Linear transformation

A linear transformation is a function between two vectors spaces that preserves the operation of vector addition and scalar multiplication. This means that if T is a linear transformation from a vector space V to a vector space W, then for any vectors.

2 Important properties $u, v \in V$ and any scaler c

1) Additivity $T(u+v) = T(u) + T(v)$

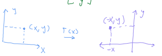
2) Homogecity $T(cu) = c\,T(u)$

$\left. T: V \to W \right\} \Rightarrow$ Linear Transformation

Example Reflection

The reflection transformation T across the Y axis maps a vector

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \quad T(x) = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Transformation can be expresed as

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{T(x) = A \cdot \vec{x}}_{\text{Linear transformation}} = \underset{x2}{\begin{bmatrix} x \\ y \end{bmatrix}} \cdot \underset{2 \times 2}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} (-1)\cdot x + 0 \cdot y \\ (0)\cdot x + 1 \cdot y \end{bmatrix} = \underset{1 \times 2}{\begin{bmatrix} -x \\ y \end{bmatrix}}$$

1) checking additivity

let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vector in R^2

$$T(u+v) = T(u) + T(v)$$

$$u+v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

$$T(u+v) = A(u+v) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -(u_1 + u_2) \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} -u_1 - u_2 \\ u_2 + v_2 \end{bmatrix}$$

$$T(u) = Au = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix}$$

$$T(v) = Av = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} \quad LHS = RHS$$

$$RHS \Rightarrow T(u) + T(v) = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -u_1 - v_1 \\ u_2 + v_2 \end{bmatrix}$$

Left-hand side (LHS) = Right-Hand side (RHS)

2) Checking homogenity

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ and c be a scalar

Homogenity Requirement

$$T(cu) = c\,T(u)$$

$$cu = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

$$T(cu) = A(cu) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} \Rightarrow LHS$$

$$c\,T(u) = c\,(A \cdot u) = c \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= c \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} \Rightarrow RHS$$

$$LHS = RHS$$

Examples That dont follow Linear Transformation

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad T(x) = x + b \qquad T: \underset{u}{R^2} \to \underset{v}{R^2}$$

vector $\Rightarrow$ fixed vector

$$T(x) = x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1) Check Additivity

$$T(u+v) = T(u) + T(v)$$

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right) =$$

$$T\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} \right)$$

$$T\Big(\underbrace{\begin{bmatrix} 6 \\ 2 \end{bmatrix}}_{x} \Big) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \to LHS$$

$$T(u) + T(v) =$$

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$T(u) + T(v) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Rightarrow RHS$$

$$LHS \neq RHS$$

Check homogencity

$$T(cu) = c\,T(u)$$

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad c = 2$$

$$T(cu) = T \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 7 \end{bmatrix} \Rightarrow LHS$$

$$c\,T(u) = 2\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Rightarrow RHS$$

$$LHS \neq RHS$$

$T(x) = \vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to$ Not linear transformation

Fails both additivy and homogenity properties