phi_{(i,j-1)}^2 + \frac{1}{4}\lambda\phi_{(i,j-1)}^4 \\
&\vdots \\
&+ \frac{1}{2}\left[\left(\phi_{(i,j)} - \phi_{(i-1,j)}\right)^2 + \left(\phi_{(i-1,j+1)} - \phi_{(i-1,j)}\right)^2\right] + \frac{1}{2}m^2\phi_{(i-1,j)}^2 + \frac{1}{4}\lambda\phi_{(i-1,j)}^4 \\
&+ \frac{1}{2}\left[\left(\phi_{(i+1,j)} - \phi_{(i,j)}\right)^2 + \left(\phi_{(i,j+1)} - \phi_{(i,j)}\right)^2\right] + \frac{1}{2}m^2\phi_{(i,j)}^2 + \frac{1}{4}\lambda\phi_{(i,j)}^4 \\
&\vdots
\end{aligned}
$$

If we update the field ϕ at site $x = (i,j)$, that is, $\phi_x \to \phi'_x$, the action becomes

$$
\begin{aligned}
S[\phi'] = \cdots& \\
&+ \frac{1}{2}\left[\left(\phi_{(i+1,j-1)} - \phi_{(i,j-1)}\right)^2 + \left(\phi'_{(i,j)} - \phi_{(i,j-1)}\right)^2\right] + \frac{1}{2}m^2\phi^2_{(i,j-1)} + \frac{1}{4}\lambda\phi^4_{(i,j-1)} \\
&\vdots \\
&+ \frac{1}{2}\left[\left(\phi'_{(i,j)} - \phi_{(i-1,j)}\right)^2 + \left(\phi_{(i-1,j+1)} - \phi_{(i-1,j)}\right)^2\right] + \frac{1}{2}m^2\phi^2_{(i-1,j)} + \frac{1}{4}\lambda\phi^4_{(i-1,j)} \\
&+ \frac{1}{2}\left[\left(\phi_{(i+1,j)} - \phi'_{(i,j)}\right)^2 + \left(\phi_{(i,j+1)} - \phi'_{(i,j)}\right)^2\right] + \frac{1}{2}m^2\phi'^2_{(i,j)} + \frac{1}{4}\lambda\phi'^4_{(i,j)} \\
&\vdots
\end{aligned}
$$

Therefore the change in the action $\Delta S = S[\phi'] - S[\phi]$ is

$$
\begin{aligned}
\Delta S = &\frac{1}{2}\left[\left(\phi'_{(i,j)} - \phi_{(i,j-1)}\right)^2 - \left(\phi_{(i,j)} - \phi_{(i,j-1)}\right)^2\right] \\
&+ \frac{1}{2}\left[\left(\phi'_{(i,j)} - \phi_{(i-1,j)}\right)^2 - \left(\phi_{(i,j)} - \phi_{(i-1,j)}\right)^2\right] \\
&+ \frac{1}{2}\left[\left(\phi_{(i+1,j)} - \phi'_{(i,j)}\right)^2 - \left(\phi_{(i+1,j)} - \phi_{(i,j)}\right)^2\right] \\
&+ \frac{1}{2}\left[\left(\phi_{(i,j+1)} - \phi'_{(i,j)}\right)^2 - \left(\phi_{(i,j+1)} - \phi_{(i,j)}\right)^2\right] \\
&+ \frac{1}{2}m^2\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) + \frac{1}{4}\lambda\left(\phi'^4_{(i,j)} - \phi^4_{(i,j)}\right) \\
&= \frac{1}{2}\left[\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) - 2\phi_{(i,j-1)}\left(\phi'_{(i,j)} - \phi_{(i,j)}\right)\right] \\
&+ \frac{1}{2}\left[\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) - 2\phi_{(i-1,j)}\left(\phi'_{(i,j)} - \phi_{(i,j)}\right)\right] \\
&+ \frac{1}{2}\left[\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) - 2\phi_{(i+1,j)}\left(\phi'_{(i,j)} - \phi_{(i,j)}\right)\right] \\
&+ \frac{1}{2}\left[\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) - 2\phi_{(i,j+1)}\left(\phi'_{(i,j)} - \phi_{(i,j)}\right)\right] \\
&+ \frac{1}{2}m^2\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) + \frac{1}{4}\lambda\left(\phi'^4_{(i,j)} - \phi^4_{(i,j)}\right) \\
&= 2\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) \\
&- 2\left(\phi'_{(i,j)} - \phi_{(i,j)}\right)\left(\phi_{(i-1,j)} + \phi_{(i,j-1)} + \phi_{(i+1,j)} + \phi_{(i,j+1)}\right) \\
&+ \frac{1}{2}m^2\left(\phi'^2_{(i,j)} - \phi^2_{(i,j)}\right) + \frac{1}{4}\lambda\left(\phi'^4_{(i,j)} - \phi^4_{(i,j)}\right)
\end{aligned}
$$

Metropolis Algorithm

We generate a change in the field, $\phi_x \to \phi'_x$ and accept it with probability

$$p = \min(1, \exp(-\Delta S)).$$