Grado en Física

End-of-Degree Project

Cosmological Perturbations:

Primordial Inflation Perturbations and its CMB Constraints

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Abstract

The Big Bang paradigm has two main problems, the large-scale homogeneity and the spatial flatness. These problems are easily solved with the theory of inflation, defined as an epoch of accelerated universe expansion. The simplest way to implement inflation is with a scalar field ϕ. The inflationary evolution needs a slow-roll regime, a phase where the field "rolls slowly" over the field potential, to solve the Big Bang problems. This regime can always be imposed, as inflation has a slow-roll attractor. By introducing quantum mechanics in the inflation paradigm, quantum fluctuations arise naturally and can be understood using perturbation theory. These primordial perturbations can be scalar or tensor, the last ones smaller in amplitude. From the primordial perturbations, the scalar and tensor power spectra can be obtained, which are related to the Cosmic Microwave Background (CMB) spectra. The CMB is a crucial cosmological observation, as most of the universe parameters can be obtained from its anisotropies and its polarization. In particular, with the CMB constraints of inflation we are going to test two different models: the quartic potential ($V(\phi) = \lambda\phi^4$), which will help us understand the theoretical derivations, and the natural inflation ($V(\phi) = \Lambda^4[1 + \cos\phi/f]$), which will be an example to complete our work. We will confirm that the quartic potential model is ruled out by the CMB observations and that the natural inflation has not completely been discarded by the last measurements. Still, it is expected to be ruled out by future experiments.

Contents

1 Introduction

The standard model of the Hot Big Bang is based on the evidence that the universe is expanding [1], so going backwards in time shows that the early universe expanded from an incredibly dense and hot state. Therefore, the universe was a plasma of particles, highly interacting in thermal equilibrium. These were the initial conditions of the Big Bang.

As the universe expanded, the temperature of the plasma decreased and the particles started to decouple from the thermal bath. At 100 GeV (10^{15} K), the electroweak symmetry was broken during the EW phase transition, and the particles received their masses. At 150 MeV (10^{12} K), quarks condensed into hadrons during the QCD phase transition. Temperature continued dropping and heavier elements were produced in the Big Bang nucleosynthesis (BBN).

About 370 000 years after the Big Bang, the universe cooled enough to form the first atoms. The scattering between the electrons and the photons stopped and the universe became transparent. The photons started to propagate freely and, after 13.8 billion years, this first light can be observed today as a microwave radiation, known as the *Cosmic Microwave Background* (CMB).

The CMB radiation is really homogeneous, with tiny variations in its intensity of 1 part in 100 000: the anisotropies of the CMB. The CMB is also polarized and its polarization carries essential information complementary to the temperature anisotropies. By studying the CMB perturbations we can understand our primordial universe (see fig. 1). Moreover, the CMB observations are the main evidence which we use to establish the cosmological model.

However, the standard Big Bang theory cannot resolve two main features of our universe: the large-scale homogeneity and the flatness geometry. These properties must be imposed by initial conditions, which is not desirable because of its arbitrariness. To solve these problems, the *inflationary paradigm* was proposed [2]. The main idea of inflation is that a sufficiently long period of accelerated expansion before the Big Bang gave the initial conditions for the primordial universe.

This is the memory of the second part of an annual project based on cosmological perturbations.

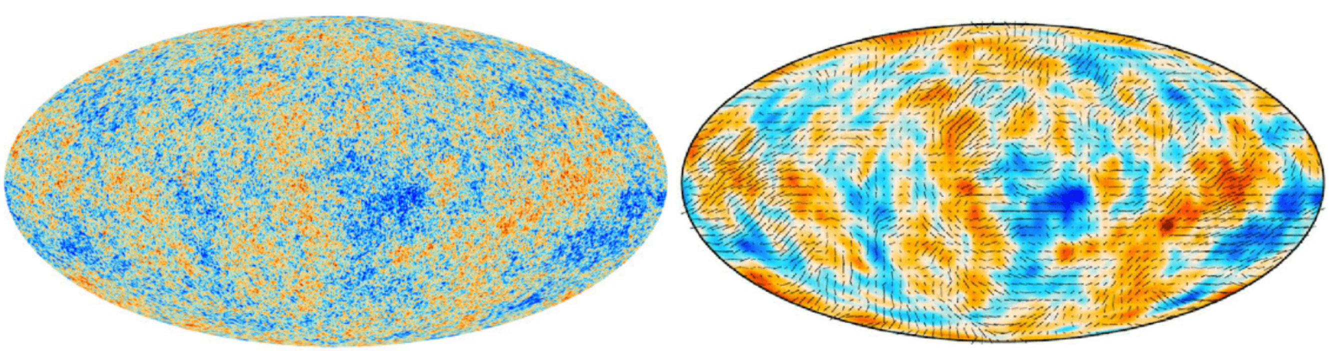

Figure 1: ***Left:*** *CMB anisotropies from the Planck experiment.* ***Right:*** *CMB polarization superimposed to the temperature fluctuations from the Planck experiment. Extracted from [3] and [4] respectively.*

During the first part, we studied the basics of general relativity and cosmology, understanding the expansion and composition of the universe. We also focused on how to extract that information from the CMB, learning how the curvature of the universe or the matter abundance got reflected on the observed spectrum.

In this second part, we will understand the physics of the inflationary paradigm. Firstly, we will briefly introduce the basic cosmology needed to understand inflation. Then, we will focus on how to describe inflation with a scalar field and, also, on how inflation perturbations are produced and its evolution. After the theoretical description, we will show how to test inflation theory with CMB anisotropies and CMB polarization. To sum up all the work, we will use a famous model of inflation, the Natural Inflation, to implement all we have derived in previous sections.

2 Preliminaries

Before starting with all our derivations, it is important to note that during this project we are going to work in natural units, so $\hbar = c = 1$.

2.1 Basic Cosmology

For large cosmic scales, the universe obeys a basic principle, the cosmological principle, which states that the universe is homogeneous and isotropic [1, 5] . Taking this into account, the metric needed for an expanding universe is the Robertson-Walker metric (FLRW) [6]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{1}$$

with

$$\gamma_{ij} \equiv \delta_{ij} + K\frac{x_i x_j}{1 - K(x_k x^k)}, \quad \text{for} \quad K \equiv \begin{cases} 0 & \text{Euclidean} \\ 1 & \text{Spherical} \\ -1 & \text{Hyperbolic} \end{cases}$$

and $a(t)$ is the scale factor. As we are going to compute the evolution of $a(t)$, it is useful to define the Hubble parameter as $H \equiv \dot{a}/a$, where the overdot denotes a time derivative.

From the requirements of isotropy and homogeneity, the universe components will be described as perfect fluids, expressed as an energy-momentum tensor:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - Pg_{\mu\nu} \ , \tag{2}$$

where ρ and P are the energy density and the pressure of the fluid respectively and u^μ is the four-velocity of the fluid with respect an observer. For a comoving observer (an observer moving with the expansion), the four-velocity is $u^\mu = (1, 0, 0, 0)$, so the energy-momentum tensor can be expressed as

$$T^{\mu}{}_{\nu} = g^{\mu\lambda}T_{\lambda\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix} . \tag{3}$$

The energy-momentum tensor has to be covariant conserved during the evolution of the universe, so $\nabla_{\mu}T^{\mu}{}_{\nu} = 0$. This equation for $\nu = 0$ (see appendix A.1) leads to the *continuity equation*

$$\dot{\rho} + 3H(\rho + P) = 0 . \tag{4}$$

Most cosmological fluids can be parametrized in terms of an equation of state $w = P/\rho$ like matter ($w = 0$), radiation ($w = 1/3$) or vacuum energy ($w = -1$). With this assumption, eq. (4) takes the form $\rho \propto a^{-3(1+w)}$.

To relate the energy-momentum to the FLRW metric and compute the dynamics of the system, the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5}$$

have to be used, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$; $R_{\mu\nu}$ is the Ricci tensor, which can be computed from the metric and its derivatives; and $R = g^{\mu\nu}R_{\mu\nu}$. In appendix A.1 we will derive this expressions carefully to get the *Friedmann equations*,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} , \tag{6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) . \tag{7}$$

Equation (6) is normally written in terms of the Hubble parameter H, so

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} . \tag{8}$$

It is also useful to substitute the Newton's constant for the reduced Planck mass $M_{Pl} \equiv (8\pi G)^{-1/2}$, so the Friedmann equations can finally be written like

$$H^2 = \frac{1}{3M_{Pl}^2}\rho - \frac{K}{a^2} , \tag{9}$$

$$\dot{H} + H^2 = -\frac{1}{6M_{Pl}^2}(\rho + 3P) . \tag{10}$$

2.2 Useful variables

We are also going to introduce two other useful variables to parameterize the time evolution, the *conformal time* $\mathrm{d}\tau = \mathrm{d}t/a(t)$ and the number of *e-folds* $\mathrm{d}N = H\mathrm{d}t$, which can also be defined as

$a = a_i e^{N-N_i}$. These variables can be related as $\mathrm{d}N = aH\mathrm{d}\tau$. Whenever we differentiate respect any of these variables we can write [7]:

$$\begin{aligned} \dot{f}(t) &= \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau) \ , \\ \ddot{f}(t) &= \frac{1}{a^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}f(\tau) - aH\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau)\right) \end{aligned} \tag{11}$$

or

$$\begin{aligned} \dot{f}(t) &= Hf'(N) \ , \\ \ddot{f}(t) &= H^2\left(f''(N) - \varepsilon f'(N)\right) \ , \end{aligned} \tag{12}$$

where the $'$ denotes derivative respect e-folds and $\varepsilon \equiv -\dot{H}/H^2$.

3 Inflation

3.1 Problems of the Big Bang Paradigm

The Big Band paradigm has two main problems: the horizon problem and the flatness problem [1, 6]. Both of them can be resolved with inflation.

To describe the *horizon problem*, we have to introduce the particle horizon, which is the comoving distance from which an observer can receive signals, expressed as

$$d_h = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} = \int_{\ln a_i}^{\ln a} (aH)^{-1}\mathrm{d}\ln a \ , \tag{13}$$

where $(aH)^{-1}$ is the comoving Hubble radius and t_i is the time when $a_i = 0$. Using Friedmann equation (9) and the continuity equation (4), we can write

$$(aH)^{-1} \propto \rho^{-1/2}a^{-1} = a^{(1+3w)/2} \ , \tag{14}$$

so, for ordinary matter sources ($1 + 3w > 0$), the comoving Hubble radius increases with time. Therefore, points in space that were not in causal contact at t_i cannot communicate with each other at any time after. This implies a serious problem, as the CMB radiation we receive is practically homogeneous and isotropic, it does not make sense if patches of space were causally disconnected at recombination (the moment when the photons decoupled).

The *flatness problem* comes from the observations that the curvature parameter today is $|\Omega_{K,0}| < 0.005$, where the curvature parameter is defined as $\Omega_K \equiv -K/(aH)^2$. If we express the curvature parameter evolution as

$$\Omega_K(t) = \frac{(a_0H_0)^2}{(aH)^2}\Omega_{K,0} \ , \tag{15}$$

we observe that the curvature in the past had to be even smaller (because $(aH)^{-1}$ is growing). To give an example, at the Big Bang nucleosynthesis the curvature parameter had to be $|\Omega_K(t_{BBN})| < 10^{-16}$, a really improbable value for arbitrary initial conditions.

A simple solution for both problems is a phase of decreasing Hubble radius before the Big Bang,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(aH)^{-1} < 0\ . \tag{16}$$

This simple statement is the equivalent to an accelerated expansion $\ddot{a} > 0$ (using $aH = \dot{a}$), which is basically the definition of *inflation*. Alternatively, a shrinking Hubble radius can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = \frac{1}{a}(1-\varepsilon)\ , \tag{17}$$

therefore, we require $\varepsilon < 1$ to have inflation, a statement that later will be really useful.

We can solve the Big Bang problems with inflation because it would lead to the particle horizon being larger and the patches in space to be connected. This would resolve the horizon problem. Also, with a sufficient long period of decreasing Hubble radius, the curvature naturally tends to flatten. However, the duration of inflation cannot be arbitrary, we at least need all the observed fluctuations of the CMB inside the particle horizon at early times, so

$$(a_0H_0)^{-1} < (a_IH_I)^{-1}\ , \tag{18}$$

where a_o is the factor scale at present time and a_I is the factor scale at the beginning of inflation. We can assume that after inflation the universe was radiation-dominated ($H \propto a^{-2}$). We can also write $(a_0H_0)/(a_eH_e) \sim T_0/T_e$, where we have used that $T \propto a^{-1}$ and the subscript e denotes the end of inflation. At the end of inflation it occurs a process called reheating (see section 3.5), so $T_e \approx T_R$. Knowing that $T_0 = 2.7K$, we can obtain

$$(a_IH_I)^{-1} > (a_0H_0)^{-1} \sim 10^{28}\frac{T_R}{10^{15}\mathrm{GeV}}(a_eH_e)^{-1}\ , \tag{19}$$

where we have used 10^{15} GeV as a reference value. If we express the duration of inflation in e-folds $N_{tot} = \ln a_e/a_I$ and we assume $H_I \sim H_e$ during inflation, we can finally get

$$N_{tot} > 64 + \ln\left(T_R/10^{15}\mathrm{GeV}\right)\ , \tag{20}$$

so we need about $N_{tot} \sim 60$ of inflation to solve the Big Bang problems.

If is also interesting to see how conformal time is modified by introducing inflation. Integrating eq. 13 we can define

$$\tau_i \equiv \frac{2H_0^{-1}}{1+3w}a_i^{\frac{1}{2}(1+3w)}\ , \tag{21}$$

which for ordinary matter sources $\tau_i \to 0$ as $a_i \to 0$. If the universe has a phase of decreasing Hubble radius, then using eq. (14) we obtain $1 + 3w < 0$, so the singularity $a_i \to 0$ is pushed to negative conformal time, $\tau_i \to -\infty$. This implies that there was more conformal time that we previously thought between the singularity and the CMB decoupling.

3.2 Scalar Field Dynamics

The simplest way to implement inflation is by the evolution of a scalar field $\phi(t, \vec{x})$, called the *inflaton* [1, 8]. Related to this field there is a potential energy density, which is denoted by $V(\phi)$. The field also carries kinetic energy density $\dot{\phi}^2/2$. There are two ways to approach the dynamics of a scalar field like the inflaton: by its action or by its energy-momentum tensor. We are going to use the second one.

The energy-momentum tensor for a scalar field is defined as

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right) \ . \tag{22}$$

We are only interested in the evolution of a homogeneous field, so $\phi = \phi(t)$. In this case we can derive the energy density (ρ_ϕ) from the time-time component $T^0{}_0 = \rho_\phi$, and the pressure (P_ϕ) from the space-space component $T^i{}_j = -P_\phi \delta^i{}_j$, obtaining

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \tag{23}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \tag{24}$$

Introducing these expressions into the Friedmann equations (9, 10) we can write

$$H^2 = \frac{1}{3M_{Pl}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \ , \tag{25}$$

$$\dot{H} = -\frac{1}{2M_{Pl}^2}\dot{\phi}^2 \ . \tag{26}$$

Having obtained the energy density and the pressure from the energy-momentum tensor, we can use the continuity equation (4) (or $\nabla_\mu T^\mu{}_0 = 0$) to obtain the *scalar field equation*

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \ . \tag{27}$$

This equation gives us the evolution of the inflaton and it is interesting to observe that it is coupled with (25), as the field determines the Hubble parameter and the Hubble parameter induces friction in the evolution of the field.

Also, it is relevant to write this expression in terms of the number of e-folds. Using (12) we find

$$\phi'' + (3 - \varepsilon)\phi' + \frac{1}{H^2}\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \ , \tag{28}$$

which is the scalar field equation respect the number of e-folds. This equation will be more relevant as it is more useful to express the inflation in terms of N.

3.3 Slow-roll Inflation

As we have said, inflation occurs if $\varepsilon < 1$. This parameter can be expressed in terms of the inflation field using (26) as

$$\varepsilon = \frac{1}{2M_{Pl}^2}\left(\frac{\dot{\phi}}{H}\right)^2 . \tag{29}$$

To satisfy the condition for ε, it is required that $\dot{\phi}^2 \ll H^2$ so the field "rolls slowly". It is now clear why this approach is called slow-roll approximation. Taking this into account, eq. (25) can be written as

$$H^2 \approx \frac{V}{3M_{Pl}^2} , \tag{30}$$

which is known as the first slow-roll condition.

Moreover, in order for the slow-roll behavior to persist, the acceleration of the field also has to be small. To achieve this condition we can define a second slow-roll parameter

$$\eta = \varepsilon - \frac{1}{2}\frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}N} = -\frac{\ddot{\phi}}{H\dot{\phi}} , \tag{31}$$

that has to satisfy $\eta \ll 1$, so the scalar field equation (27) can be expressed as

$$3H\dot{\phi} \approx -\frac{\mathrm{d}V}{\mathrm{d}\phi} , \tag{32}$$

which is known as the second slow-roll condition.

Having done these approximations, it is convenient to compute the slow-roll parameters in terms only of the potential and its derivatives, to judge whether a given potential can lead to slow-roll or not. These are called the potential slow-roll parameters:

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{\mathrm{d}V/\mathrm{d}\phi}{V}\right)^2, \qquad \eta_V \equiv M_{Pl}^2\frac{\mathrm{d}^2V/\mathrm{d}\phi^2}{V} , \tag{33}$$

where $\varepsilon \approx \epsilon_V$ and $\eta \approx \eta_V - \epsilon_V$ in the slow-roll approximation.

To compute the evolution of the inflation field using the scalar field equation (27 or 28), we need the initial condition of ϕ_0 and its initial velocity $\dot{\phi}_0$. The initial condition is usually obtained from the total number of e-folds during inflation, and the initial velocity can be obtained from the second slow-roll condition.

Inflation ends when $\varepsilon \approx 1$, so the total number of e-folds is

$$N_{tot} \equiv \int_{\phi_0}^{\phi_e} \frac{\mathrm{d}N}{\mathrm{d}\phi}\mathrm{d}\phi = \int_{\phi_0}^{\phi_e} \frac{1}{\sqrt{2\varepsilon}}\frac{|\mathrm{d}\phi|}{M_{Pl}} , \tag{34}$$

where ϕ_e is the value of the field where inflation stops and the absolute value is needed to ensure $N_{tot} > 0$. In the slow-roll regime, we can write the integral as

$$N_{tot} \approx \int_{\phi_0}^{\phi_e} \frac{1}{\sqrt{2\epsilon_V}}\frac{|\mathrm{d}\phi|}{M_{Pl}} . \tag{35}$$

For a potential given, ϵ_V can be calculated and the total number of e-folds can be obtained as a function of the initial value of the field. By choosing a value for N_{tot}, the value of ϕ_0 can be derived from it.

Assuming that at the initial moment the field is in the slow-roll regime, the initial velocity of the field is easy to compute combining (30) and (32), getting

$$\dot{\phi}_0 = -\sqrt{2\epsilon_V} H M_{Pl} \quad \text{or} \quad \phi_0' = -\sqrt{2\epsilon_V} M_{Pl} \ . \tag{36}$$

Any other initial velocity can be possible, but in the next section we will show that the inflaton has an attractor that makes the field to rapidly move to the slow-roll regime, so the approximation we have done is valid.

3.4 The Quartic Potential

The quartic potential is particularly interesting as it has a simple form and was relevant in the WMAP (Wilkinson Microwave Anisotropy Probe) analysis [9, 10], one of the first CMB analysis. It is defined as

$$V(\phi) = \lambda\phi^4 \ . \tag{37}$$

The first step before computing the inflaton evolution is to calculate the potential slow-roll parameters (33). In our case they are written as

$$\epsilon_V = 8\frac{M_{Pl}^2}{\phi^2}, \qquad \eta_V = 12\frac{M_{Pl}^2}{\phi^2} \ , \tag{38}$$

so with eq. (36) we can obtain that the initial velocity of the field (respect e-folds) is $\phi_0' = -4M_{Pl}^2/\phi_0$ for the slow-roll regime. Also, by defining the end of inflation when $\epsilon_V(\phi_e) = 1$, it is straightforward

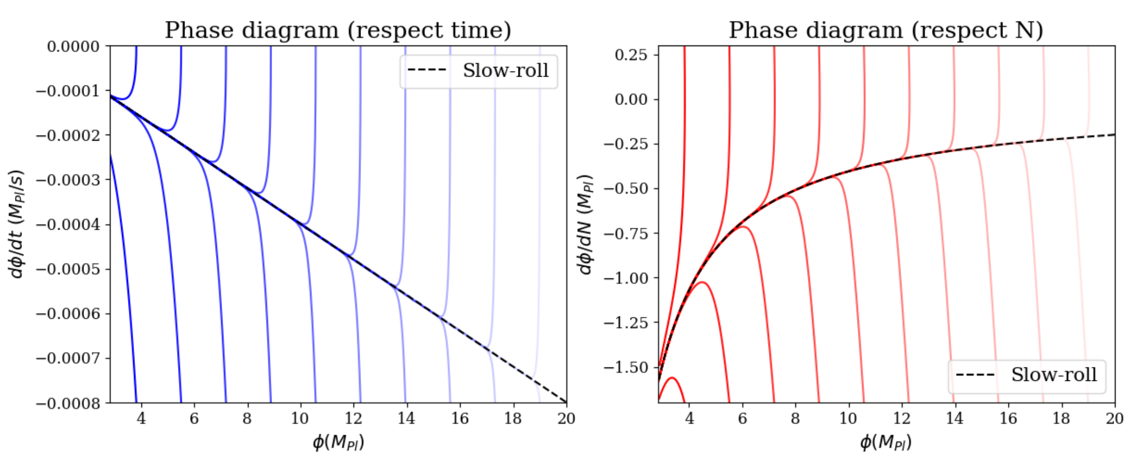

Figure 2: *Phase diagram of the evolution of the inflation field, respect time (left) and respect e-folds (right). In dashed lines is the slow-roll solution.*

that $\phi_e = 2\sqrt{2}M_{Pl}$. From eq. (35) the total number of e-folds in this case is

$$N_{tot} = \int_{\phi_0}^{\phi_e} \frac{\phi}{4M_{Pl}^2}|\mathrm{d}\phi| = \frac{1}{8}\frac{\phi_0^2 - \phi_e^2}{M_{Pl}^2} \ . \tag{39}$$

If we want to have near 55 e-folds before the end of inflation, the initial value of the field must be $\phi_0 \approx 20M_{Pl}$.

This potential is also convenient, as we can implicitly show the *slow-roll attractor* of inflation by computing the scalar field equation (27) and (28) for different initial velocities and plotting the phase diagram (see fig. 2). In this figure it is shown that for arbitrary initial conditions, the field rapidly goes to the slow-roll regime, due to the friction term in the scalar field equation. This property is really useful because it implies that the initial conditions need not be fine-tuned to get slow-roll regime. Therefore, in every model of inflation a initial slow-roll regime phase can be assumed.

3.5 Reheating

During inflation most of the energy density is carried by the inflaton. When inflation ends, the energy of the field has to be transferred to the particles of the Standard Model, a process called reheating [1, 11]. After the end of inflation, the field reaches the bottom of the potential and starts to oscillate, following a different equation of state w than in the slow-roll approximation, usually $w = 1/3$ (radiation domination) or $w = 0$ (matter domination).

The quartic potential previously described can be used to illustrate this process (see fig. 3). On the left panel, we see the inflation field evolution which oscillates when inflation ends as explained

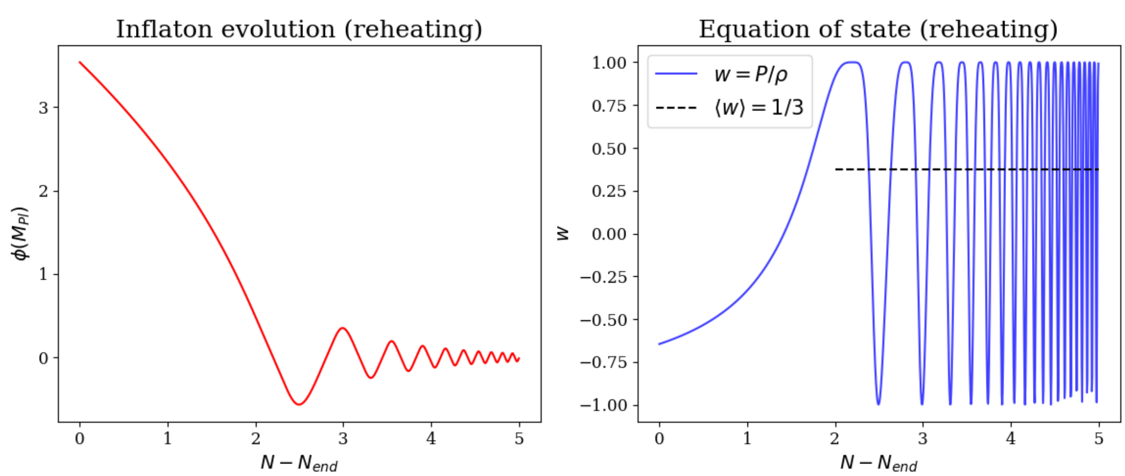

Figure 3: ***Left***: *Inflation evolution during reheating process for 5 e-folds after inflation.* ***Right***: *Equation of state during reheating. In dashed lines is the average equation of state for the oscillations, analogous to radiation domination era.*

before. On the right panel, we plot the equation of state where the average equation of state (omitting the transition phase) is the same as the one on a radiation domination era ($w = 1/3$).

4 Perturbation Theory

So far, we have considered that the universe is homogeneous for simplicity. However, to understand the large-scale structure of the universe it is important to consider the inhomogeneities and their evolution. This can be done with perturbation theory, where the perturbations can be separated in scalar, vector and tensor perturbations. We are only going to focus on scalar (section 4.1) and tensor perturbations (section 4.2), as the vector perturbations are not created by inflation. To compute this theory, we have to perturb the metric and the energy-momentum tensor and expand them at linear order in perturbations. We are going to considered that the background will only depend on time evolution so

$$g_{\mu\nu}(t,\vec{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x}) \ , \tag{40}$$

$$T_{\mu\nu}(t,\vec{x}) = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(t,\vec{x}) \ . \tag{41}$$

During this section we are also going to considered that $M_{Pl} = 1$ for simplicity, so the Einstein equations (5) will be written as

$$G_{\mu\nu} = T_{\mu\nu} \ . \tag{42}$$

Starting with the perturbed metric, the line element can be expressed as [12]

$$\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - 2a(t)B_i\mathrm{d}x^i\mathrm{d}t - a^2(t)\left[(1-2\Psi)\delta_{ij} + 2E_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j \ , \tag{43}$$

where Φ and Ψ are scalars and are called lapse and spatial curvature perturbation respectively; B_i is a 3-vector called spatial shear; and E_{ij} is a rank-2 symmetric and traceless tensor.

To separate the scalar and tensor perturbations it is convenient to perform a scalar-vector-tensor (SVT) decomposition [1, 12], defining the scalar, vector and tensor perturbations as having helicities 0, ±1 and ±2 respectively. For a 3-vector, the decomposition is simply splitting into a helicity scalar (the gradient of a scalar) and a helicity vector (divergence-less vector)

$$B_i = B_i^S + B_i^V = \partial_i B + \hat{B}_i \ , \tag{44}$$

where $\partial_i B$ is the gradient and $\partial^i \hat{B}_i = 0$. Similarly, a tensor can be decomposed into a helicity scalar that contains the trace of the tensor and traceless helicity vector and tensor components

$$\gamma_{ij} = \gamma_{ij}^S + \gamma_{ij}^V + \gamma_{ij}^T \ . \tag{45}$$

Because we have defined E_{ij} as traceless, it will only have vector and tensor components

$$E_{ij} = E_{ij}^V + E_{ij}^T = \partial_{(i}\hat{E}_{j)} + h_{ij}/2 \ . \tag{46}$$

4.1 Scalar Fluctuations

4.1.1 Dynamics in Spatially-Flat Gauge

For scalar perturbations we are going to express eq. (43) in terms of only the scalar components, so

$$\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - 2a(t)\partial_i B \mathrm{d}x^i \mathrm{d}t - a^2(t)\left[(1-2\Psi)\delta_{ij} + 2\partial_{(i}\hat{E}_{j)}\right]\mathrm{d}x^i \mathrm{d}x^j \ . \tag{47}$$

The metric perturbations are not uniquely defined, they depend on our gauge choice. To simplify the calculations we are going to work in spatially flat gauge, that is $\Psi = E = 0$, so the line element is [13]

$$\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - 2a(t)\partial_i B \mathrm{d}x^i \mathrm{d}t - a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \ . \tag{48}$$

To obtain the fluctuations of the inflation field, we have to expand the energy-momentum tensor and solve the continuity equation ($\nabla_\mu T^\mu{}_0 = 0$). To do this, we have to define the inflation field as

$$\phi(t,\vec{x}) = \bar{\phi}(t) + \delta\phi(t,\vec{x}) \ . \tag{49}$$

Inserting this expression into the energy-momentum tensor, we can derive the continuity equation at linear order. To simplify this equation the Einstein equations can be used (see appendix A.2 for all the derivation), so the *inflation field perturbation* can be expressed as

$$(\ddot{\delta\phi}) + 3H(\dot{\delta\phi}) - \frac{\nabla^2\delta\phi}{a^2} + \underbrace{\left[\left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\bar{\phi}} - \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\bar{\phi}}^2\right)\right]\delta\phi}_{\text{Metric perturbations}} = 0 \ . \tag{50}$$

The inflation fluctuations are treated in Fourier space, as the modes are easy to track independently and are decoupled at linear order. We define

$$\delta\phi_k(t) = \int \mathrm{d}^3\vec{x}\ \delta\phi(t,\vec{x})e^{-i\vec{k}\vec{x}} \ , \tag{51}$$

so eq. (50) in Fourier space can be written like

$$(\ddot{\delta\phi_k}) + 3H(\dot{\delta\phi_k}) + \frac{k^2}{a^2}\delta\phi_k + \left[\left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\bar{\phi}} - \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\bar{\phi}}^2\right)\right]\delta\phi_k = 0 \ , \tag{52}$$

where we have used $\nabla^2 = -k^2$ in Fourier space. This equation is difficult to interpret, so to simplify the expression, a good approach is to try to eliminate the friction term $(\dot{\delta\phi_k})$ [1]. We define a new variable, $u_k = a\delta\phi_k$, and differentiating respect τ using eq. (11) we can obtain

$$(\ddot{\delta\phi_k}) + 3H(\dot{\delta\phi_k}) = \frac{1}{a^3}\left(\frac{\mathrm{d}^2u_k}{\mathrm{d}\tau^2} - (aH)^2(2-\varepsilon)u_k\right) \ . \tag{53}$$

The metric perturbations term can also be simplified by defining the variable $z = a\dot{\bar{\phi}}/H$. Differentiating respect conformal time twice we can obtain

$$\frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = -(aH)^2 \left\{ \frac{1}{H^2}\left[\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right|_{\bar{\phi}} - \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\bar{\phi}}^2 \right) \right] - (2-\varepsilon) \right\} , \tag{54}$$

so combining eq. (52) with eqs. (53) and (54) we get the *Mukhanov-Sasaki equation*

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + \left(k^2 - \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} \right) u_k = 0 . \tag{55}$$

This equation is easier to understand than eq. (52), as it is just a harmonic oscillator with a time-dependent frequency. During slow-roll inflation, H and $\dot{\bar{\phi}}$ can be considered approximately constant so

$$\frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} \approx \frac{1}{a}\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} \approx 2(aH)^2 , \tag{56}$$

which is the inverse of the comoving Hubble radius (14).

With this expression, it is easy to define two regions for the perturbations. At early times, the comoving Hubble radius was large ($aH \ll 1$) and all the modes were inside the horizon. In this limit we have $k^2 \gg (\mathrm{d}^2 z/\mathrm{d}\tau)/z$, so eq. (55) reduces to

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + k^2 u_k = 0 , \tag{57}$$

which is the subhorizon limit. This is a harmonic oscillator with fixed frequency $\omega = k$, so modes inside the horizon will oscillate as $u_k \propto e^{\pm ik\tau}$ (see fig. 4).

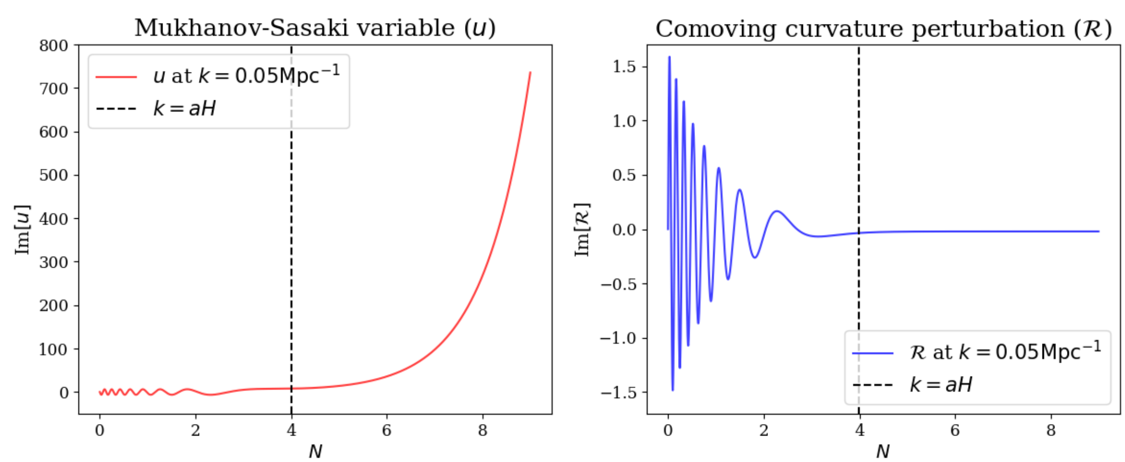

Figure 4: ***Left***: *Evolution of the Mukhanov-Sasaki variable (u) for a mode with* $k = 0.05$ Mpc^{-1}. ***Right***: *Comoving curvature perturbation* ($\mathcal{R} = u/z$) *for a mode with* $k = 0.05$ Mpc^{-1}. *In both images at the left of $k = aH$ is the subhorizon limit, for which the perturbation oscillates. At the right of $k = aH$ is the superhorizon limit, for which $u \propto z$ and $\mathcal{R}$ is frozen.*

The comoving horizon will shrink over time and modes will eventually exit the horizon. In this limit we now have $k^2 \ll (\mathrm{d}^2 z/\mathrm{d}\tau)/z$ and eq. (55) will reduce to

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} - \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} u_k = 0 \ , \tag{58}$$

which is the superhorizon limit. In this case, the solution has a growing mode $u_k \propto z$.

To understand how modes evolve in superhorizon scales, it is better to use another variable called the *comoving curvature perturbation* ($\mathcal{R}$). This variable is gauge-invariant, so it is really useful for tracking the inflation perturbations, as it does not matter in which gauge you are working. In spatially-flat gauge, it is defined as

$$\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi = \frac{u}{z} \ . \tag{59}$$

This variable has its own equation of motion, which can be derived by introducing its definition into eq. (55). If we express it with respect e-folds we can get

$$\mathcal{R}'' + (3 + \varepsilon - 2\eta)\mathcal{R}' + \left(\frac{k}{aH}\right)^2 \mathcal{R} = 0 \ . \tag{60}$$

However, it is more practical to compute $\mathcal{R}$ resolving the equation of motion for u and then changing the variable. Evaluating $\mathcal{R}$ in the limits described above, we can get a similar behavior for the subhorizon scales but completely different for superhorizon scales (see fig. 4). Recalling eq. (58) and (59), we can see that $\mathcal{R} \propto$ constant on superhorizon limit, so on large scales, modes will have frozen $\mathcal{R}$.

4.1.2 Quantum Fluctuations

To completely understand the inflation perturbations, it is crucial to know how these fluctuations were produced. When quantum mechanics are introduced in the inflation theory, fluctuations arise naturally as a consequence of the uncertainty principle [1, 6]. We can interpret that the inflaton fluctuations $\delta\phi$ originated a variation in the time δt at which inflation ends across the space. These differences led to different local densities and, therefore, to the CMB temperature fluctuation δT.

As we have seen in the previous section, at early times (subhorizon scales), the inflation field can be described as a collection of harmonic oscillators. So, quantum fluctuations induce a non-zero variance in the zero-point fluctuations that can be calculated. Because $\mathcal{R}$ will froze outside the horizon, it is more useful to compute the fluctuations of the comoving curvature perturbation, as it will be observable at later times.

It is known how a harmonic oscillator can be quantized, so to quantize the inflation field, we will follow the same procedure [8]. We start by promoting $u(\tau, \vec{x})$ and its conjugate momentum $\pi(\tau, \vec{x})$ to quantum operators $\hat{u}$ and $\hat{\pi}$, so they have to satisfy the commutation relation

$$[\hat{u}(\tau, \vec{x}), \hat{\pi}(\tau, \vec{x}')] = i\delta_D(\vec{x} - \vec{x}') \ , \tag{61}$$

where δ_D is the Dirac delta function. As we are treating the perturbations in Fourier space, the commutation relation, in this case, is

$$[\hat{u}_k(\tau), \hat{\pi}_{k'}(\tau)] = i(2\pi)^3 \delta_D(\vec{k} + \vec{k}') \ . \tag{62}$$

It is convenient to express $\hat{u}_k$ in terms of time-independent operators, as we are working on the Heisenberg picture. Therefore,

$$\hat{u}_k(\tau) = u_k(\tau)\hat{a}_{\vec{k}} + u_k^*(\tau)\hat{a}^\dagger_{-\vec{k}} \ , \tag{63}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{-\vec{k}}$ are the annihilation and creation operators used in the quantum oscillator, which are time-independent. The complex mode function $u_k(\tau)$ satisfies the classical Mukhanov-Sasaki equation (55). Inversely transforming $\hat{u}_k(\tau)$, we can finally write $\hat{u}(\tau, \vec{x})$ as

$$\hat{u}(\tau, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\hat{a}_{\vec{k}} u_k(\tau) e^{i\vec{k}\vec{x}} + \hat{a}^\dagger_{\vec{k}} u_k^*(\tau) e^{-i\vec{k}\vec{x}} \right] \ . \tag{64}$$

The final step is to choose the normalization of the mode functions, so we can get the commutation relation for the creation and annihilation operators

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3 \delta_D(\vec{k} - \vec{k}') \ . \tag{65}$$

To obtain a solution for $\hat{u}_k$, a vacuum state that satisfies $\hat{a}_{\vec{k}}(\tau_i)|0\rangle = 0$ needs to be defined. A natural way to define the vacuum state is like the ground state of the Hamiltonian at the beginning of inflation, when $\tau \to -\infty$. At this initial time, all the modes were inside the horizon (subhorizon scales), so we have to resolve eq. (57) which is a harmonic oscillator. Imposing normalization, we obtain the *Bunch-Davies vacuum*

$$u_k(\tau) \xrightarrow[\tau \to -\infty]{} \frac{1}{\sqrt{2k}} e^{-ik\tau} \ . \tag{66}$$

For the slow-roll approximation, an exact solution can be obtained with the Bunch-Davies vacuum. Combining (55) and (56), the Mukhanov-Sasaki equation takes the form (we omit the subscript k for simplicity)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \left(k^2 - \frac{2}{\tau^2} \right) u = 0 \ , \tag{67}$$

where $2/\tau^2$ is obtained knowing that for a constant H, the scale factor in conformal time is $a(\tau) = -1/(H\tau)$. This equation has an exact solution and imposing the Bunch-Davies initial condition we finally get

$$u(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) \ . \tag{68}$$

To obtain the complete solution (without an slow-roll approximation), we have to separate the real and imaginary part of the solution. To numerically compute u, it would be useful to express eq. (55) in terms of N to get

$$u'' + (1-\varepsilon)u' + \left(\frac{k}{aH}\right)^2 u + \left[\frac{1}{H^2}\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\bigg|_{\bar{\phi}} - 2\varepsilon(3+\varepsilon-2\eta) - (2-\varepsilon)\right] u = 0 \ , \tag{69}$$

which has to be solved for the real and imaginary part. The initial conditions defined with the Bunch-Davies vacuum can be written as [14]

$$\mathrm{Re}[u] = \frac{1}{\sqrt{k}}, \quad \mathrm{Im}[u] = 0, \quad \mathrm{Re}\left[\frac{\mathrm{d}u}{\mathrm{d}N}\right] = 0, \quad \mathrm{Im}\left[\frac{\mathrm{d}u}{\mathrm{d}N}\right] = -\frac{\sqrt{k}}{\sqrt{2}k_i} \ , \tag{70}$$

where $k_i \equiv (aH)_i$ is a factor that comes from $u' = (aH)^{-1}\mathrm{d}u/\mathrm{d}\tau$. It is sufficient to use $k_i = k/100$.

4.1.3 Power Spectrum

With all the previous derivations, we can finally predict the quantum statistics of $\hat{u}$ to obtain the zero-point fluctuations of inflation. The expectation value $\langle\hat{u}\rangle$ vanishes due to the annihilation and creation operators. However, the variance of the inflation field receives a nonzero quantum contribution [1] and can be computed as

$$\langle|\hat{u}|^2\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\langle 0|(u_k^*\hat{a}^\dagger_{-\vec{k}} + u_k\hat{a}_{\vec{k}})(u_{k'}\hat{a}_{\vec{k}'} + u_{k'}^*\hat{a}^\dagger_{-\vec{k}'})|0\rangle \ . \tag{71}$$

Using eq. (65) and expressing the integral in Fourier spherical coordinates, we can finally obtain

$$\langle|\hat{u}|^2\rangle = \int \mathrm{d}\ln k \frac{k^3}{2\pi^2}|u_k(\tau)|^2 \ , \tag{72}$$

where the *dimensionless power spectrum* can be defined as

$$\Delta_u^2(k,\tau) \equiv \frac{k^3}{2\pi^2}|u_k(\tau)|^2 \ . \tag{73}$$

It is convenient to work with $\mathcal{R}$, as it is better defined at horizon crossing (it froze). Using eq. (59) and writing $z = a\sqrt{2\varepsilon}$, the comoving curvature power spectrum is

$$\Delta_{\mathcal{R}}^2(k,\tau) \equiv \frac{k^3}{2\pi^2}|\mathcal{R}|^2 = \frac{k^3}{(2\pi)^2\varepsilon}\frac{|u_k|^2}{a^2} \ . \tag{74}$$

The power spectrum can be obtained for slow-roll approximation introducing the Bunch-Davies mode function modulus (68) to get

$$\Delta_{\mathcal{R}}^2 = \frac{k^3}{(2\pi)^2\varepsilon}\frac{H^2}{2k^3}\left(1+\left(\frac{k}{aH}\right)^2\right) = \frac{H^2}{8\pi^2\varepsilon}\left(1+\left(\frac{k}{aH}\right)^2\right) \ , \tag{75}$$

where the approximation often used is that the power spectrum at horizon crossing is

$$\Delta_{\mathcal{R}}^2 = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{Pl}^2}\bigg|_{k=aH} \quad . \tag{76}$$

We have added the Planck mass for completeness. The time dependence of H and ε in the slow-roll regime leads to a small scale dependence of $\Delta_{\mathcal{R}}^2$, so it can be approximated as a power law

$$\Delta_{\mathcal{R}}^2 = A_s\left(\frac{k}{k_*}\right)^{n_s-1} \quad . \tag{77}$$

A_s is known as the *scalar amplitude*, n_s is known as the *spectral index* and k_* is a pivot scale where these variables have to be computed: $A_s \equiv \Delta_{\mathcal{R}}^2(k_*)$ and the spectral index can be obtained as

$$n_s - 1 \equiv \frac{\mathrm{d}\ln\Delta_{\mathcal{R}}^2}{\mathrm{d}\ln k}\bigg|_{k=aH} \approx \frac{1}{H}\frac{\mathrm{d}\ln\Delta_{\mathcal{R}}^2}{\mathrm{d}t} = -4\varepsilon_* + 2\eta_* \ , \tag{78}$$

where it is written as a function of the slow-roll parameters (in the pivot scale).

4.2 Tensor Fluctuations

For tensor perturbations the procedure is analogous to the scalar case. First, we express eq. (43) only in terms of the tensor components to obtain

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\delta_{ij} + h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j \ , \tag{79}$$

where the only tensor perturbation is h_{ij} that can only be gravitational waves. Obtaining a spectrum of primordial gravitational waves is one of the cleanest prediction of inflation.

As in the scalar case, the evolution of h_{ij} needs to be computed using the continuity equation and the Einstein fields equations (see appendix A.3). Perturbations satisfy the wave equation

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 0 \ , \tag{80}$$

or, in Fourier space (defined as in the scalar case),

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0 \ . \tag{81}$$

It is convenient to align the z-axis with the momentum of the mode ($\vec{k} = (0,0,k)$), so the perturbation can be decomposed into the gravitational wave polarizations as

$$\frac{a}{\sqrt{2}}h_{ij} \equiv \begin{pmatrix} u_+ & u_\times & 0 \\ u_\times & -u_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{82}$$

where u_+ and $u_\times$ describe each polarization mode. It is important to note that $h_{ij}^2 = 4(u_+^2 + u_\times^2)/a^2$. The variables u_λ (with $\lambda = +, \times$) satisfy the equation

$$\frac{\mathrm{d}^2 u_\lambda}{\mathrm{d}\tau^2} + \left(k^2 - \frac{1}{a}\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2}\right) u_\lambda = 0 \ , \tag{83}$$

which is similar to the Mukhanov-Sasaki equation (in fact, is the Mukhanov-Sasaki equation derived for the slow-roll approximation in the scalar case). As we have seen, imposing the Bunch-Davies vacuum, we can obtain an exact solution for this equation to compute the power spectrum $\Delta_u^2(t)$. Moreover, we need to multiply the spectrum by 2 because of the two polarization modes. The power spectrum of tensor fluctuations will finally be

$$\Delta_h^2(k) = \frac{4}{a^2} 2\Delta_u^2(k,\tau) \approx \frac{2}{\pi^2}\frac{H^2}{M_{Pl}^2}\bigg|_{k=aH} \ , \tag{84}$$

where the prefactor $4/a^2$ comes from h_{ij}^2 and $\Delta_u^2(k,\tau)$ can be computed as $\Delta_u^2(k,\tau) = z^2\Delta_{\mathcal{R}}^2(k,\tau)$. It is interesting to note that the tensor power spectrum only depends on H, so the measure of Δ_h^2 is a direct measure of the expansion rate. We can also express the power spectrum as a power law

$$\Delta_h^2 = A_t \left(\frac{k}{k_*}\right)^{n_t} \ , \tag{85}$$

where now the tensor amplitude A_t and the *tensor spectral index* n_t are defined. The spectral index is also defined as

$$n_t \equiv \frac{\mathrm{d}\ln\Delta_h^2}{\mathrm{d}\ln k}\bigg|_{k=aH} \approx \frac{1}{H}\frac{\mathrm{d}\ln\Delta_h^2}{\mathrm{d}t} = -2\varepsilon_* \ , \tag{86}$$

in terms of the slow-roll parameters as in the scalar case. Experimentally, n_t is hard to distinguish from zero and has not been detected yet. Moreover, A_t is normally expressed respect the scalar amplitude A_s by defining the *tensor-to-scalar ratio*,

$$r \equiv \frac{A_t}{A_s} = 16\varepsilon_* \ . \tag{87}$$

4.3 The Quartic Potential Revisited

Going back to the quartic potential, all of our inflation perturbations theory can be illustrated with this model. Firstly, we need to define a pivot scale k_* to compute our parameters in that value. This scale is usually $k_* = 0.05$ Mpc^{-1} and from the time this scale exits the horizon $N_* \approx 55$ e-folds of inflation are needed. As we have seen in the inflation background (section 3.4), the number of e-folds respect the inflation field (39) are

$$N_{tot} = \frac{1}{8}\frac{\phi_0^2 - \phi_e^2}{M_{Pl}^2} \approx \frac{1}{8}\frac{\phi_0^2}{M_{Pl}^2} \ , \tag{88}$$

so $N_* \approx \phi_*^2/8M_{Pl}^2$. With this expression, the potential slow-roll parameters (38) can be written as

$$\epsilon_V \approx \frac{1}{N_*}, \qquad \eta_V \approx \frac{3}{2}\frac{1}{N_*} \ . \tag{89}$$

As we are working in the slow-roll approximation, it is convenient to define the scalar amplitude and the scalar spectral index in terms of the potential slow-roll parameters to obtain

$$A_s = \frac{1}{24\pi^2 \epsilon_{V,*}} \frac{V(\phi_*)}{M_{Pl}^4} = \frac{\lambda}{192\pi^2}\left(\frac{\phi_*}{M_{Pl}}\right)^6 , \tag{90}$$

$$n_s - 1 = -6\epsilon_{V,*} + 2\eta_{V,*} \ , \tag{91}$$

where we have used eq. (30) to express H^2 in A_s.

In section 3.4, the parameter λ could be arbitrary, as we did not compare it with any observational constraints. In this case, the observational values for scalar amplitude and spectral index are $A_s = (2.100 \pm 0.030) \cdot 10^{-9}$ and $n_s = 0.9665 \pm 0.0038$ for $k_* = 0.05$ Mpc^{-1} [15] (we will see where these results come from in the next section). To meet the value of A_s, the parameter λ has to be $\lambda \approx 4.6 \cdot 10^{-14}$.

The inflation perturbations can be computed with approximations and with the exact solution. The first approach is to assume that the slow-roll conditions are valid while the time inflation lasts. With these assumptions, the equations needed for scalar and tensor power spectrum are (76) and (84). The exact solution has to be numerically computed, by solving the Mukhanov-Sasaki equation (69) numerically and then introducing its modulus into the power spectrum using eqs. (74) and (84). To get the power spectrum respect k (see fig. 5), we take into account the relation $k = aH = a_* e^N H$, where a_* is the scale factor at k_*.

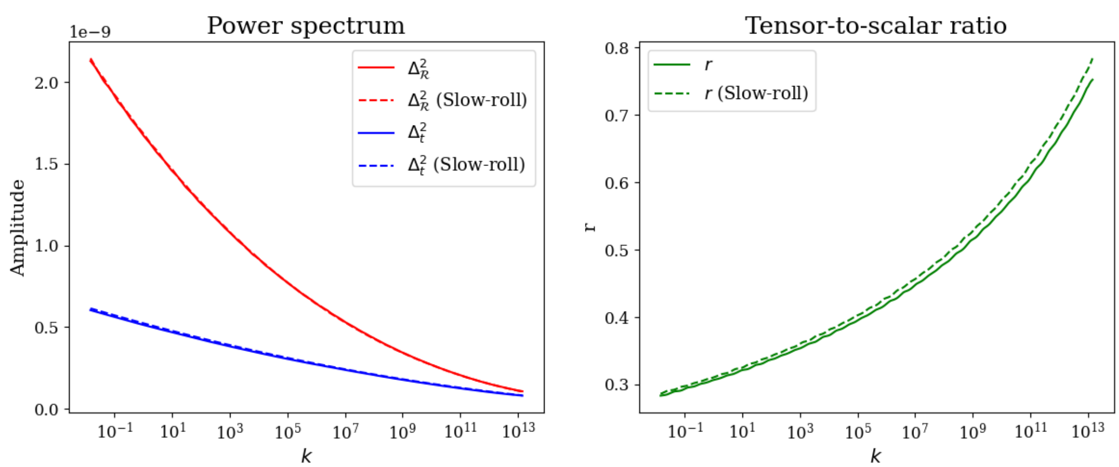

Figure 5: ***Left***: *Power spectrum for scalar and tensor perturbations for the quartic potential, in the slow-roll approximation and without approximations.* ***Right***: *Tensor-to-scalar ratio for the quartic potential, in the slow-roll approximation and without approximations.*

In the left panel of fig. 5, we can observe that the slow-roll approximation and the numerically approach are practically equal, so we can conclude that for the quartic potential the slow-roll approximation is valid. In the right panel, the tensor-to-scalar ratio is plotted, which also has a good correlation between the slow-roll approximation and the numerical computation.

With slow-roll approximations, we can also obtain a value for n_s and r for this model, to see its validity. The spectral index can be calculated with eqs. (89) and (91), and r can be obtained with eq. (87). The values of these variables are

$$n_s = 1 - \frac{3}{N_*} \approx 0.945 \ , \qquad r = 16\epsilon_{V,*} = \frac{16}{N_*} \approx 0.29 \ . \tag{92}$$

A large value for r is inconsistent with the experimental observations [16], also n_s is smaller than expected [17], as we will see later, so this model has to be ruled out.

5 Cosmic Microwave Background

The CMB is the main observational evidence that we have to determine whether our theories of inflation are possible or not. As the fluctuations got imprinted in the CMB when they were still small, they can be described by the linear perturbation theory [1]. These perturbations evolved and are observed at the present time, so it is important to understand their evolution.

The metric perturbations originated in inflation coupled gravitationally to all matter perturbations in the primordial plasma [8]. In this plasma, electrons and baryons were coupled via Coulomb interaction while electrons and photons were coupled via Thomson scattering. At recombination, Thomson scattering became inefficient and the photons started to propagate freely from the decoupling to our times, capturing them in the CMB.

The CMB anisotropies we observe can be described by the measured temperature and polarization of these photons. The temperature fluctuation is easy to describe, because we can measure the deviation of a point in the sky from the average CMB temperature today, which is $T_0 = 2.7$ K [18]. The deviation will, therefore, be

$$\Theta(\hat{n}) \equiv \frac{\delta T(\hat{n})}{T_0} \ . \tag{93}$$

The CMB polarization is produced by the scattering of the photons in the primordial plasma. As the primordial plasma had fluctuations, the polarization was not isotropic. This can be split in two polarization patterns, the *E-mode* and the *B-mode*. The E-mode is created by polarizations parallel or perpendicular to the wavevector $\vec{k}$ and the B-mode is created by a 45 degrees rotation of E-mode polarization directions.

5.1 CMB Power Spectra

The CMB fluctuations are analyzed statistically, comparing points in the sky with the same angle separation. This gives the two-point correlation function, which in the case of temperature anisotropies is

$$C(\theta) \equiv \langle \Theta(\hat{n})\Theta(\hat{n}') \rangle \ , \tag{94}$$

where we have $\cos\theta = \hat{n}\cdot\hat{n}'$. As we observe the CMB on a spherical surface, the temperature fluctuations can be expanded in spherical harmonics and eq. (94) can be expressed as

$$C(\theta) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta) \ , \tag{95}$$

where P_l are the Legendre polynomials and C_l is defined as the angular temperatue power spectrum. This was obtained from the two-point function of the multipole moments as

$$\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'} \ . \tag{96}$$

To see the spherical expansion of the two-point correlation function see appendix B.1.

E- and B-modes can also be analogously expanded in spherical harmonics to obtain the polarization power spectra

$$\begin{aligned} \langle a_{lm} a^{E*}_{l'm'} \rangle =& C_l^{TE} \delta_{ll'} \delta_{mm'} \ , \\ \langle a^E_{lm} a^{E*}_{l'm'} \rangle =& C_l^{EE} \delta_{ll'} \delta_{mm'} \ , \\ \langle a^B_{lm} a^{B*}_{l'm'} \rangle =& C_l^{BB} \delta_{ll'} \delta_{mm'} \ , \end{aligned} \tag{97}$$

where all the other possible combinations are 0 by parity conservation. Observing these power spectra definitions, C_l is better defined as C_l^{TT}.

The angular power spectra are interesting, as we can relate them to the initial curvature perturbations $\Delta^2_{\mathcal{R}}(k)$ for scalar perturbations and to $\Delta^2_h(k)$ for the tensor case. This is done by defining the transfer function $\Theta_l(k)$, which relates the initial perturbations to the power spectrum of the CMB by capturing the evolution of the fluctuations in the plasma and the evolution of the photons after decoupling [8]. The angular power spectra are, therefore,

$$C_l^{(s)} = 4\pi \int \mathrm{d}\ln k \ |\Theta_l^{(s)}(k)|^2 \Delta^2_{\mathcal{R}}(k) \ , \qquad C_l^{(t)} = 4\pi \int \mathrm{d}\ln k \ |\Theta_l^{(t)}(k)|^2 \Delta^2_h(k) \ . \tag{98}$$

The derivation of this expression is in appendix B.2. This definition of C_l can be applied for the temperature fluctuations and also for the polarization modes, but the transfer function will be different for each case.

The angular power spectra can be visualized in fig. 6. We are not going to derive precisely where these spectra come from, we are only going to describe the main features of them.

For temperature fluctuations, the spectrum can be separated into three different regimes. The first region ($l \lesssim 50$ or large-scales) is produced by the fluctuations that entered the horizon after

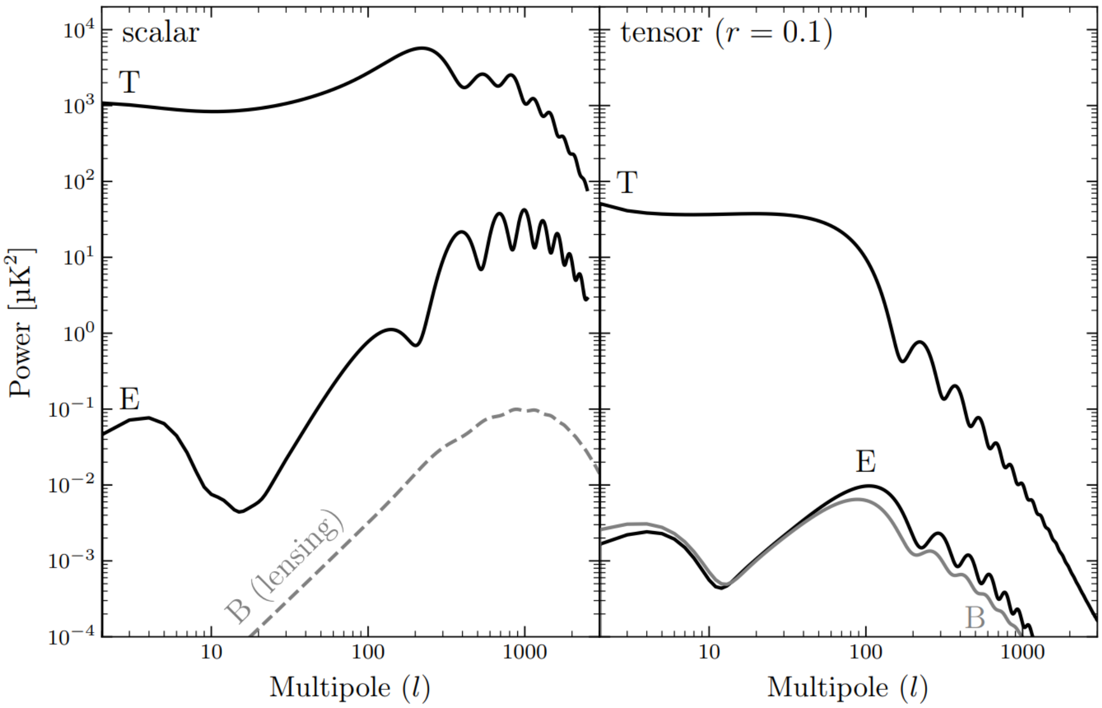

Figure 6: *CMB power spectrum for temperature fluctuations and polarization modes (E-mode and B-mode), for scalar perturbations (left) and for tensor perturbations (right). As the tensor perturbations had not been observed with precision, it is assumed a tensor-to-scalar ratio* $r = 0.1$. *Extracted from [1].*

decoupling, and therefore, they are useful to obtain the initial conditions of the universe. The region between $l \sim 50$ and $l \sim 1000$ is the sound-waves regime, where the perturbations entered the horizon before recombination and were tightly coupled with the primordial plasma, producing the baryon acoustic oscillations (BAO). The last region ($l \gtrsim 1000$ or small scales) is the damping scale, where the random diffusion of the photons led to a damping of the wave amplitudes. For inflation, the most important region to obtain the parameters A_s and n_s is the first one.

The polarization modes of the CMB spectrum provide complementary information to obtain the relevant parameters of the universe. E-mode can produce scalar and tensor perturbations, but B-mode cannot produce scalar primordial fluctuations, so the measure of B-mode is a key signature of primordial gravitational waves. The B-mode observed in the scalar spectrum is due to the gravitational lensing at times after the recombination, so they are not evidence of primordial fluctuations. One of the main objectives of the experimental cosmology is to detect B-mode primordial polarization, as it has not been detected currently (the B-mode from lensing has been detected).

There is an effect present in the power spectrum that we have not mentioned, the reionization after the decoupling. The ultraviolet light from the first stars reionized the universe, and it adds a $e^{-2\tau}$ factor to the power spectrum, where τ is the optical depth. This effect is only relevant on low multipoles ($l < 10$), but it degenerates the measurement of A_s, so what it is actually measured is

$A_s e^{-2\tau}$. To break this degeneration, the optical depth can be measured in the polarization power spectra at low multipoles, as it is responsible for the peak at $l < 10$ in the spectra.

5.2 Observational Constraints

There have been lots of CMB experiments, like WMAP, COBE or Planck [19]. Planck data currently provides the best contraints on the CMB anisotropies [17], except on small angular scales, where BICEP (Background Imaging of Cosmic Extragalactic Polarization) has given better results [20]. However, a few months ago the ACT (Atacama Cosmology Telescopy) has released new results especially relevant for the tensor-to-scalar ratio [16]. We will use the Planck results in combination with the ACT measurements.

As we have said, primordial B-mode polarization has not been detected, so the Planck power spectra obtained are C_l^{TT}, C_l^{TE} and C_l^{EE}. With these spectra, the values for A_s and n_s can be measured, obtaining [17]

$$\begin{aligned} A_s &= (2.100 \pm 0.030) \cdot 10^{-9} \ , \\ n_s &= 0.9665 \pm 0.0038 \ . \end{aligned} \tag{99}$$

The tensor-to-scalar ratio can only be bounded with the current experiments. Combining the ACT result with Planck and BICEP data [16] we obtain

$$r_{0.05} < 0.038 \ , \tag{100}$$

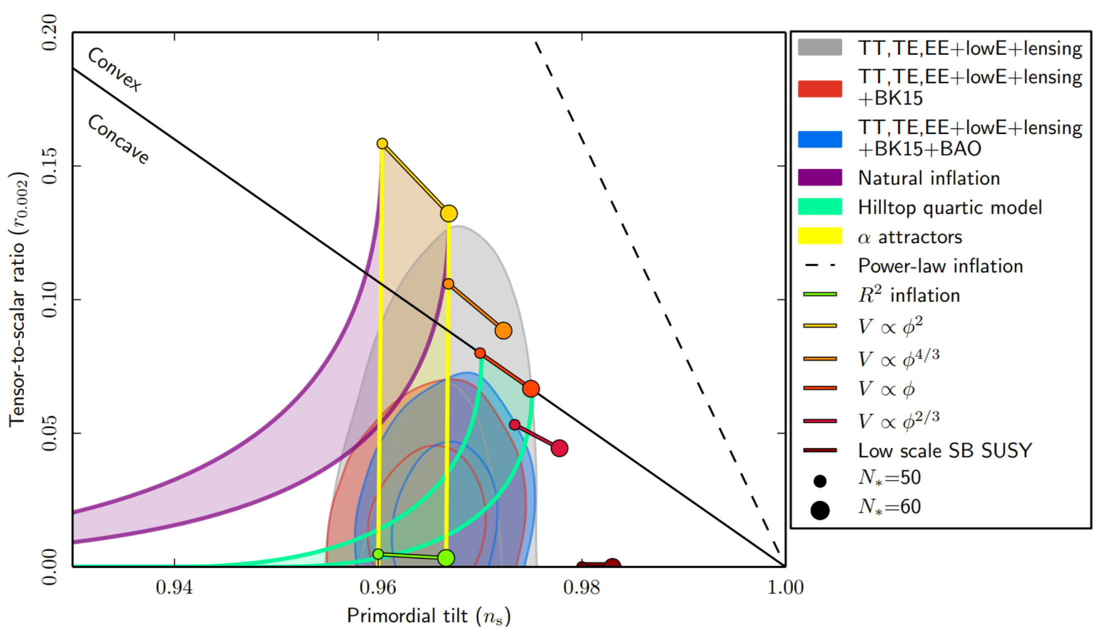

Figure 7: *Planck results for n_s and r in combination also with other experiments like BICEP for $k_* = 0.002$ Mpc^{-1}. The experimental values are compared with some famous theoretical inflation models, for a number of e-folds of inflation between $N_* = 50$ and $N_* = 60$. Extracted from [17].*

where the subscript denotes the scale at which it is bounded ($k_* = 0.05$). Plotting the primordial tilt n_s against the tensor-to-scalar ratio we can get fig. 7 for the Planck data, where the theoretical predictions for some famous inflation models are also plotted.

6 Natural Inflation

To finish our work, we are going to apply all the main features of the cosmological perturbations to a theoretical model in accordance with the CMB constraints, the *natural inflation*. This model proposes that axions (particles described in gauge theory and particle physics) can naturally give rise to an inflationary epoch, at the Grand Unified Theory (GUT) scale [21]. The scalar potential generated is

$$V(\phi) = \Lambda^4[1 + \cos(\phi/f)] \ , \tag{101}$$

where f and Λ are two parameters needed to describe the axions. For appropriately chosen values of the variables, $f \sim m_{Pl}$ and $\Lambda \sim m_{GUT} \sim 10^{16}$ GeV, the field ϕ can drive inflation [22]. It is important to note that $m_{Pl} \equiv \sqrt{1/G}$ is the Planck mass, different from the reduced Planck mass M_{Pl}. With the potential of eq. (101), we can calculate the potential slow-roll parameters (33) as

$$\epsilon_V = \frac{m_{Pl}^2}{16\pi}\left(\frac{\mathrm{d}V/\mathrm{d}\phi}{V}\right)^2 = \frac{1}{16\pi}\left(\frac{m_{Pl}}{f}\right)^2 \frac{\sin^2(\phi/f)}{[1+\cos(\phi/f)]^2} \ , \tag{102}$$

$$\eta_V = \frac{m_{Pl}^2}{8\pi}\frac{\mathrm{d}^2V/\mathrm{d}\phi^2}{V} = \frac{1}{8\pi}\left(\frac{m_{Pl}}{f}\right)^2 \frac{\cos(\phi/f)}{1+\cos(\phi/f)} \ . \tag{103}$$

As in the quartic case, to compute the perturbations parameters, we need to obtain the value of the field at which the scale k_* exited the horizon. To do this, we have to compute the number of e-folds N_* needed. Inflation ends when $\epsilon_V(\phi_e) = 1$, so with eq. (35) the number of e-folds of inflation is

$$N_* = -\frac{8\pi f}{m_{Pl}^2}\int_{\phi_e}^{\phi_*} \frac{1+\cos(\phi/f)}{\sin(\phi/f)}\mathrm{d}\phi = -16\pi\left(\frac{f}{m_{Pl}}\right)^2 \ln\left[\frac{\sin(\phi_*/2f)}{\sin(\phi_e/2f)}\right] \ . \tag{104}$$

The value ϕ_e can be obtained solving the transcendental equation $\epsilon_V = 1$ numerically. With this value, ϕ_* will therefore be

$$\phi_* = 2f \arcsin\left(\sin(\phi_e/2f)\exp\left\{-\frac{N_*}{16\pi}\left(\frac{m_{Pl}}{f}\right)^2\right\}\right) \ . \tag{105}$$

The last step is to combine the potential slow-roll equations (102) and (103) with the slow roll expressions of n_s (91) and r (87), finally obtaining

$$n_s = 1 - \frac{1}{8\pi}\left(\frac{m_{Pl}}{f}\right)^2 \frac{4-[1-\cos(\phi_*/f)]^2}{[1+\cos(\phi_*/f)]^2} \ , \tag{106}$$

$$r = \frac{1}{\pi}\left(\frac{m_{Pl}}{f}\right)^2 \frac{\sin^2(\phi_*/f)}{[1+\cos(\phi_*/f)]^2} \ . \tag{107}$$

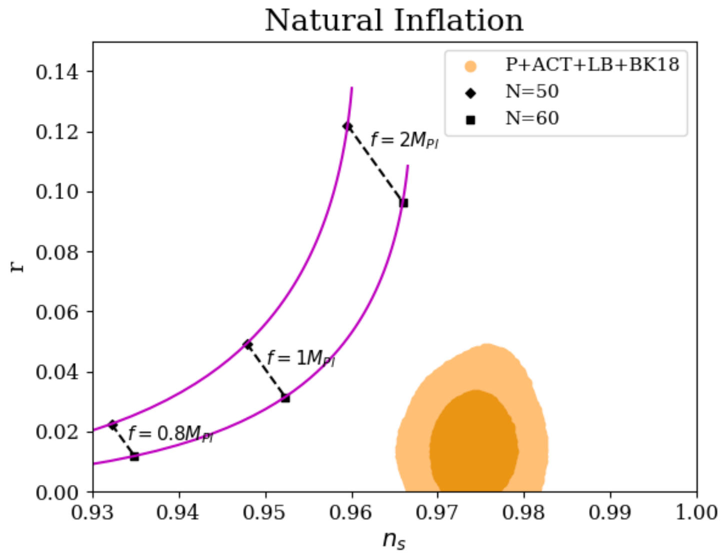

Figure 8: *Theoretical prediction for the range of values* $f = [0.8, 2.5]$ m_{Pl} *and* $N_* = [50, 60]$ *e-folds for the spectral index* (n_s) *and the tensor-to-scalar ratio* (r) *for the natural inflation model. We have also plotted the constraints given by the combination of Planck (P), ACT and BICEP (BK18) data, extracted from [16].*

Varying the parameter f and the number of e-folds between $N_* = 50$ and $N_* = 60$, we can obtain the theoretical predictions for the natural inflation model of n_s and r (see fig. 8). Along with the predictions, the constraints of the latest CMB data (ACT in combination with Planck) are plotted. Looking at fig. 7, the natural inflation is on the edge of the observations, but with the last results in fig. 8 the model is practically ruled out, so other models will be needed in order to agree with the CMB constraints.

7 Conclusions

The theory of inflation is capable of solving the main problems of the Big Bang paradigm with an extremely elegant idea: the universe suffered a period of accelerated expansion before the initial conditions of the Big Bang. As we have seen, this simple statement has a lot of implications.

Expressing inflation with a scalar field is really useful. A slow-roll regime can be defined for the inflation evolution where the equations are simplified. Moreover, to completely describe the inflationary epoch, it is crucial to study the initial perturbations and their evolution, as all the phenomenology seen in the present is due to these perturbations. In this description, the quantum mechanics cannot be forgotten, as the quantum fluctuations are responsible for the origin of perturbations during inflation. Also, the inflation slow-roll regime really helps us with the primordial perturbations because the main variables can be easily obtained with this approximation. For example, the scalar and tensor power spectrum or the primordial tilt.

Luckily, inflation is not a great purely theoretical idea, it can be tested with observations, in particular with the Cosmic Microwave Background, one of the most important observables to understand the evolution of the universe. A better comprehension of the CMB is the key to finally understand the inflation theory, because more inflation models can be tested to comprehend how our universe was originated. Furthermore, the discovery of primordial gravitational waves will be a great accomplishment, as it would give us yet another evidence of the inflationary epoch.

During our project, we have described two main models, the quartic potential and the natural inflation. The first model was ruled out by the observations made some years ago and the natural inflation is practically discarded by the latest experimental values. These results are a sign that more complex models will be needed in the future and that the inflation theory is not a simple task, as a lot of fields in physics have to be considered to fully understand the evolution of our universe.

A Appendix: Metric derivations

A.1 FLRW metric

To compute the covariant derivatives and the Ricci tensor in a curve spacetime, the Christoffel symbols need to be calculated, defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \ . \tag{A.1}$$

For the FLRW metric defined in eq. (1), the nonzero components are

$$\begin{aligned} \Gamma^0_{ij} &= a^2 H \gamma_{ij} \ , \\ \Gamma^i_{0j} &= H \delta^i_j \ , \\ \Gamma^i_{jk} &= \frac{1}{2}\gamma^{il}(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}) \ , \end{aligned} \tag{A.2}$$

where the other symbols are related by symmetry ($\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$). The covariant derivative of a mixed tensor $T^{\mu}{}_{\nu}$ is defined from the Christoffel symbols as

$$\nabla_\sigma T^{\mu}{}_{\nu} = \partial_\sigma T^{\mu}{}_{\nu} + \Gamma^{\mu}_{\sigma\alpha} T^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{\sigma\nu} T^{\mu}{}_{\alpha} \ . \tag{A.3}$$

The Ricci tensor is also defined in terms of the Christoffel symbols as

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Gamma^{\lambda}_{\nu\lambda} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} \ . \tag{A.4}$$

To obtain the continuity equation, we need to express eq. (A.3) as $\nabla_\mu T^{\mu}{}_{\nu}$ and choose $\nu = 0$, so the expression takes the form

$$\nabla_\mu T^{\mu}{}_{0} = \partial_\mu T^{\mu}{}_{0} + \Gamma^{\mu}_{\mu\alpha} T^{\alpha}{}_{0} - \Gamma^{\alpha}_{\mu 0} T^{\mu}{}_{\alpha} \ . \tag{A.5}$$

The only component of $T^{\mu}{}_{\nu}$ that contributes in the first two terms is $T^0{}_0 = \rho$. In the last term, the only Christoffel symbol that appears is $\Gamma^i_{j0} = H\delta^i_j$ and because $T^{\mu}{}_{\nu}$ is diagonal will only contribute as $\Gamma^i_{i0} T^i{}_i$. The equation is then

$$\nabla_\mu T^{\mu}{}_{0} = \partial_0 T^0{}_0 + \Gamma^{\mu}_{\mu 0} T^0{}_0 - \Gamma^i_{i0} T^i{}_i = \dot\rho + 3H\rho + 3HP \ , \tag{A.6}$$

so the continuity equation reads $\dot\rho + 3H(\rho + P) = 0$.

The Ricci tensor in this metric will only have R_{00} and R_{ij} components because the R_{i0} vanish due to isotropy. First, we will derive R_{00} setting $\mu = 0$ and $\nu = 0$:

$$R_{00} = \partial_\lambda \Gamma^{\lambda}_{00} - \partial_0 \Gamma^{\lambda}_{0\lambda} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{00} - \Gamma^{\sigma}_{0\lambda}\Gamma^{\lambda}_{0\sigma} = -\partial_0 \Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i} \ . \tag{A.7}$$

Summing over the spatial indices we get

$$R_{00} = -\frac{\mathrm{d}(3H)}{\mathrm{d}t} - 3H^2 = -3\frac{\ddot{a}}{a} \ . \tag{A.8}$$

For the R_{ij} component, we set $\mu = i$ and $\nu = j$. Following the same procedure as above we can get

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right] g_{ij} \ . \tag{A.9}$$

The next step in order to compute the Friedmann equations is to calculate the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. This is simple because it only has two terms, so

$$R = g^{00}R_{00} + g^{ij}R_{ij} = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] \ . \tag{A.10}$$

Finally, inserting these expressions into $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the nonzero components of the Einstein tensor $G^{\mu}{}_{\nu} = g^{\mu\lambda}G_{\lambda\nu}$ are

$$G^{0}{}_{0} = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] \ , \tag{A.11}$$

$$G^{i}{}_{j} = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right]\delta^i_j \ . \tag{A.12}$$

Equating these with the energy-momentum tensor, the Einstein equations are written as

$$3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] = 8\pi G\rho \ , \tag{A.13}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = -8\pi GP \ . \tag{A.14}$$

Combining these final expressions we get the Friedmann equations (9, 10).

A.2 Scalar perturbations metric

As we have done in the previous section, to compute the continuity equation and the Einstein equations, we need to calculate the Christoffel symbols first. For the metric defined in eq. (48), the nonzero components (at first order) are

$$\begin{aligned}
\Gamma^0_{00} &= \underbrace{\dot{\Phi}}_{O(1)} \\
\Gamma^0_{0i} &= \underbrace{\partial_i\Phi + aH\partial_i B}_{O(1)} \ , \\
\Gamma^0_{ij} &= \underbrace{a^2 H\delta_{ij}}_{O(0)} - \underbrace{a(2aH\Phi\delta_{ij} + \partial_i\partial_j B)}_{O(1)} \ , \\
\Gamma^i_{00} &= \underbrace{\frac{1}{a^2}\partial^i\Phi + \frac{1}{a}\left(H\partial^i B + \partial^i \dot{B}\right)}_{O(1)} \ , \\
\Gamma^i_{j0} &= \underbrace{H\delta^i_j}_{O(0)} \ , \\
\Gamma^i_{jk} &= \underbrace{-aH\partial^i B \ \delta_{jk}}_{O(1)} \ .
\end{aligned} \tag{A.15}$$

To obtain the continuity equation we calculate $\nabla_\mu T^\mu{}_0 = 0$, so we also need an expression for $T^\mu{}_\nu$. Using the definition of the inflaton (49), the energy-momentum tensor can be expressed as

$$\begin{aligned}
T_{\mu\nu} =& \partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi} + V(\bar{\phi})\right) + \\
& 2\partial_{(\mu}\bar{\phi}\partial_{\nu)}\delta\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_{(\alpha}\bar{\phi}\partial_{\beta)}\delta\phi + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\bar{\phi}}\delta\phi\right) \ ,
\end{aligned} \tag{A.16}$$

where we have used

$$V(\phi) = V(\bar{\phi}) + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\bar{\phi}}\delta\phi \ .$$

Splitting (A.16) in components and contracting the first index, an expression of the form $T^\mu{}_\nu = g^{\mu\rho}T_{\rho\nu}$ can be obtained:

$$\begin{aligned}
T^0{}_0 &= \underbrace{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})}_{O(0)} + \underbrace{\dot{\bar{\phi}}(\dot{\delta\phi}) + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\bar{\phi}}\delta\phi - \Phi\dot{\bar{\phi}}^2}_{O(1)} \ , \\
T^0{}_i &= \underbrace{\dot{\bar{\phi}}\partial_i(\dot{\delta\phi})}_{O(1)} \ , \\
T^i{}_0 &= \underbrace{-\left[\partial_j B + \frac{\partial_i(\delta\phi)}{a\dot{\bar{\phi}}}\right]\frac{\delta^{ij}}{a}\dot{\bar{\phi}}^2}_{O(1)} \ , \\
T^i{}_j &= \underbrace{-\delta^i_j\left[\frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi})\right]}_{O(0)} - \underbrace{\delta^i_j\left[\dot{\bar{\phi}}(\dot{\delta\phi}) - \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\bar{\phi}}\delta\phi - \Phi\dot{\bar{\phi}}^2\right]}_{O(1)} \ .
\end{aligned} \tag{A.17}$$

Using eq. (A.6) we find

$$\begin{aligned}\nabla_\mu T^\mu{}_0 =& \left(\partial_0 T^0{}_0\right)^{(0)} + \left(\partial_0 T^0{}_0\right)^{(1)} + \partial_i T^i{}_0 + \Gamma^0_{00} T^0{}_0 + \left(\Gamma^i_{i0} T^0{}_0\right)^{(0)} + \left(\Gamma^i_{i0} T^0{}_0\right)^{(1)} \\ &+ \Gamma^0_{0i} T^i{}_0 + \Gamma^j_{ji} T^i{}_0 - \Gamma^0_{00} T^0{}_0 - \Gamma^0_{i0} T^i{}_0 - \underbrace{\Gamma^i_{00} T^0{}_i}_{O(2)} - \left(\Gamma^i_{j0} T^j{}_i\right)^{(0)} - \left(\Gamma^i_{j0} T^j{}_i\right)^{(1)} \\ =& \left(\partial_0 T^0{}_0\right)^{(0)} + \left(\Gamma^i_{i0} T^0{}_0\right)^{(0)} - \left(\Gamma^i_{j0} T^j{}_i\right)^{(0)} \\ &+ \left(\partial_0 T^0{}_0\right)^{(1)} + \partial_i T^i{}_0 + \left(\Gamma^i_{i0} T^0{}_0\right)^{(1)} - \left(\Gamma^i_{j0} T^j{}_i\right)^{(1)} \ .\end{aligned} \tag{A.18}$$

The zeroth order is simply the scalar field equation (27), as it should be. The first order part is

$$(\ddot{\delta\phi}) + 3H(\dot{\delta\phi}) - \frac{\nabla^2 \delta\phi}{a^2} + \left.\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right|_{\bar\phi} \delta\phi = \underbrace{\frac{\nabla^2 B}{a}\dot{\bar\phi} + \dot{\bar\phi}\dot\Phi + 2\Phi\left(\ddot{\bar\phi} + 3H\dot{\bar\phi}\right)}_{\text{metric perturbations}} . \tag{A.19}$$

where $\nabla^2 \equiv \partial_i \partial^i$.

For the Einstein equations, we compute the Ricci tensor and the Ricci scalar. The Ricci tensor (A.4) split in components is

$$\begin{aligned} R_{00} =& \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^i_{i0}\Gamma^0_{00} - \Gamma^j_{0i}\Gamma^i_{0j} \ , \\ =& \underbrace{-3(\dot H + 3H^2)}_{O(0)} + \underbrace{\frac{1}{a^2}\nabla^2\Phi + 3H\dot\Phi + \frac{1}{a}\left(H\nabla^2 B + \nabla^2 \dot B\right)}_{O(1)} \ , \\ R_{i0} =& -\partial_0 \Gamma^j_{ji} + \Gamma^j_{j0}\Gamma^0_{0i} - \Gamma^j_{00}\Gamma^0_{ij} + \Gamma^j_{jk}\Gamma^k_{0i} - \Gamma^j_{ik}\Gamma^k_{0j} \\ =& \underbrace{2H\partial_i\Phi + a\left(\dot H + 3H^2\right)\partial_i B}_{O(1)} \ , \\ R_{ij} =& \partial_0\Gamma^0_{ij} - \partial_i \Gamma^0_{j0} + \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{jk} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^k_{k0}\Gamma^0_{ij} - \Gamma^k_{i0}\Gamma^0_{jk} - \Gamma^0_{ik}\Gamma^k_{j0} \\ =& \underbrace{a^2(\dot H + 3H^2)\delta_{ij}}_{O(0)} - \underbrace{a^2\left(2(\dot H + 3H^2)\Phi + H\dot\Phi\right)\delta_{ij} - \partial_i\partial_j\Phi}_{O(1)} \\ & \underbrace{-aH\nabla^2 B\delta_{ij} - a\left(2H\partial_i\partial_j B + \partial_i\partial_j \dot B\right)}_{O(1)} \ , \end{aligned} \tag{A.20}$$

and the Ricci scalar is

$$\begin{aligned} R =& g^{00}R_{00} + \underbrace{2g^{i0}R_{i0}}_{O(2)} + g^{ij}R_{ij} \\ =& \underbrace{-6(\dot H + 2H^2)}_{O(0)} + \underbrace{6\left[2(\dot H + 2H^2)\Phi + H\dot\Phi\right] + \frac{2}{a^2}\left(\nabla^2\Phi + 3aH\nabla^2 B + a\nabla^2 \dot B\right)}_{O(1)} \ . \end{aligned} \tag{A.21}$$

The zeroth order is equal to eq. (A.10) (with K=0), so the zeroth order Einstein equations would give the Friedmann equations, as it should be.

From now on, we will only work with the first order Ricci tensor and Ricci scalar for simplicity. We are only going to compute ${G^0}_0$ and ${G^0}_i$, as are the only ones needed to simplify eq. (A.19). We get

$$\begin{aligned} {G^0}_0 =& g^{00}G_{00} + \underbrace{g^{0i}G_{i0}}_{O(2)} = g^{00}R_{00} - \frac{1}{2}R \\ =& -2H\left(3H\Phi + \frac{1}{a}\nabla^2 B\right) \ , \end{aligned} \tag{A.22}$$

$$\begin{aligned} {G^0}_i =& g^{00}G_{0i} + g^{0j}G_{ji} = g^{00}R_{0i} + g^{0j}R_{ji} \\ =& 2\partial_i\left(H\Phi\right) \ . \end{aligned} \tag{A.23}$$

Equating these expressions with $({T^0}_0)^{(1)}$ and ${T^i}_0$ is easy to obtain the relevant Einstein equations

$$H\Phi = \frac{1}{2}\dot{\bar{\phi}}\delta\phi \ , \tag{A.24}$$

$$-\frac{\nabla^2 B}{a} = 3H\Phi + \frac{1}{2H}\left(\dot{\bar{\phi}}(\dot{\delta\phi}) + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\bar{\phi}}\delta\phi - \Phi\dot{\bar{\phi}}^2\right) \ , \tag{A.25}$$

so combining eq. (A.19) with eqs. (A.24) and (A.25) we can obtain the final expression for the inflation field perturbations (50).

A.3 Tensor perturbations metric

With the metric defined in eq. (79) we compute the nonzero Christoffel symbols, which are

$$\begin{aligned} \Gamma^0_{ij} &= \underbrace{a^2 H\delta_{ij}}_{O(0)} - \underbrace{a\left(\frac{1}{2}\dot{h}_{ij} + Hh_{ij}\right)}_{O(1)} \ , \\ \Gamma^i_{j0} &= \underbrace{H\delta^i_j}_{O(0)} + \underbrace{\frac{1}{2}\delta^{il}\dot{h}_{jl}}_{O(1)} \ , \\ \Gamma^i_{jk} &= \underbrace{\frac{1}{2}\delta^{il}\left(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}\right)}_{O(1)} \ . \end{aligned} \tag{A.26}$$

In this case we are only going to compute the Einstein equation for G_{ij}, as is the only one needed to obtain the tensor perturbations. It is important to note that h_{ij} is defined traceless and transverse so ${h^i}_i = 0$ and $\partial^i h_{ij} = 0$. Moreover, the Ricci scalar at linear order cannot receive contribution from tensor perturbations, so

$$R = -6(\dot{H} + 2H^2) \ , \tag{A.27}$$

which is the Ricci scalar for a non perturbed FLRW metric. Therefore, to calculate G_{ij} we only need to compute R_{ij}, getting

$$\begin{aligned} R_{ij} =& \partial_0\Gamma^0_{ij} + \partial_k\Gamma^k_{ij} + \Gamma^k_{k0}\Gamma^0_{ij} - \Gamma^k_{i0}\Gamma^0_{jk} - \Gamma^0_{ik}\Gamma^k_{j0} \\ =& \underbrace{a^2(\dot{H}+3H^2)\delta_{ij}}_{O(0)} - \underbrace{\frac{a^2}{2}\left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij}\right) + a^2(\dot{H}+3H^2)h_{ij}}_{O(1)} \ . \end{aligned} \tag{A.28}$$

With R_{ij} is straightforward to calculate $G_{ij} = R_{ij} - 1/2Rg_{ij}$, which at first order is

$$\delta G_{ij} = \frac{a^2}{2}\left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij}\right) - a^2(2\dot{H}+3H^2)h_{ij} \ . \tag{A.29}$$

In order to obtain the Einstein equation, we need to compute lastly $\delta G_{ij} = \delta T_{ij}$. To do this we recall the definition of T_{ij} given in eq. (2) and perturbed it at first order

$$\delta T_{ij} = \delta P a^2 \delta_{ij} + a^2 \bar{P} h_{ij} \ , \tag{A.30}$$

which the only tensor component is $\delta T_{ij} = a^2\bar{P}h_{ij}$. Combining the Friedmann equations (9) and (10) we can obtain an expression for $\bar{P}$ (with $M_{Pl} = 1$),

$$\bar{P} = -2\dot{H} + 3H^2 \ , \tag{A.31}$$

so equating (A.29) with δT_{ij} we finally obtain

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 0 \ . \tag{A.32}$$

B Appendix: CMB derivations

B.1 Expansion in spherical harmonics

To obtain the two-point correlation function, $\Theta(\hat{n})$ needs to be expanded in terms of the spherical harmonics [1], so

$$\Theta(\hat{n}) = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\hat{n}) \ , \tag{B.1}$$

where Y_{lm} are the spherical harmonics and a_{lm} are the multipole moments. The two-point correlation function of the multipole moments is defined in (96), so introducing this expression into eq. (94) we have

$$C(\theta) = \sum_{l}\sum_{m}\sum_{l'}\sum_{m'}\langle a_{lm}a^*_{l'm'}\rangle Y_{lm}(\hat{n})Y^*_{l'm'}(\hat{n}') = \sum_{l} C_l \sum_{m} Y_{lm}(\hat{n})Y^*_{lm}(\hat{n}') \ . \tag{B.2}$$

We can express the spherical harmonics in terms of the Legendre polynomials using the relation

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_m Y_{lm}(\hat{n})Y^*_{lm}(\hat{n}') \ , \tag{B.3}$$

so eq. (B.2) takes the form

$$C(\theta) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta) \ , \tag{B.4}$$

which is eq. (95).

B.2 Transfer Function

We are going to derive the angular power spectrum in terms of a transfer function only for the temperature scalar case [8], but it is analogous for the other cases. We start expressing $\Theta(\hat{n})$ in Fourier space,

$$\Theta(\hat{n}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}_*}\Theta(\eta_*, \vec{k}, \hat{n}) \ , \tag{B.5}$$

where the subscript $*$ is the moment of last-scattering of the photons (when they decoupled). The exponential term can be expanded in terms of the Legendre polynomials to obtain

$$\Theta(\hat{n}) = \sum_l i^l(2l+1)\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\Theta_l(k)\mathcal{R}(\vec{k})P_l(\hat{k}\cdot\hat{n}) \ , \tag{B.6}$$

where the transfer function is defined as $\Theta_l(k) \equiv \Theta_l(\eta_*, \vec{k})/\mathcal{R}(\vec{k})$. Here we have introduced the curvature perturbation $\mathcal{R}$, so now we can obtain the power spectrum $\Delta^2_{\mathcal{R}}$ from the two-point correlation. We write

$$\begin{aligned}\langle\Theta(\hat{n})\Theta(\hat{n}')\rangle = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{\mathrm{d}^3\vec{k}'}{(2\pi)^3}\sum_l\sum_{l'} i^{l+l'}(2l+1)(2l'+1)\Theta_l(k)\Theta_{l'}(k') \\ \langle\mathcal{R}(\vec{k})\mathcal{R}(\vec{k}')\rangle P_l(\hat{k}\cdot\hat{n})P_{l'}(\hat{k}'\cdot\hat{n}')\end{aligned} \tag{B.7}$$

and with

$$\langle\mathcal{R}(\vec{k})\mathcal{R}(\vec{k}')\rangle = \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}(2\pi)^3\delta_D(\vec{k}+\vec{k}') \tag{B.8}$$

we can obtain

$$\begin{aligned}\langle\Theta(\hat{n})\Theta(\hat{n}')\rangle = \frac{1}{4\pi}\int \frac{\mathrm{d}k}{k}\Delta^2_{\mathcal{R}}(k)\sum_l\sum_{l'} i^{l+l'}(2l+1)(2l'+1)\Theta_l(k)\Theta_{l'}(k) \\ \int \mathrm{d}^2\hat{k}\ P_l(\hat{k}\cdot\hat{n})P_{l'}(\hat{k}\cdot\hat{n}') \ .\end{aligned} \tag{B.9}$$

Using the relation

$$\int \mathrm{d}^2\hat{k}\ P_l(\hat{k}\cdot\hat{n})P_{l'}(\hat{k}\cdot\hat{n}') = \frac{4\pi}{2l+1}P_l(\hat{n}\cdot\hat{n}')\delta_{ll'} \ , \tag{B.10}$$

eq. (B.7) can finally be expressed as

$$\langle \Theta(\hat{n})\Theta(\hat{n}')\rangle = \sum_l \frac{2l+1}{4\pi} \underbrace{\left[4\pi \int \mathrm{d}\ln k \; \Theta_l(k)^2 \Delta^2_{\mathcal{R}}(k)\right]}_{=C_l} P_l(\cos\theta) \ , \tag{B.11}$$

so we have obtained the expression for the angular power spectrum of eq. (98).

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