Linear Predictors + Perceptron

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Lecture Overview

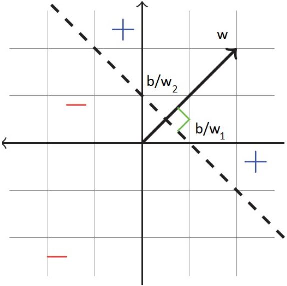
1. Linear Predictors

2. Perceptron

Halfspaces

The class of halfspace classifiers separate instances using a family of hyperplanes $\mathbb{L} = \{L_n\}_{n=1}^{\infty}$ where $L_n = \{H = [f : d] \mid f \equiv \overline{w} \in \mathbb{R}^n,\ d \in R\}$ to separate classes, i.e. hyperplane classifier $H = [f : d]$ where $f \equiv \overline{w} \in \mathbb{R}^n$ computes class y for an instance $\overline{x} \in \mathbb{R}^n$ as $y = \mathbf{sign}(\overline{w} \bullet \overline{x} - d)$.

Solving hyperplane equation $\overline{w} \bullet \overline{x} = d$ gives that hyperplane intercepts with i^{th} axis at $\frac{d}{w_i}$.



If we are looking for a linear predictor that is ERM predictor w.r.t realizable PAC learning case, then for a given sample set $S = \{(\overline{x}_1, y_1), \ldots, (\overline{x}_m, y_m)\}$ we need to find $\overline{w}$ and $d \in \mathbb{R}$ such that for every $i \in \{1, \ldots, m\}$ we have $\mathbf{sign}(\overline{w} \bullet \overline{x}_i - d) = y_i$. Equivalently $y_i \cdot (\overline{w} \bullet \overline{x}_i - d) > 0, \quad \forall i = 1, \ldots, m$

There must be a solution for the system of m inequalities $y_i \cdot (\overline{w} \bullet \overline{x}_i - d) > 0, \quad \forall i = 1, \ldots, m$ because we assume that it is realizable case. To solve we could use Linear programming, but inequalities are strict so LP is inapplicable.

Halfspaces

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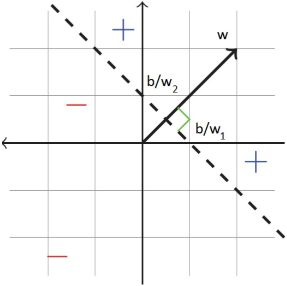

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To find equivalent system with $\geq$ constraints, suppose $\overline{w}^*, d^*$ is a solution. Let then $\gamma = \min_i(y_i(\overline{w}^* \bullet \overline{x}_i - d^*))$ and let $\overline{w}^\dagger = \frac{\overline{w}^*}{\gamma}$ and $d^\dagger = \frac{d^*}{\gamma}$. Then $\frac{1}{\gamma} y_i \cdot (\overline{w}^* \bullet \overline{x}_i - d^*) = y_i \cdot (\overline{w}^\dagger \bullet \overline{x}_i - d^\dagger) \geq 1, \quad \forall i = 1, \ldots, m$. Vector satisfying these conditions can be found by solving LP with dummy objective $(\min 0)$.

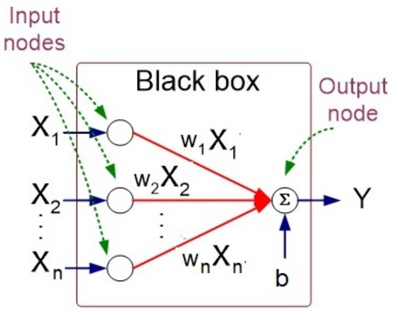
Lecture Overview

1. Linear Predictors

2. Perceptron

Perceptron Device

- The device is an assembly of inter-connected nodes,
- In the simplest variant an input node i does multiplication of an input by a constant w_i and outputs it,
- Output node sums values of its input and compares against threshold b. If the sum is less than b then it outputs 1, otherwise it outputs -1.

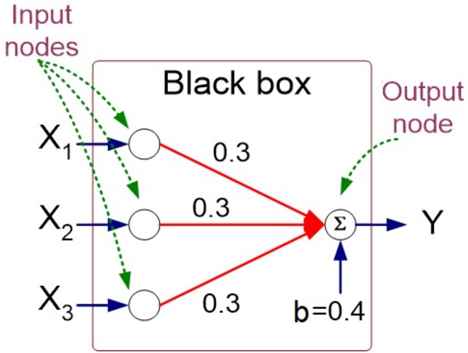


In other words perceptron computes $y = \mathbf{sign}(\overline{w} \bullet \overline{x} - b)$.

Perceptron Device

Example: Majority classifier

X_1	X_2	X_3	Y
1	0	0	-1
1	0	1	1
1	1	0	1
1	1	1	1
0	0	1	-1
0	1	0	-1
0	1	1	1
0	0	0	-1



$$y = \mathbf{sign}\left[\begin{pmatrix}0.3\\0.3\\0.3\end{pmatrix}^T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - 0.4)\right] = \mathbf{sign}\left[0.3x_1 + 0.3x_2 + 0.3x_3 - 0.4\right].$$

Training Perceptron - Desiderata

Find a weight vector $\overline{w}$ and threshold b defining a hyperplane h such that

- all positive examples $\overline{x}_i^+$ from the training set (i.e. the feature vectors from the training set with $y = 1$) are on the positive side of the hyperplane h (i.e. $\overline{w} \bullet \overline{x}_i^+ - b > 0$),
- all negative examples (i.e. feature vectors with $y = -1$) are on the negative side of the hyperplane$h$ (i.e. $\overline{w} \bullet \overline{x}_j^- - b < 0$)

We simplify the task by noticing that $\overline{w} \bullet \overline{x} + b = \overline{w}' \bullet \overline{x}'$ where $\overline{w}' = (\overline{w}^T,\ b)^T$ and $\overline{x}' = (\overline{x}^T,\ 1)^T$.

- In the example $\overline{w}' = (\overline{w}^T b)^T = (0.3, 03, 0.3, -0.4)^T$ and $\overline{x}' = (\overline{x}^T,\ 1)^T = (x_1, x_2, x_3, 1)^T$ which makes

$$\overline{w}' \bullet \overline{x}' = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ -0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = 0.3x_1 + 0.3x_2 + 0.3x_3 - 0.4.$$

Training Perceptron - Take 1

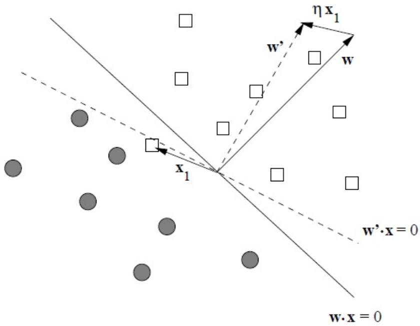
Modified Task: find $\overline{w}'$ such that $\overline{w}' \bullet (\overline{x}_i^+)' > 0$ and $\overline{w}' \bullet (\overline{x}_i^-)' < 0$ for all positive and negative eaxmples.
The following method converges to some hyperplane that separates the positive and negative examples, provided one exists.

1. Initialize the weight vector $\overline{w}$ to all 0's,
2. Pick a learning-rate parameter η, which is a small, positive real number,
3. While $\overline{w}$ keeps changing consider each training example $t = (\overline{x}, y)$ in turn:
   i. Compute $y' = \overline{w} \bullet (\overline{x}^T, 1)^T$;
   ii. If y' and y have different signs or $y' = 0$, then
      replace $\overline{w}$ by $\overline{w} + \eta \cdot y \cdot (\overline{x}^T, 1)^T$. That is, adjust $\overline{w}$ slightly in the direction of $\overline{x}$.

      else do nothing (t is properly classified).

Training Perceptron - What is Going on?

- Moving $\overline{w}$ towards $\overline{x}$ moves the orthogonal hyperplane h in such a direction that it makes it more likely that $\overline{x}$ will be on the correct side of the hyperplane, although it does not guarantee that $\overline{x}$ will then be correctly classified.
- The choice of η affects the convergence of the perceptron. If η is too small, then convergence is slow; if it is too big, then the decision boundary will dance around and again will converge slowly, if at all.



Training Perceptron - Example

from Lescovec, Rajaraman and Ullman. Mining of Massive Datasets

Perceptron to recognize spam email:

- The training set $(\overline{x}, y)$ where $\overline{x}$ is a vector of 0s and 1s, with value of x_i corresponding to the presence or absence of a word x_1 in the eemail,
- Since all values are boolean b must be 0 so to simplify computations we work with $\overline{w}$ instead of $\overline{w}'$ and with $\overline{x}$ instead of $\overline{x}'$,
- Tarining set consistst of 6 emails $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$.
- We use learning rate $\eta = 1/2$.

	and	viagra	the	of	nigeria	y
a	1	1	0	1	1	+1
b	0	0	1	1	0	−1
c	0	1	1	0	0	+1
d	1	0	0	1	0	−1
e	1	0	1	0	1	+1
f	1	0	1	1	0	−1

Step 1: $\overline{w}_1 = (0, 0, 0, 0, 0)^T$ and $y'_{\overline{a}} = \overline{w} \bullet \overline{a} = 0$; case $3.\mathrm{i}$, so $\overline{w} := \overline{w} + \eta \cdot y_{\overline{a}} \cdot \overline{a}$ or $\overline{w}_1 := (0, 0, 0, 0, 0)^T + (1/2)(+1)(1, 1, 0, 1, 1) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})^T$.

Training Perceptron - Example

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Step 2: $y'_{\overline{b}} = \overline{w}_1 \bullet \overline{b} = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})(0,0,1,1,0)^T = \frac{1}{2}$, but $y_{\overline{b}} = -1$, i.e. signs are different; case 3.i, so $\overline{w}_2 := (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})^T + (1/2)(-1)(0,0,1,1,0) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2})^T$.

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Step 4: $y'_{\overline{d}} = \overline{w}_3 \bullet \overline{d} = 1$, but $y_{\overline{d}} = -1$; case 3.i, so
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Step 5: $y'_{\overline{e}} = \overline{w}_4 \bullet \overline{e} = \frac{1}{2}$, and $y_{\overline{e}} = 1$; case $3.\text{ii}$, do nothing.

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Step 6: $y'_{\overline{f}} = \overline{w}_4 \bullet \overline{f} = -\frac{1}{2}$, and $y_{\overline{f}} = -1$; case $3.\mathrm{ii}$, do nothing.

If we now check $\overline{a}$ through $\overline{d}$, we find that this current $\overline{w}_4$ correctly classifies them all.

Perceptron Convergence

What means that for a given training dataset D there is a separating hyperplane h that passes through the origin? This means that there is $\delta > 0$ such that marginal distance of any data point in D from h is at least $\delta > 0$. Let $\overline{w}^*$ be a unit vector orthogonal to h. Then marginal distance of $\overline{x}$ from h is the length of its projection onto $\overline{w}^*$. Since $\overline{w}^*$ is a unit vector the projection is $(\overline{x} \bullet \overline{w}^*)\overline{w}^*$. So existence of separating plane for D means that $\|(\overline{x} \bullet \overline{w}^*)\overline{w}^*\| = |\overline{x} \bullet \overline{w}^*| \geq \delta$.

Theorem (Convergence)

If for a given training dataset D there is a separating hyperplane h that passes through the origin then the perceptron training converges in at most $\left(\frac{\max_{\overline{x} \in D} \|\overline{x}\|}{\delta}\right)^2$ iterations of step 3.

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Proof.

Let $\overline{w}_{n+1}$ denote the weight vector after $n+1$ iterations, let $\overline{w}^*$ be a unit vector orthogonal to h and let δ be separating margin. If we obtained $\overline{w}^{n+1}$ then $\overline{w}_n$ did not classify D correctly and $\overline{w}_{n+1} = \overline{w}_n + y\eta\overline{x}$ where $\overline{x}$ was last misclassified feature vector. Then

$$\begin{array}{rcl} \overline{w}_{n+1} \bullet \overline{w}^* & = & (\overline{w}_n + y\eta\overline{x}) \bullet \overline{w}^* \\ & = & \overline{w}_n \bullet \overline{w}^* + y\eta\overline{x} \bullet \overline{w}^* \\ & = & \overline{w}_n \bullet \overline{w}^* + \eta|\overline{x} \bullet \overline{w}^*| \\ & \geq & \overline{w}_n \bullet \overline{w}^* + \eta\delta \end{array}.$$

By Cauchy-Schwartz we have $\overline{w}_{n+1} \bullet \overline{w}^* \leq \|\overline{w}_{n+1}\|\,\|\overline{w}^*\|$, so since $\|\overline{w}\|^* = 1$ holds $\|\overline{w}_{n+1}\| \geq \overline{w}_n \bullet \overline{w}^* + \eta\delta$. □

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Notice that

$$\begin{aligned}
\|\overline{w}_{n+1}\|^2 &= \|\overline{w}_n + y\eta\overline{x}\|^2 \\
&= \|\overline{w}_n\|^2 + 2y\eta(\overline{x} \bullet \overline{w}_n) + \eta^2 \|\overline{x}\| \\
&\leq \|\overline{w}_n\|^2 + \eta^2(\max_{\overline{x}\in D} \|\overline{x}\|)^2
\end{aligned}$$

The last line is justified because $2y\eta(\overline{x} \bullet \overline{w}_n) \leq 0$ which holds because otherwise $\operatorname{sgn} y \neq \operatorname{sgn} \overline{x} \bullet \overline{w}_n$ because otherwise no update is done. $\square$

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Proof.

We proved $\|\overline{w}_{n+1}\| \geq \overline{w}_{n+1} \bullet \overline{w}^* \geq \overline{w}_n \bullet \overline{w}^* + \eta\delta$ and $\|\overline{w}_{n+1}\|^2 \leq \|\overline{w}_n\|^2 + \eta^2(\max_{\overline{x}\in D}\|\overline{x}\|)^2$. So since we started with $\overline{w}_1 = \overline{0}$ by induction after N actual updates $\|\overline{w}_N\|^2 \leq N\eta^2(\max_{\overline{x}\in D}\|\overline{x}\|)^2$ and $\|\overline{w}_N\| \geq \overline{w}_N \bullet \overline{w}^* \geq N\eta\delta$. Combining we get

$$(N\eta\delta)^2 \leq (\overline{w}_N \bullet \overline{w}^*)^2 \leq \|\overline{w}_N\|^2 \leq N\eta^2(\max_{\overline{x}\in D}\|\overline{x}\|)^2$$

thus training converges in $N \leq \left(\frac{\max_{\overline{x}\in D}\|\overline{x}\|}{\delta}\right)^2$ updates. $\square$

Dealing with Perceptron Convergence

- If the data points are linearly separable, then the perceptron will eventually find a separator hyperplane. But if margin δ is very small it'll take very long time
- If the data is not linearly separable, then the perecptron will eventually repeat a weight vector and loop infinitely

Yet if convergence rate is very slow then the two cases are indistinguishable. Need a termination strategy. Possible solutions:

- Termination strategy:
    1. Terminate after a fixed number of rounds,
    2. Terminate after normalized distance between true classification vector $\overline{y}$ and computed classification vector $\overline{y}'$ becomes small, i.e. $\frac{\|\overline{y}-\overline{y}'\|}{n} \leq \theta$ where θ is some small constant,
    3. Terminate when the number of misclassified training points stops changing.

Speedup Improvement Ideas

- Lower the training rate as the number of rounds increases. For example, start with the training rate η_0 and lower it to $\frac{\eta_0}{1+ct}$ after the t^{th} round, where c is some small constant. Works well inmost cases in practice but no convergence guarantee.
- Instead of η use flexible rate increases for different components of $\overline{w}$:
  - For a training data vector $\overline{x}$ that is in the class $y_{\overline{x}} = +1$ if current $\overline{w}$ is such that $\overline{w} \bullet \overline{x}$ is small/negative then we have to raise the weights w_i in those components x_i where value of x is positive and relatively large. Thus multiplier for w_i for these components must be $c > 1$. Usually this is implemented by taking $c = 2$
  - For a training data vector $\overline{x}$ that is in the class $y_{\overline{x}} = -1$ if current $\overline{w}$ is such that $\overline{w} \bullet \overline{x}$ is large/positive then we have to lower the weights w_i in those components x_i where value of x is positive and relatively large. Thus multiplier for w_i for these components must be < 1. Usually this is implemented by taking $c = \frac{1}{2}$

  Implementation of this idea with appropriate values of multipliers guarantees convergence and in most cases works faster than basic perceptron.