# Introduction to univariate AR state-space models

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*FISH 507 – Applied Time Series Analysis*

21 January 2019

# Points from Thursday

- Data affected by a perturbation is problematic for arima(), Arima().
- Seasonal ARIMA has effect of Jan (or Feb …) in year t on Jan (or Feb …) in year t+1. Not typical when working with population data.
- Removing the mean season is different than a seasonal difference.
- Data with multiple seasons (daily, monthly, yearly) will be problematic for standard ARIMA seasonal models.
- Linear effects of past values might be problematic.

# Weeks 1-3.5: building blocks for analysis of multivariate time-series data with observation error, structure, and missing values

- Matrix math (multivariate)
- Properties of time series data
- AR and MA models
- ➢ State-space models: observation + process model
- Model evaluation and model selection
- Fitting models with STAN (non-linear, non-Gaussian, disparate data streams)

Starting next week: we will put this all together to start analyzing ecological data sets

# univariate linear state-space model

$$x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q)$$
$$y_t = x_t + v_t, \quad v_t \sim Normal(0, r)$$

The x model is the classic “random walk”.
This model is a random walk observed with error.

# univariate linear state-space model

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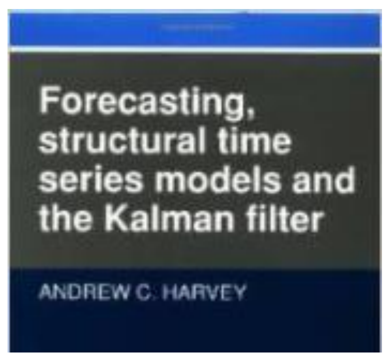
$$y_t = x_t + v_t, \quad v_t \sim Normal(0, r)$$



Many textbooks on this class of model. Used in extensively in economics and engineering

# Definition: AR-1 or AR lag-1

Value at time t is the value at time t-1 plus random error

$$x_t = x_{t-1} + u + w_t$$

$$x_{t+1} = x_t + w_t$$

$$x_t = bx_{t-1} + u + w_t$$

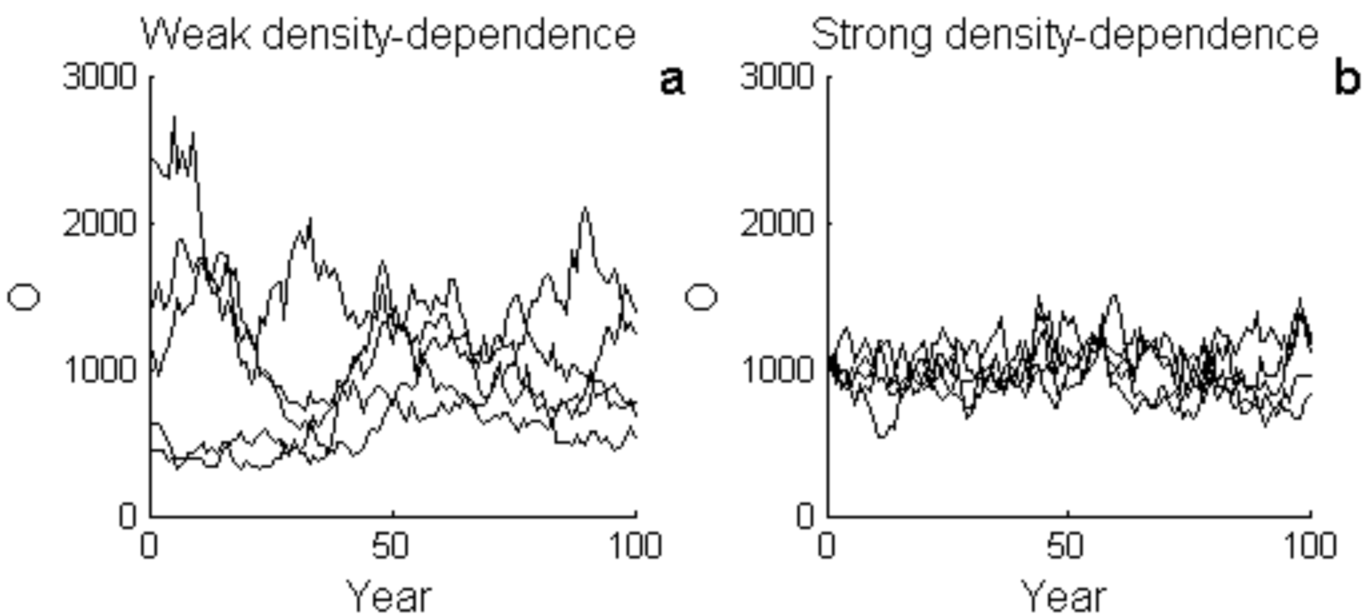
# Addition of "b" (<1) leads to process model with mean-reversion,

$$N_t = \exp(u + e_t) N_{t-1}^b$$

$$x_t = b x_{t-1} + u + e_t$$

Log-space

$$e_t \sim Normal(0, q)$$




b<1: Gompertz density-dependent process

# This model is quite hard to fit

$$N_t = \exp(u + e_t) N_{t-1}^b$$

$$x_t = b x_{t-1} + u + e_t$$

Log-space

$$e_t \sim Normal(0, q)$$

b and u are confounded = ridge likelihood = many b/u combinations that fit the data

If you have observation error, you need either long times or replication to estimate this model.

# Why is the AR-1 model so important in analysis of ecological data?

Additive random walks

- Movement, changes in gene frequency, somatic growth if growth is by fixed amounts

$$x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q)$$

Why normal? The average of many small perturbations, regardless of their distribution, is normal

Multiplicative random walks

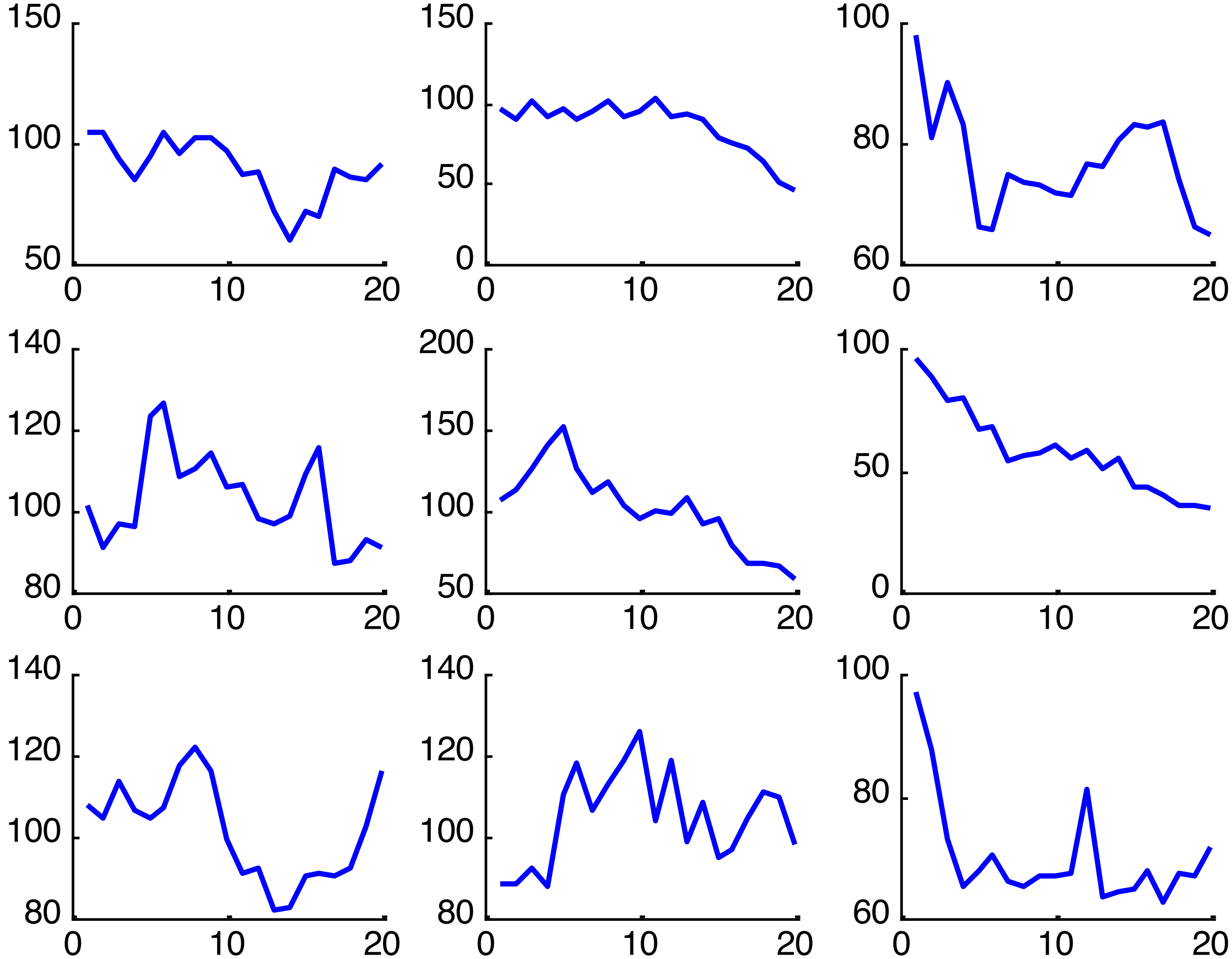
- Population growth, somatic growth if growth is by percentage

$$n_t = \lambda n_{t-1} w_t, \quad w_t \sim \log\text{-}Normal(0, q)$$

- take log and you get the linear additive model above. log-normal means that 10% increase is as likely as 10% decrease

# An AR-1 random walk can show a wide-range of trajectories, even for the same parameter values

All trajectories came from the same rw model: $x_t = x_{t-1} - 0.02 + e_t$, $e_t \sim$ Normal(mean=0.0, var=0.01)
same as the "stochastic exponential growth model": $N_t = N_{t-1} \exp(-0.02 + e_t)$



# Definition: state-space

The "state", the x, is a hidden (dynamical) variable. In this class, it is a **hidden random walk.**

Our data, y, are observations of this.

*Often state-space models include inputs (explanatory variables). and typically at least the x is multivariate, and often also y.*

The model you are seeing today is a simple univariate state-space model with no inputs.

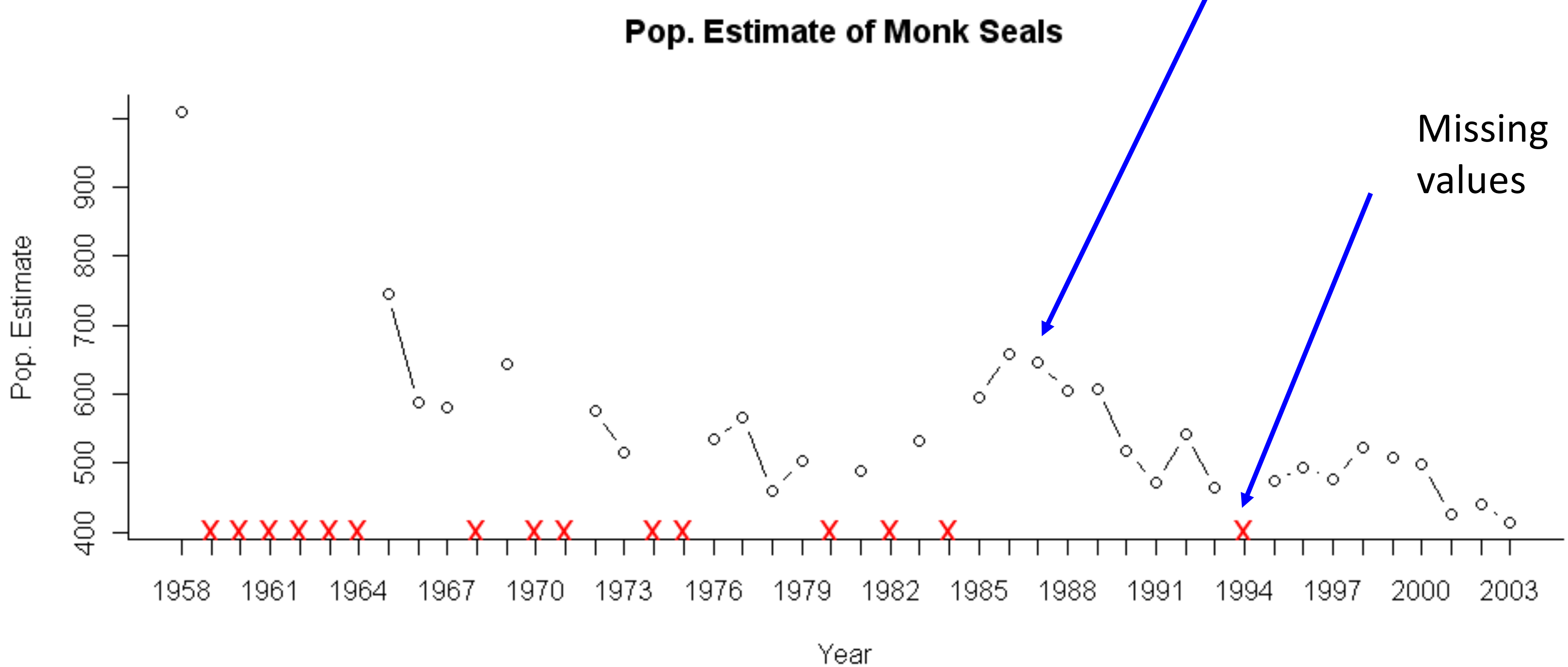
state process $x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q)$

obs process $y_t = x_t + v_t, \quad v_t \sim Normal(0, r)$

# univariate example: population count data

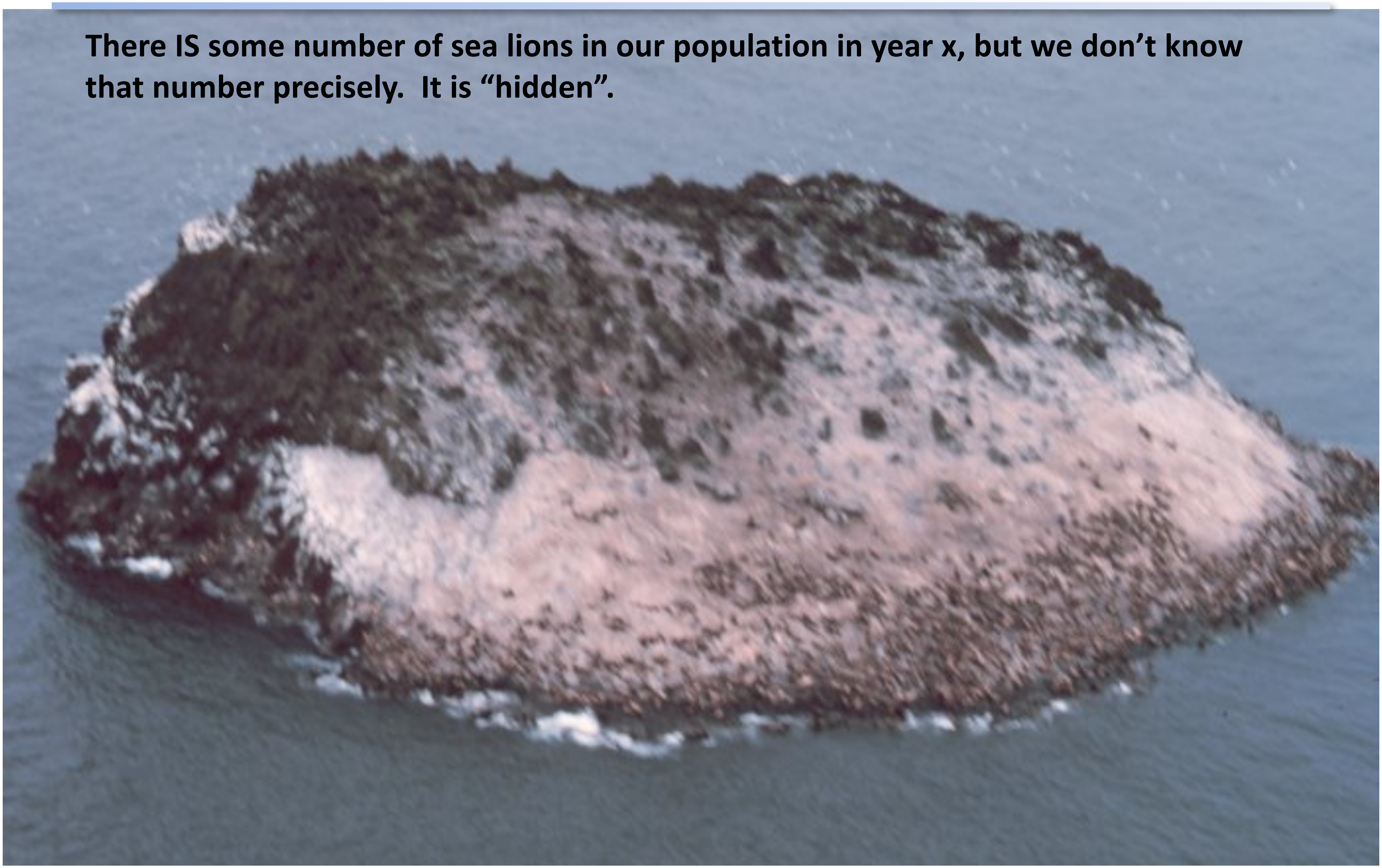
Yearly (usually) population (or subpopulation) counts

Missing values

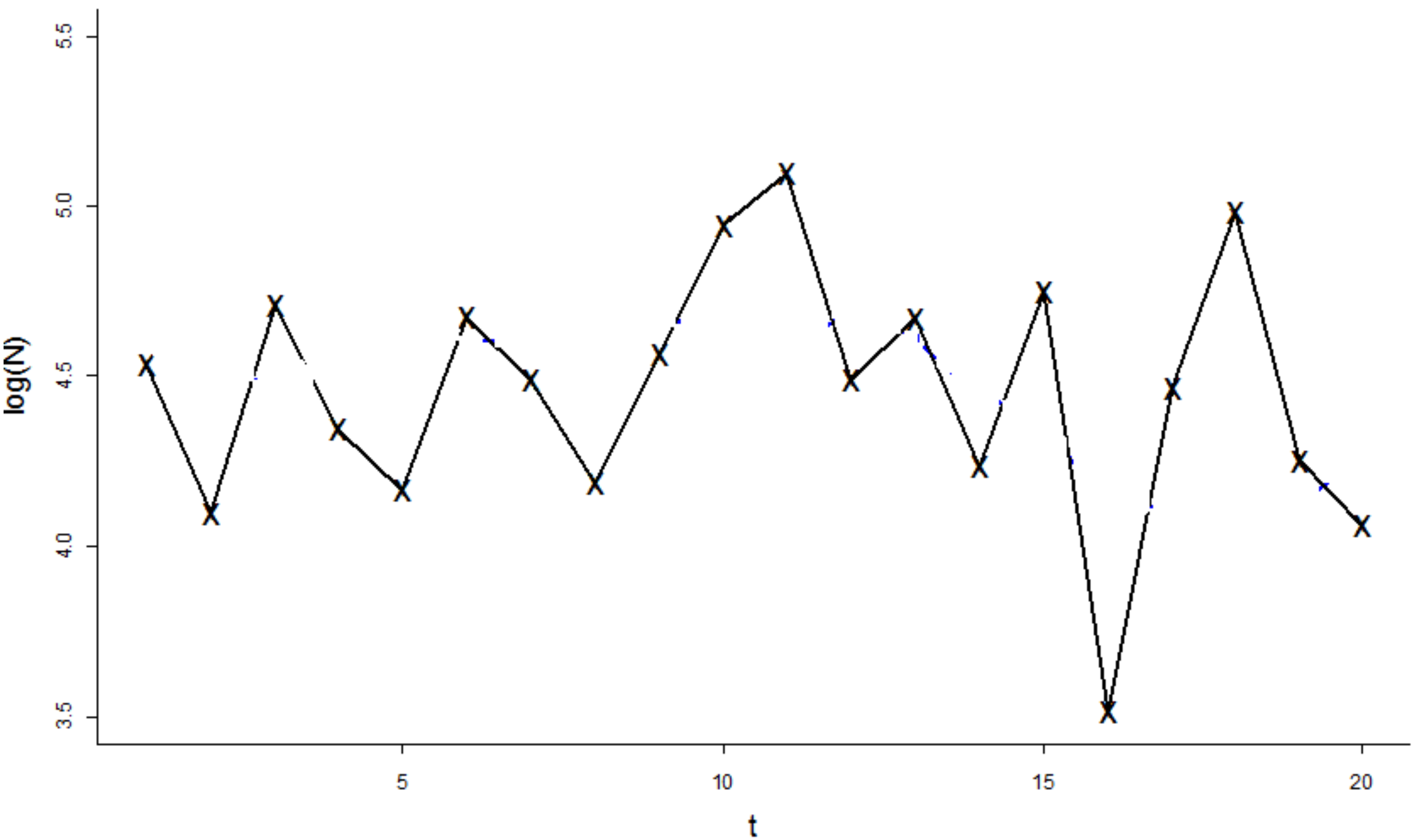


# Observation error

**There IS some number of sea lions in our population in year x, but we don't know that number precisely.  It is "hidden".**

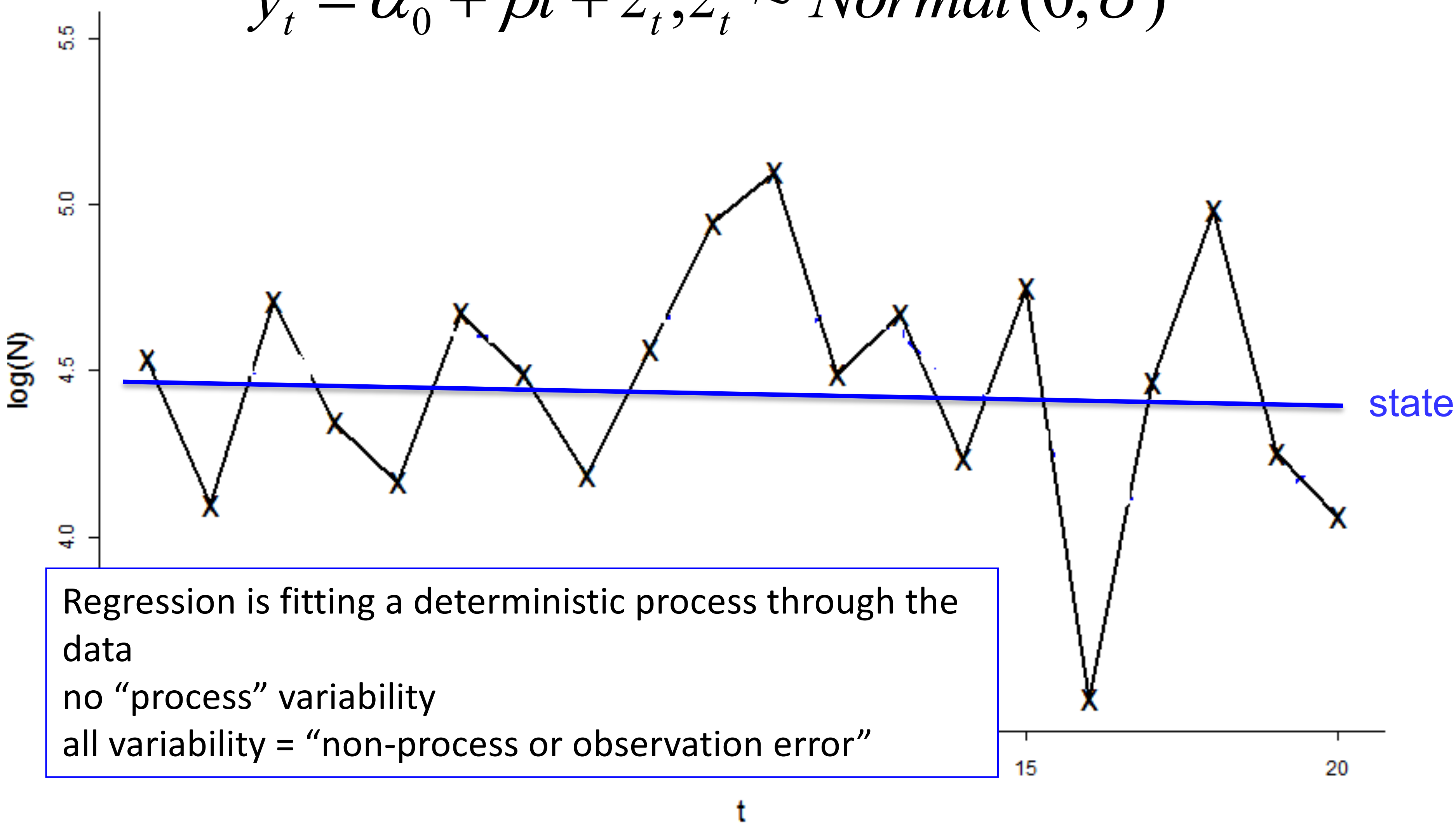


# Suppose we have the following data (population counts logged)



# A linear regression model

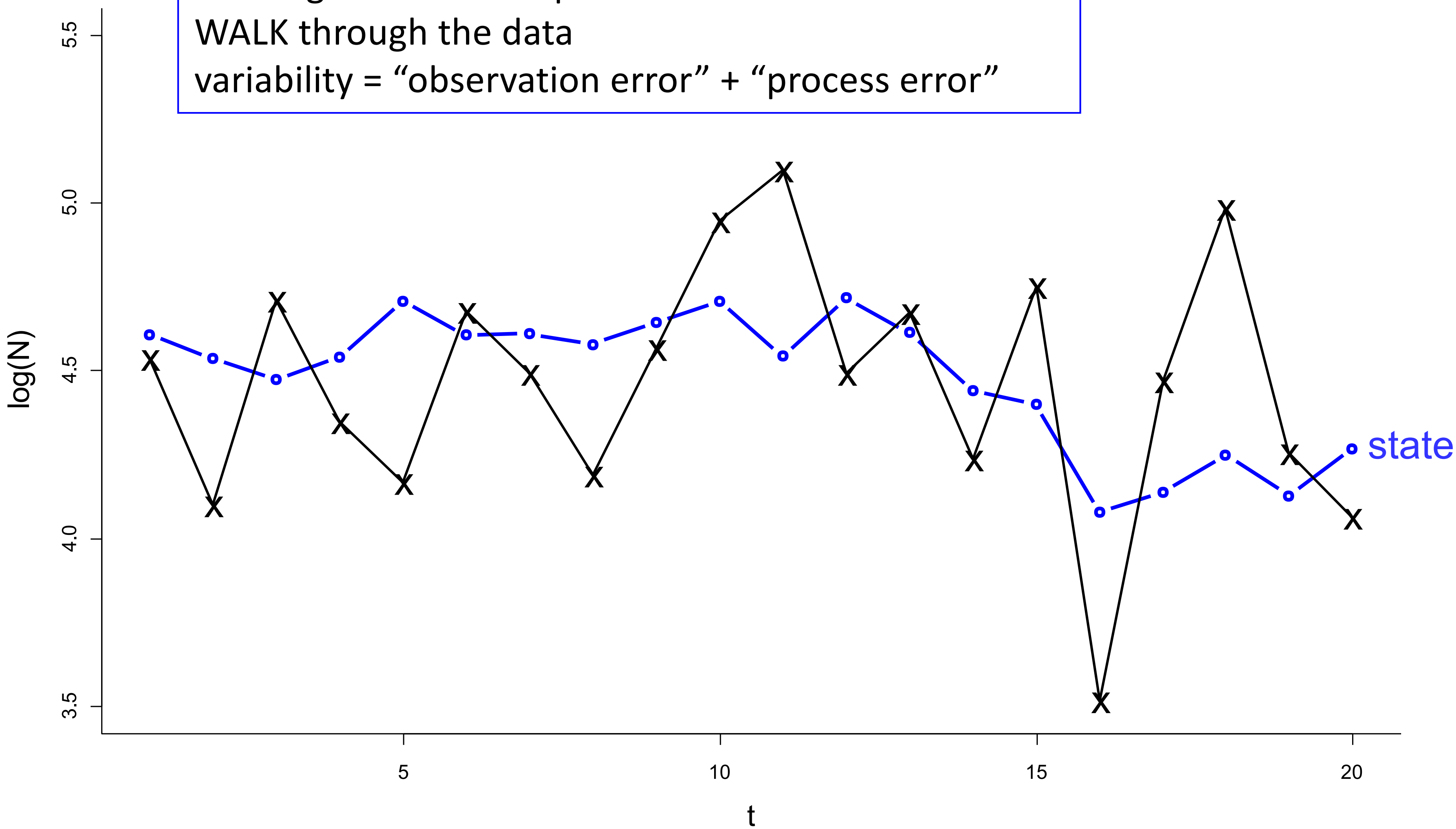
$$y_t = \alpha_0 + \beta t + z_t ; z_t \sim Normal(0, \sigma)$$



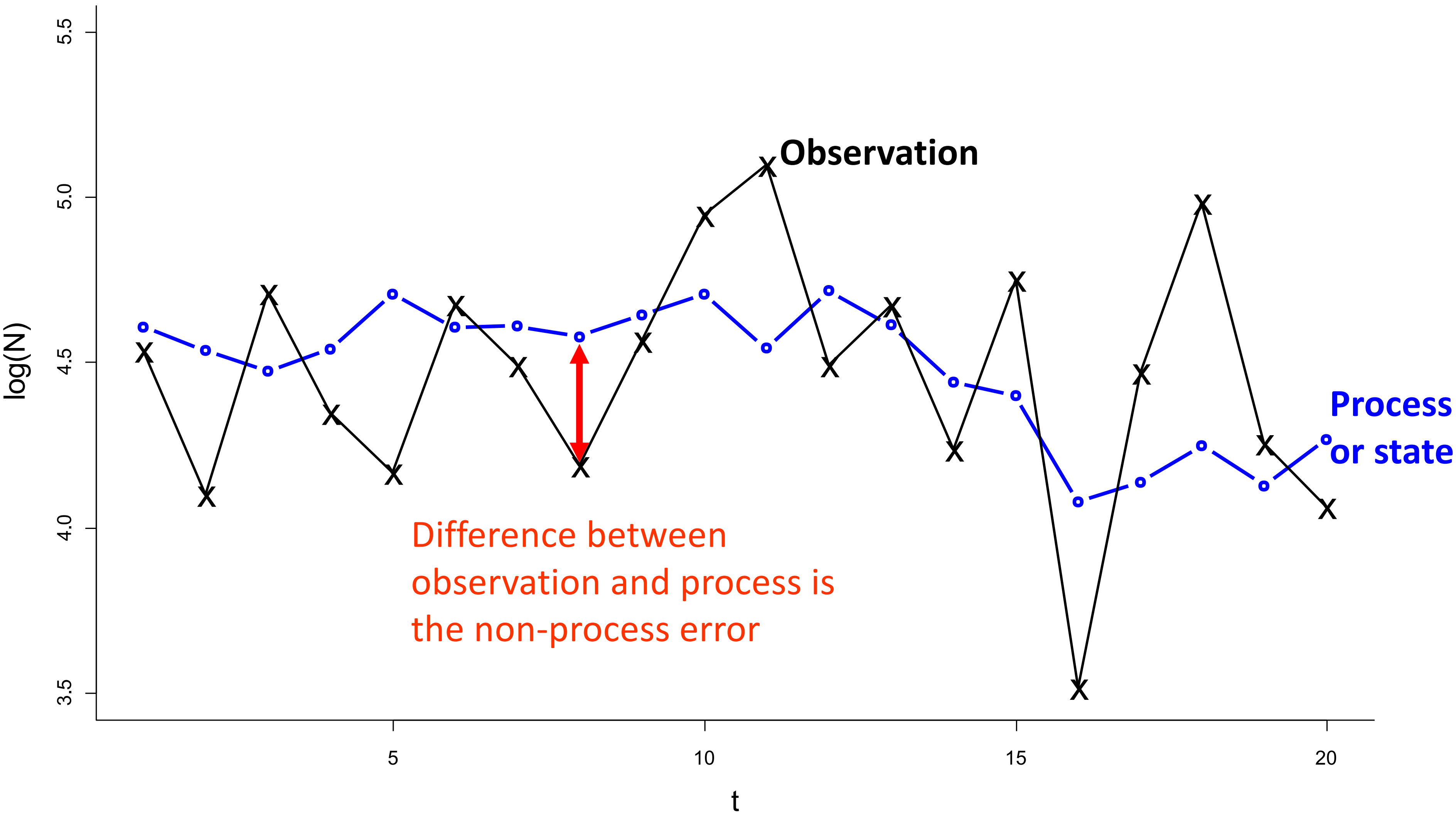

Regression is fitting a deterministic process through the data
no “process” variability
all variability = “non-process or observation error”

# Versus a state-space model

Autoregressive state-space models fit a RANDOM WALK through the data
variability = “observation error” + “process error”



# Two types of variability
# #1 observation or "non-process" variability



# Two types of variability

## #1 observation or “non-process” variability

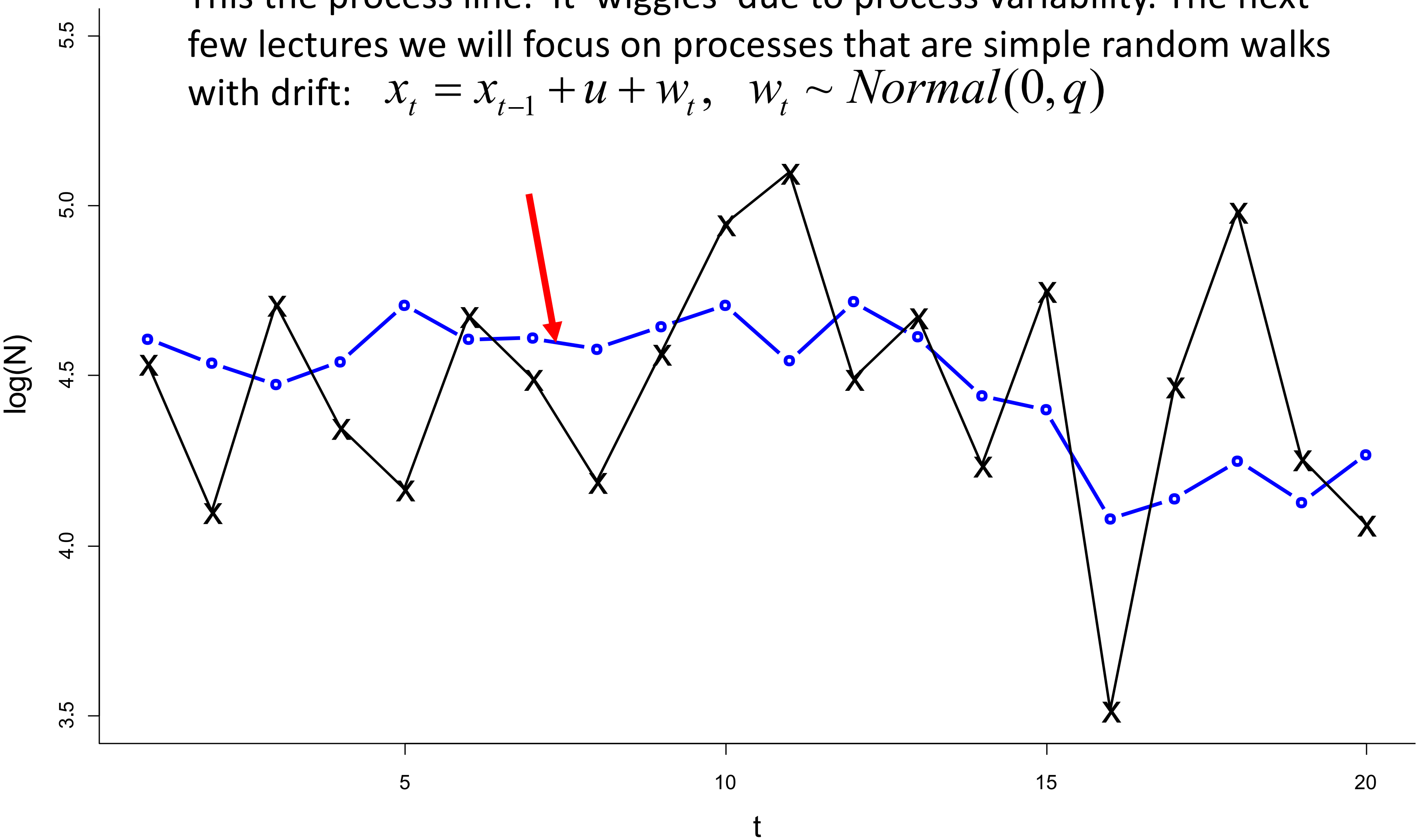
The non-process (observation) variance is often unknowable in fisheries and ecological data

- *Sightability varies due to factors that may not be fully understood or measureable*
  - Environmental factors (tides, temperature, etc.)
  - Population factors (age structure, sex ratio, etc.)
  - Species interactions (prey distribution, prey density, predator distribution or density, etc.)

- *Sampling variability*--due to how you actually count animals--is just one component of observation variance

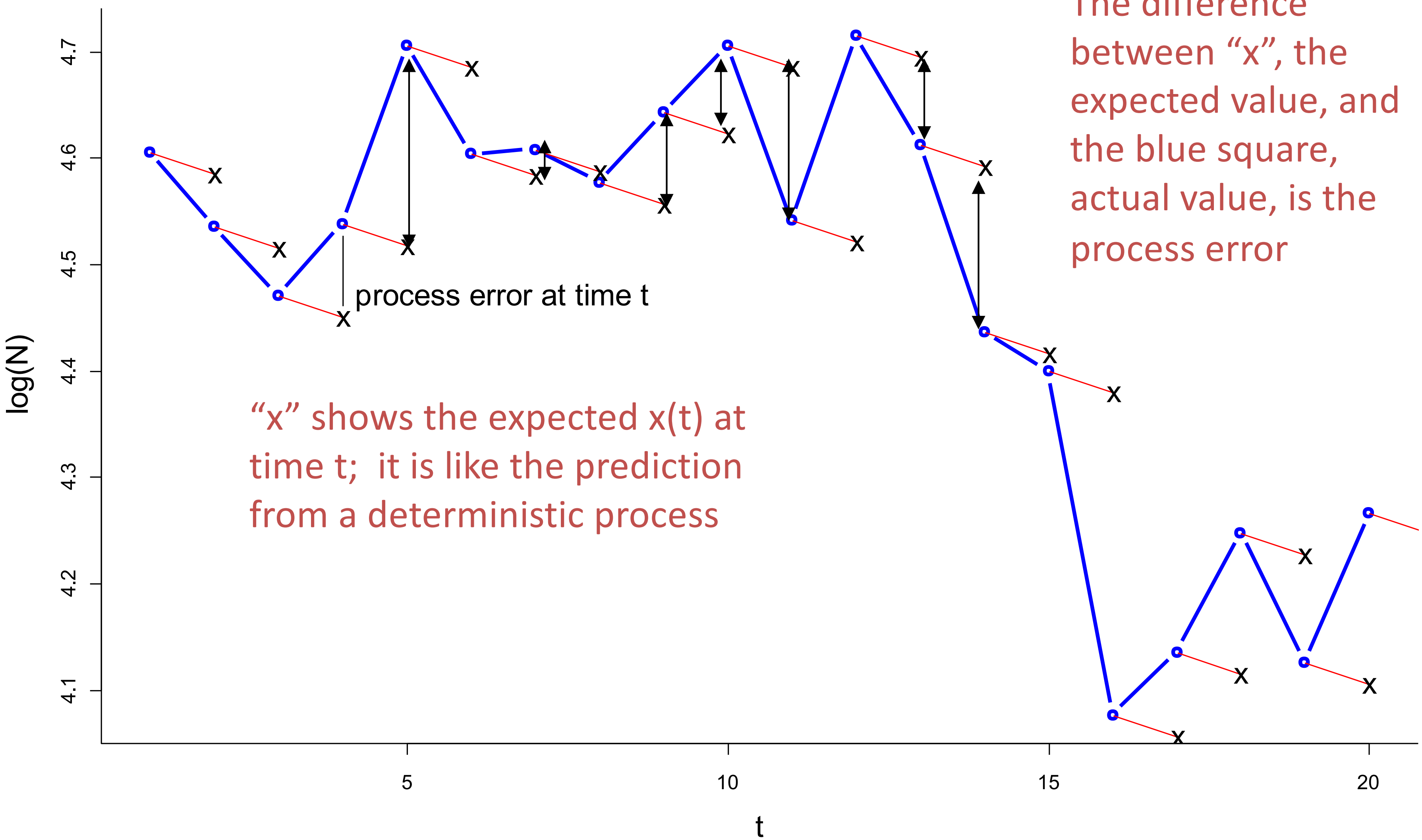
# #2 Process variability

This the process line. It 'wiggles' due to process variability. The next few lectures we will focus on processes that are simple random walks with drift: $x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0,q)$



# Process error is the difference between the expected x(t) and the actual value

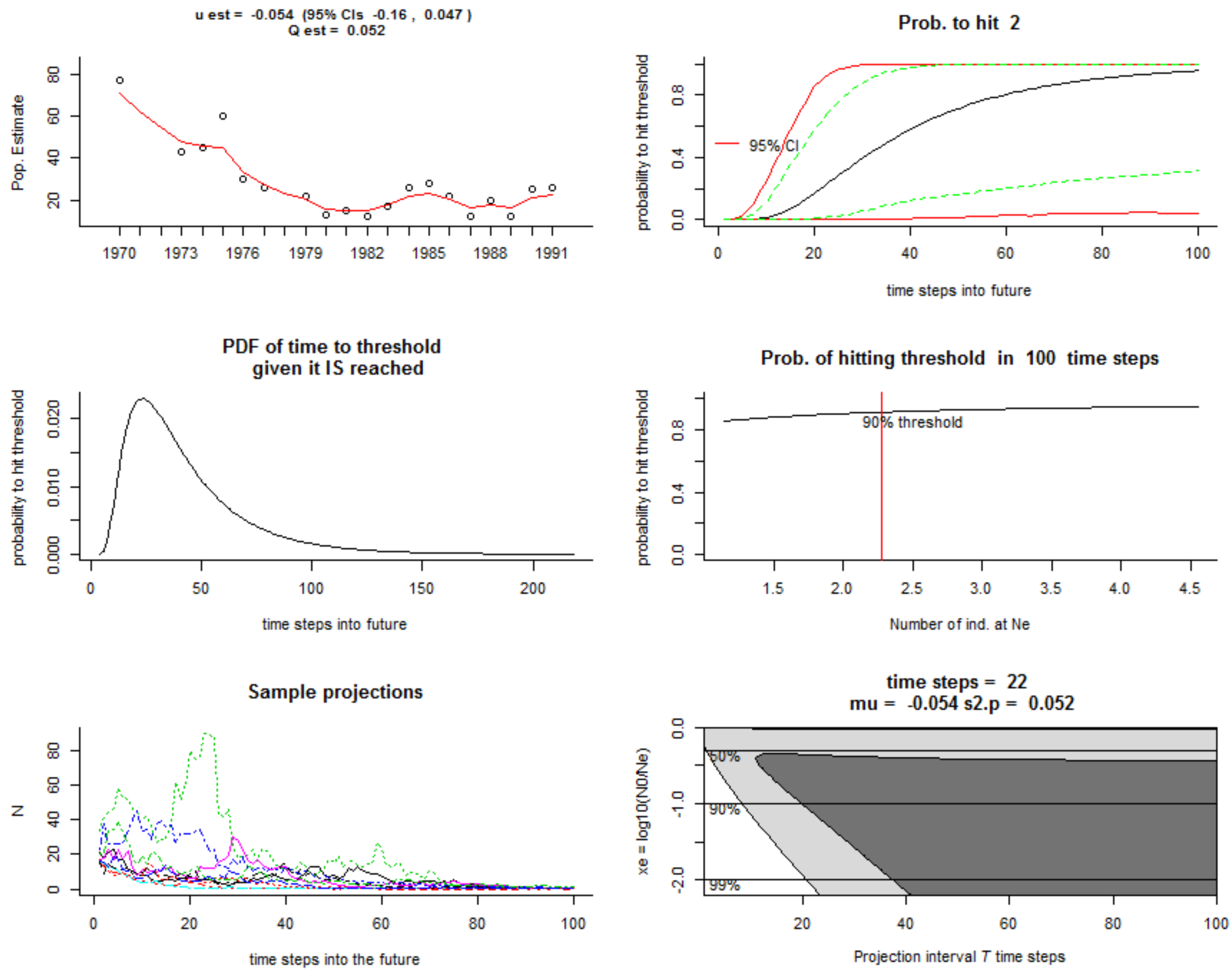
Let's say that in x(t)=x(t-1)-0.02+e(t)*



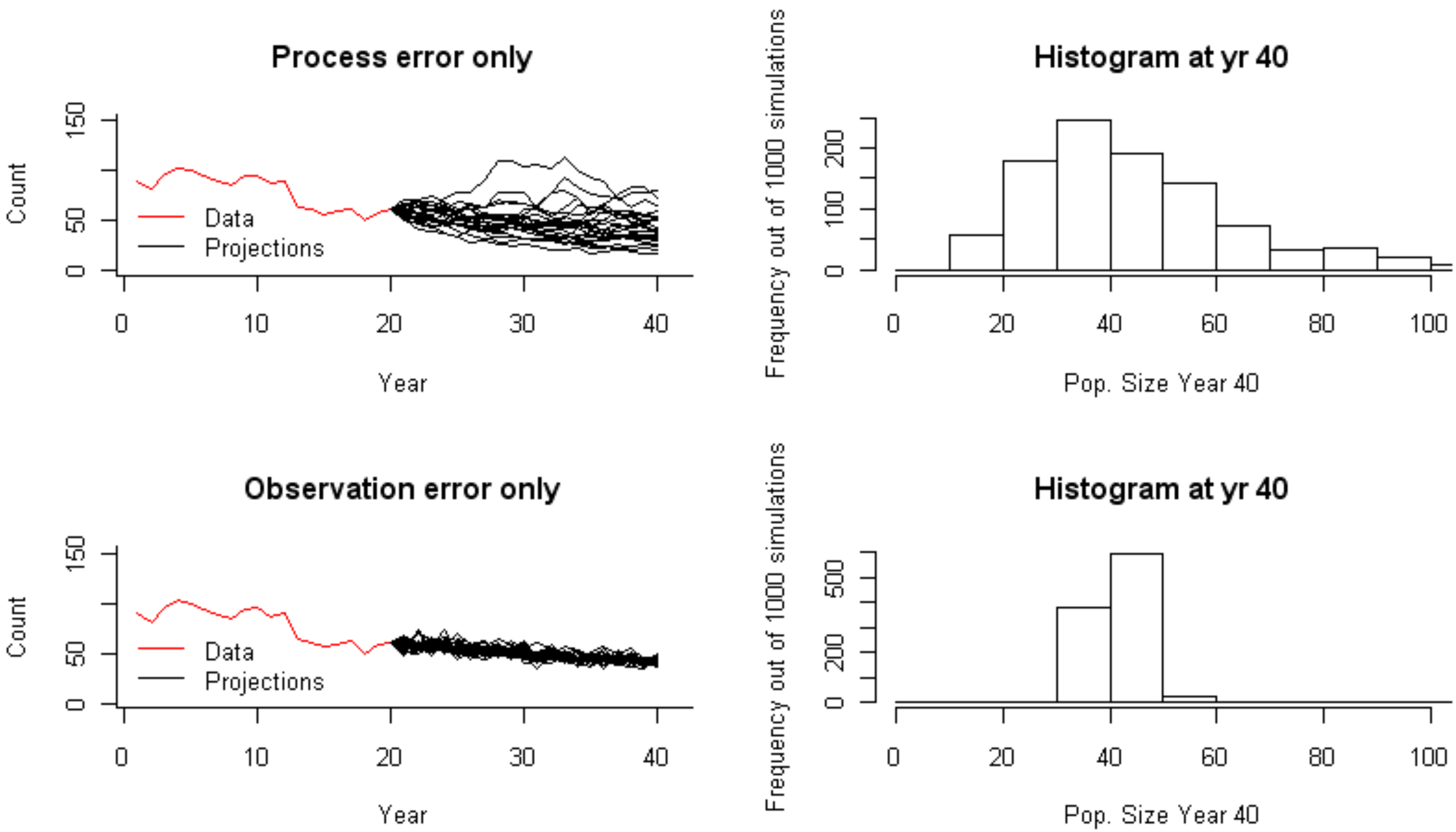

The difference between "x", the expected value, and the blue square, actual value, is the process error

*If this were a population model, that means a the mean rate of decline is ca 2% per year

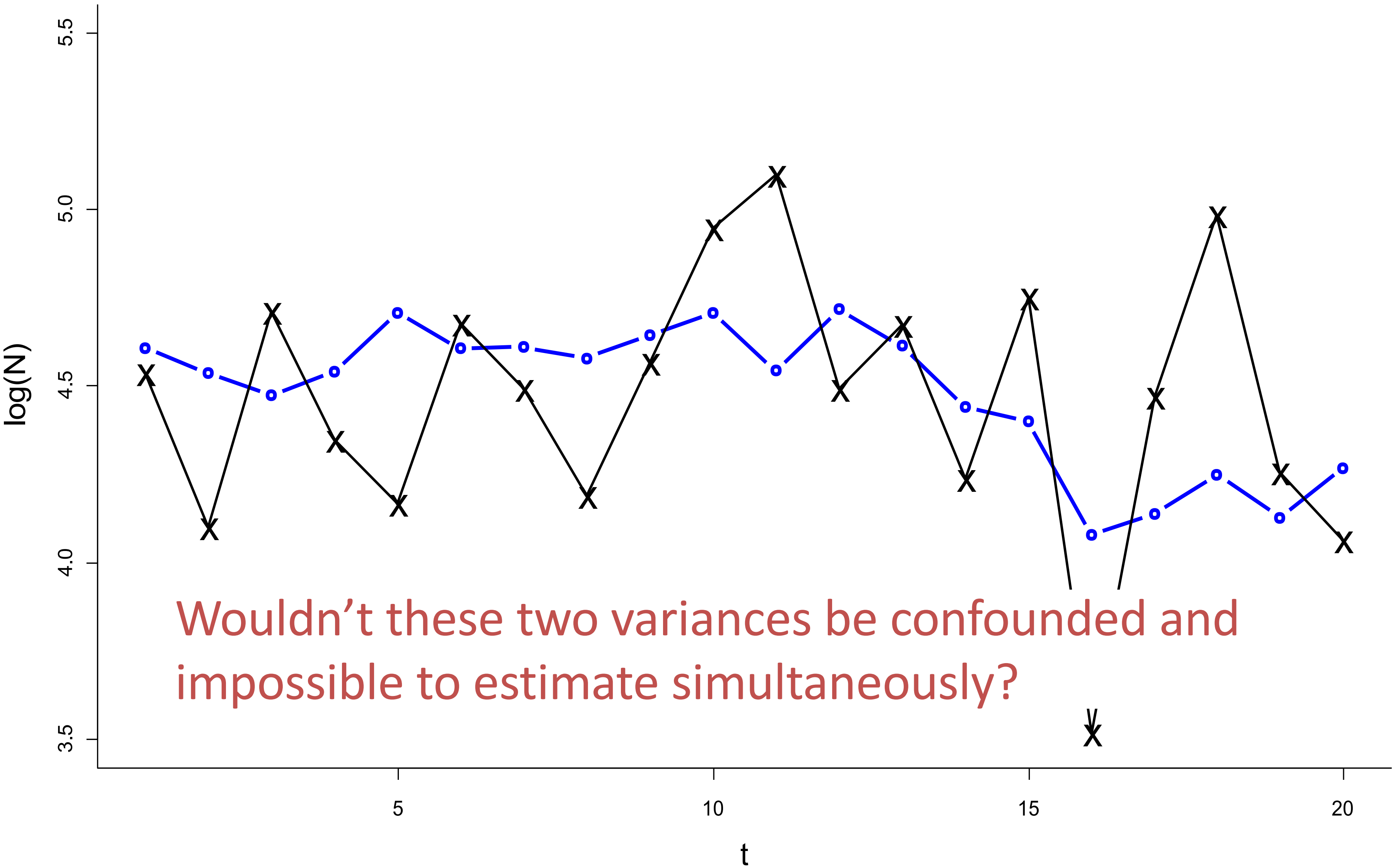
# One use of univariate state-space models is “count-based” population viability analysis (chap 6 HWS2014)



# How you model your data has a large impact on your forecasts



# How can we separate process and non-process variance?

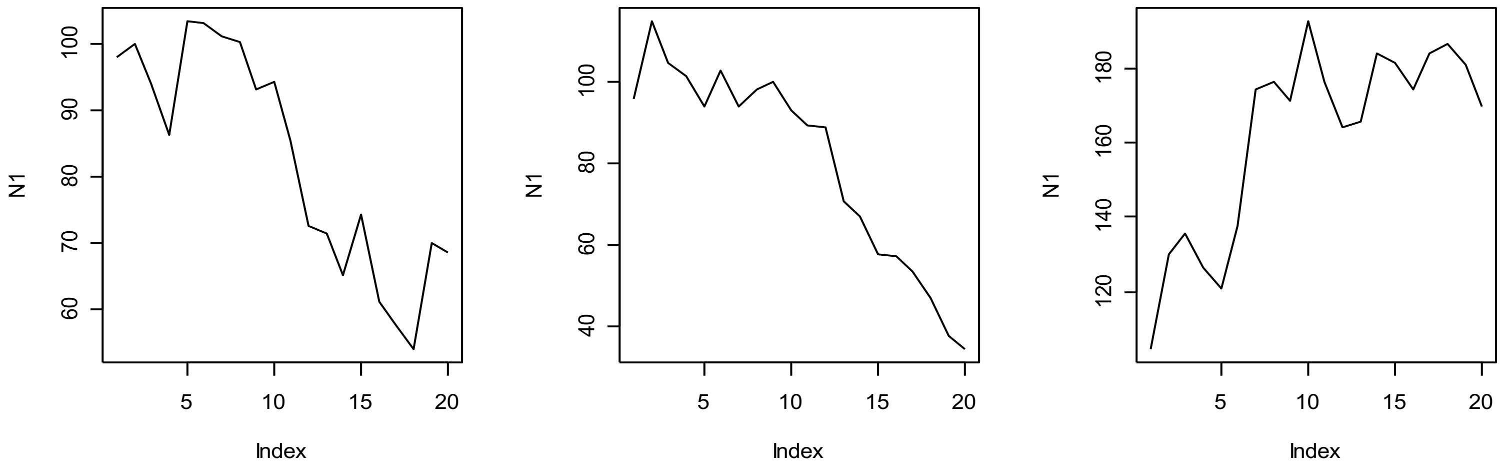



Wouldn’t these two variances be confounded and impossible to estimate simultaneously?

# How can we separate process and observation variance? They have different temporal patterns.

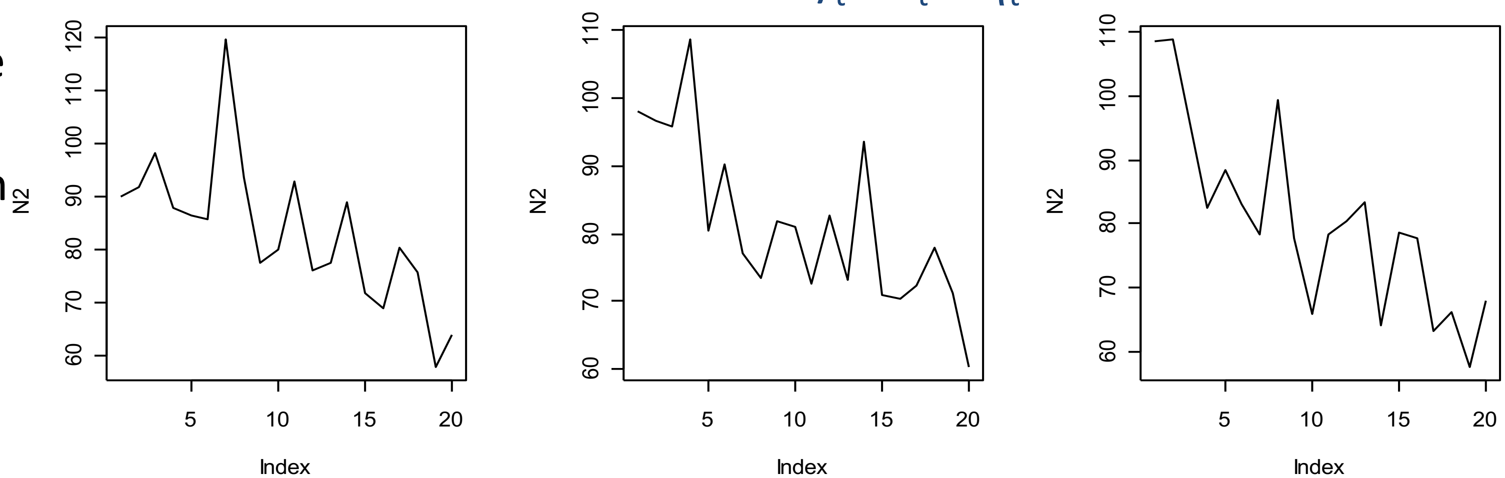
Process error: $x_t = x_{t-1} + u + e_t$

multiple sims of x(t) with same u and q




Observation error: $y_t = x_t + \eta_t$

multiple sims of y(t) with same x(t)



# An AR-1 state-space model combines a model for the hidden AR-1 process with a model for the observation process

…and allows us to separate the variances

Process model

$$x_t = x_{t-1} + u + w_t$$
$$w_t \sim Normal(0, q)$$

AR lag-1
random walk with drift
normally distributed
process errors

Observation model

$$y_t = x_t + v_t$$
$$v_t \sim Normal(0, r)$$

observation errors
normally distributed
process errors

## Kalman Filter: Estimate the x in a state-space model

A mathematical algorithm that solves for the 'optimal' (least error or maximum-likelihood) x_t given all the data (y) from time 1 to t

| | | |
|---|---|---|
| Predict: Given an x_0, predict x_1 from your model | Predict: Given an x_1, predict x_2 from your model | Predict: Given an x_2, predict x_3 from your model |
| ↓ | ↓ | ↓ |
| Update: Given y_1, update your x_1 estimate | Update: Given y_2, update your x_2 estimate | Update: Given y_3, update your x_3 estimate |

# Let’s simulate and try fitting some models

- Open up R and follow after me
- univariate_example_1.R
- univariate_example_2.R
- univariate_example_3.R

# How to write a straight-line as AR-1

- ##Preliminaries: how to write ##x=intercept+slope*t as a AR-1
- x(0)=intercept
- x(1)=x(0)+slope #this is x at t=1
- x(2)=x[1]+slope
- so..
- x(t)=x(t-1)+slope+w(t), w(t)~N(0,0)

# MARSS R Package

- Fits MARSS models (multivariate AR-1 state-space)
- General, fits any MARSS model with Gaussian errors

- But
- Maximum likelihood
- Slow. Students working with large data sets have gotten huge speed improvements by coding their models in TMB

# MARSS R Package

- Fits MARSS models (multivariate AR-1 state-space)
- MARSS model syntax

$$X(t) = \mathbf{B}\, X(t-1) + \mathbf{U} + w(t),\ w(t) \sim N(0, \mathbf{Q})$$
$$Y(t) = \mathbf{Z}\, X(t) + \mathbf{A} + v(t),\ v(t) \sim N(0,\mathbf{R})$$

- **`fit2=MARSS(y,model=mod.list)`**

- **`y`** is data; **`model`** tells MARSS what the parameters are
- The parameters are MATRICES
- You write matrices just like they appear in your model on paper
- You pass **`model`** to MARSS as a list

# MARSS model in matrix form

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \\ \eta_{4,t} \\ \eta_{5,t} \end{bmatrix}$$

X(t) = **B** X(t-1) + **U** + w(t), w(t) ~ N(0, **Q**)

Y(t) = **Z** X(t) + **A** + v(t), v(t) ~ N(0,**R**)

Let's say we want to fit this model:

```
mod.list=list(
  U=matrix("u"),
  x0=matrix(0),
  B=matrix(1),
  Q=matrix(0.1),
  Z=matrix(1),
  A=matrix(0),
  R=matrix("r"),
  tinitx=0)
```

$$x_t = x_{t-1} + u + w_t, w_t \sim N(0, \sigma^2 = 0.1)$$

$$y_t = x_t + v_t, v_t \sim N(0, r)$$

$$x_0 = 0$$

# Let’s simulate and try fitting some models

- Open up R and follow after me
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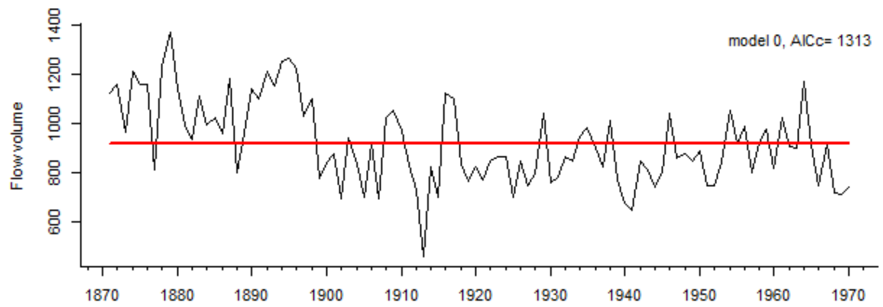
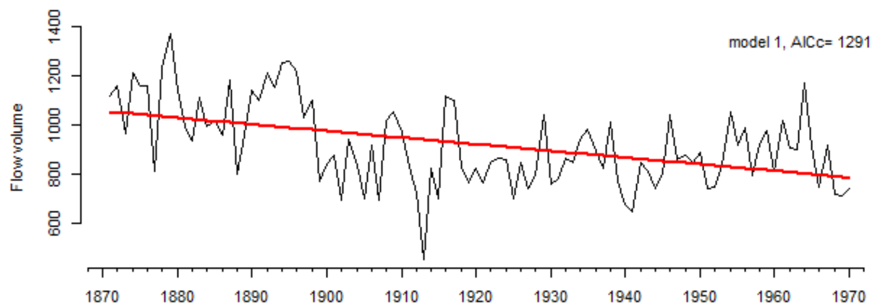
- univariate_example_2.R
- univariate_example_3.R

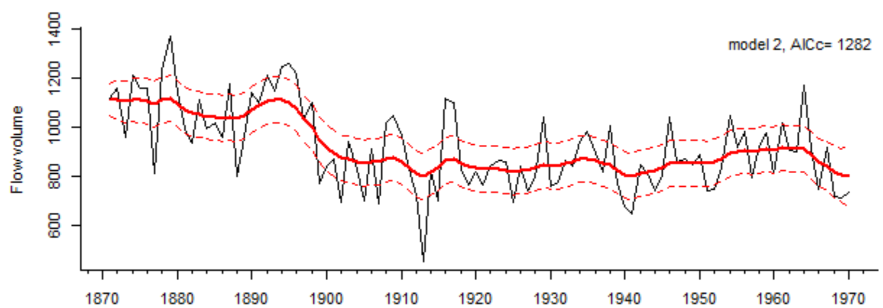
# State-space diagnostics

# Basic diagnostics

Nile River models from the lab handout



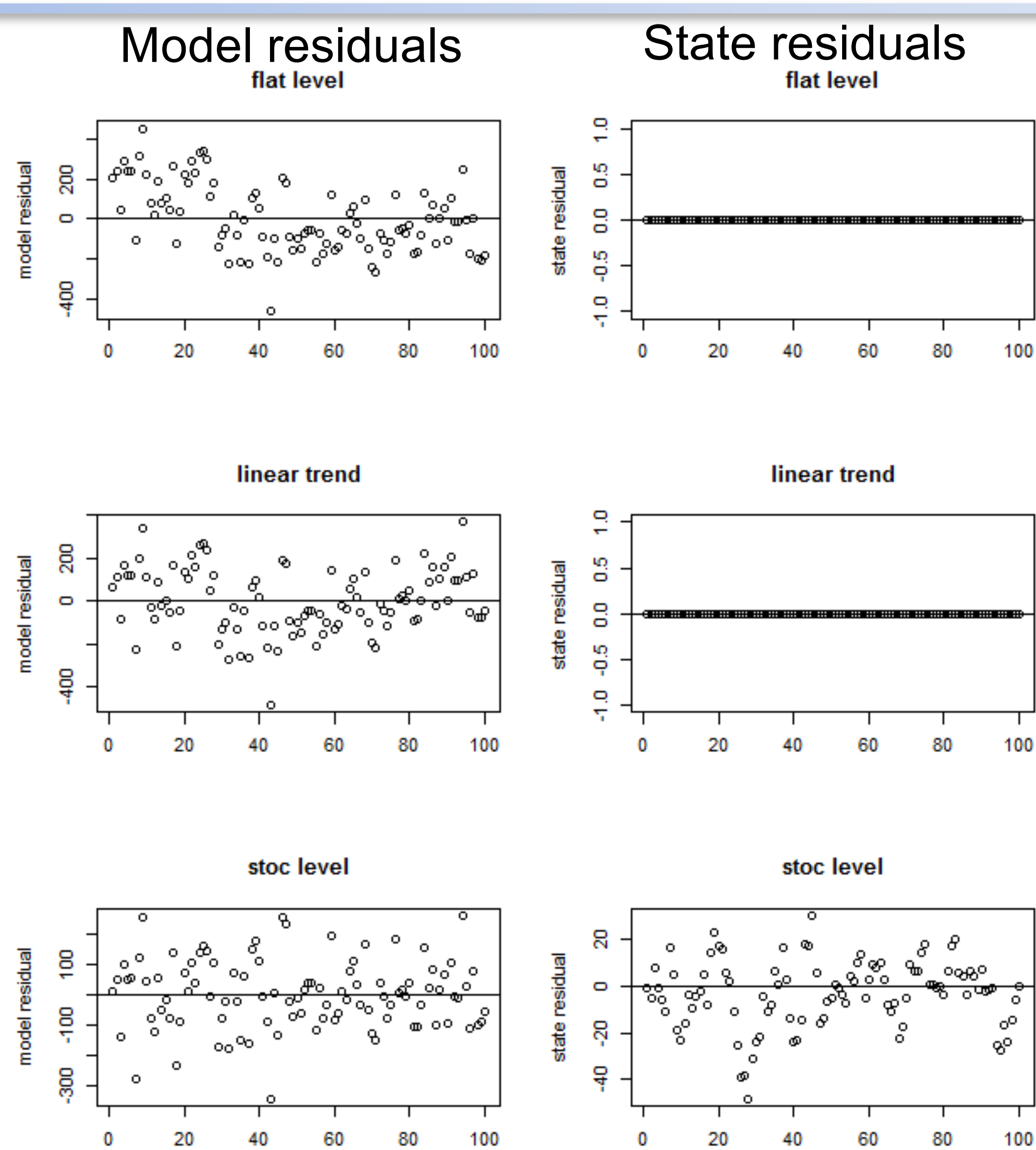



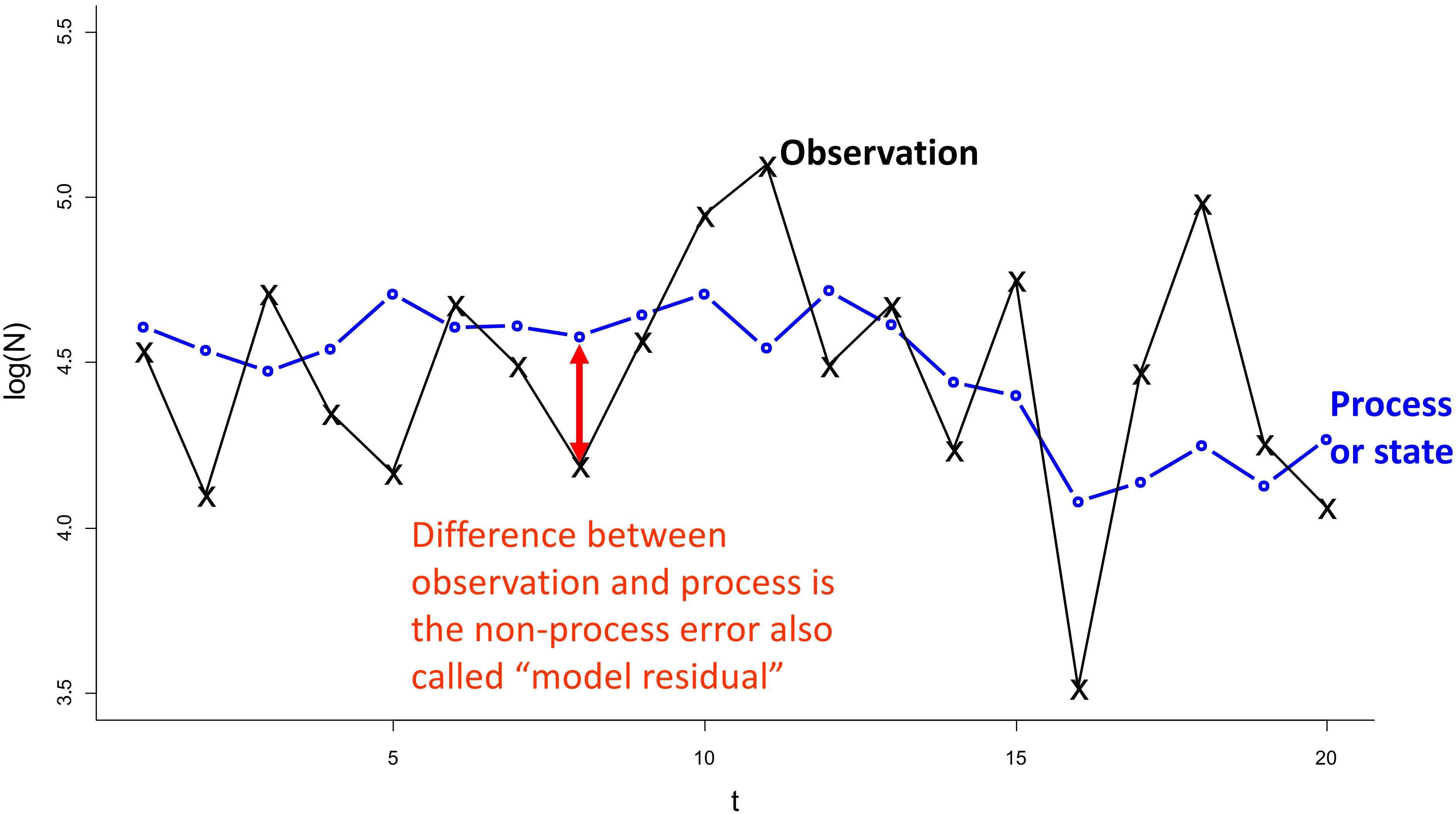
# Basic diagnostics: #1 plot the residuals

There should be no temporal trends!

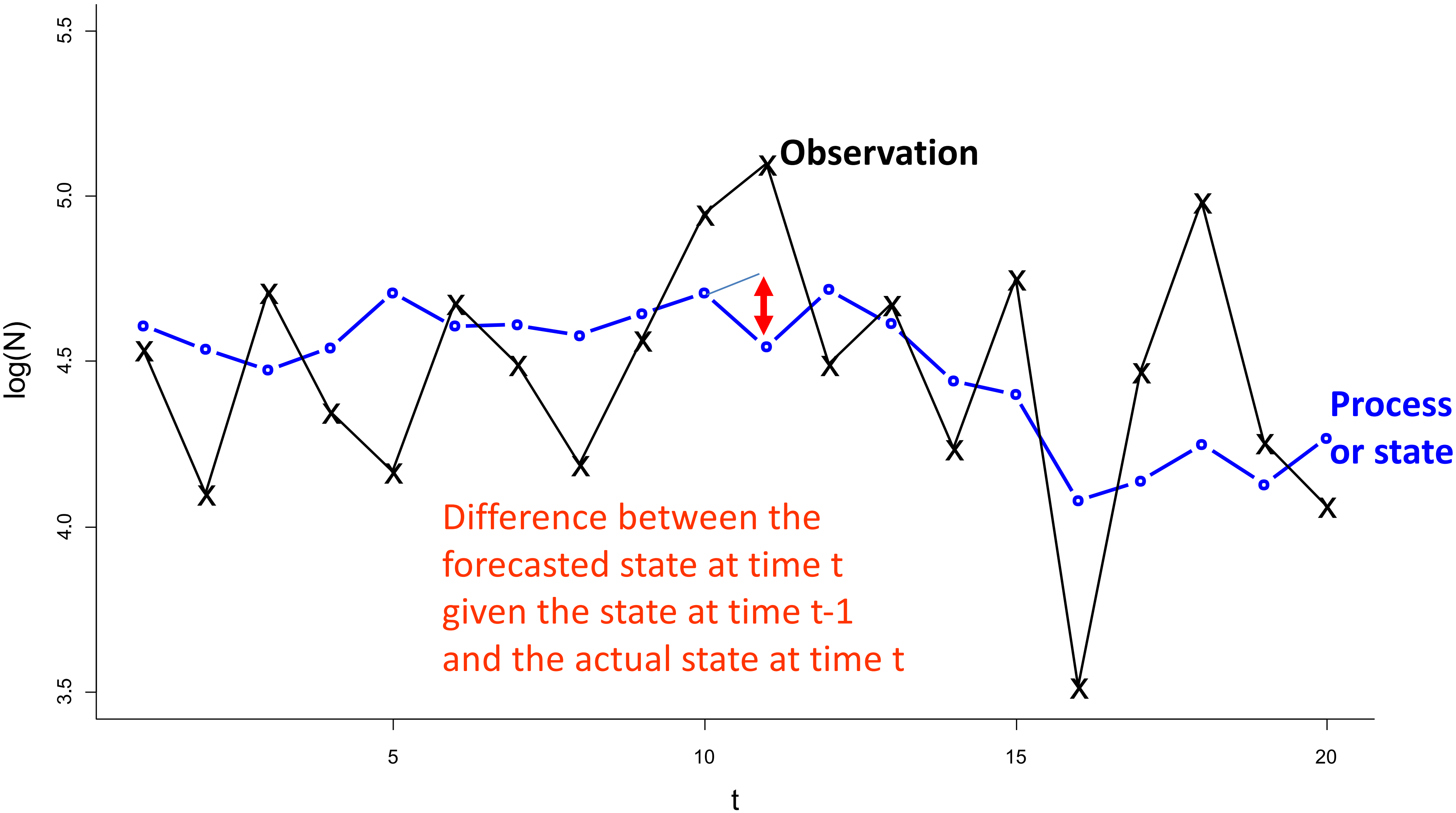
They should be centered about 0.



# non-process error or model residual

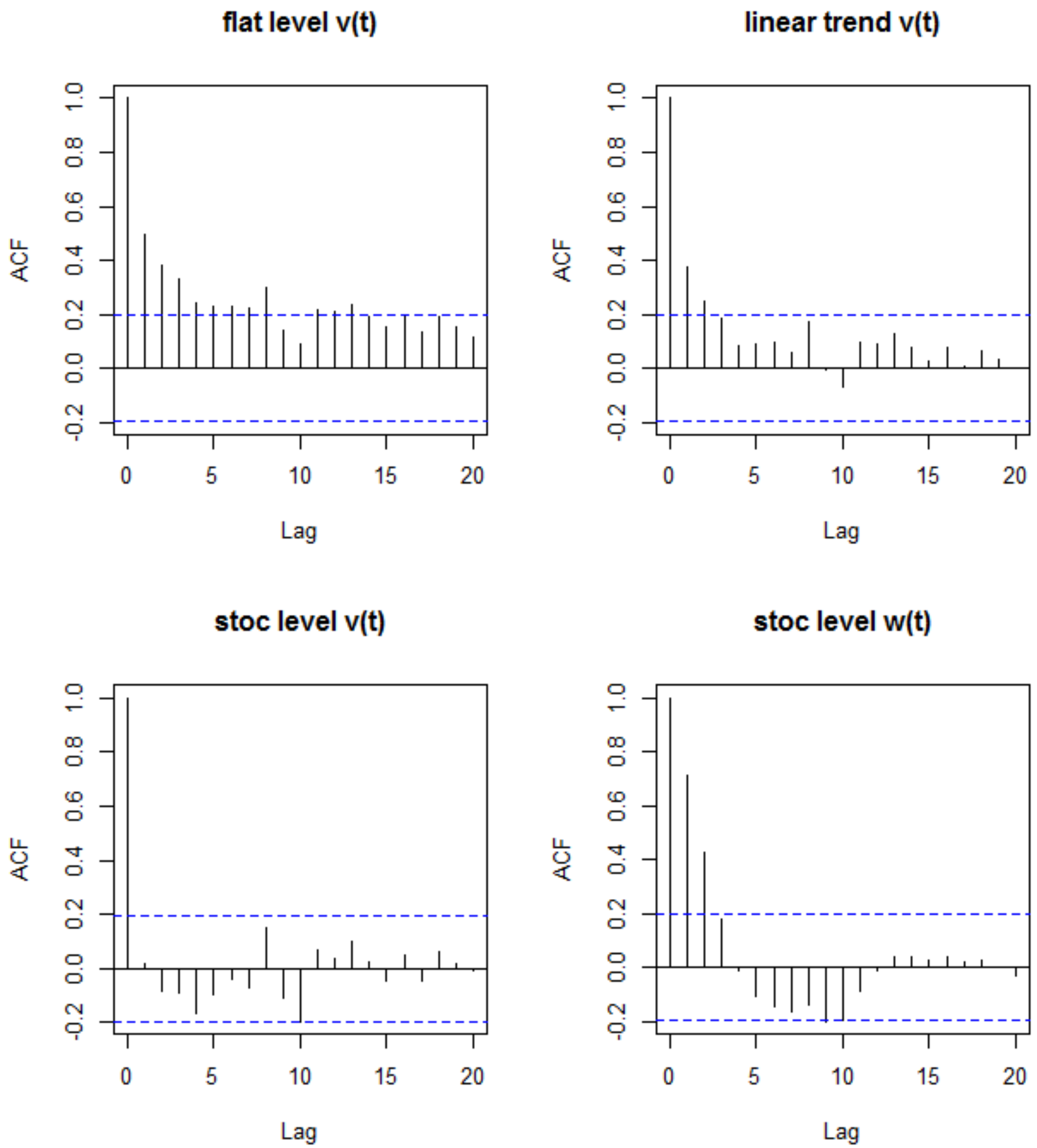


# process error or state residual



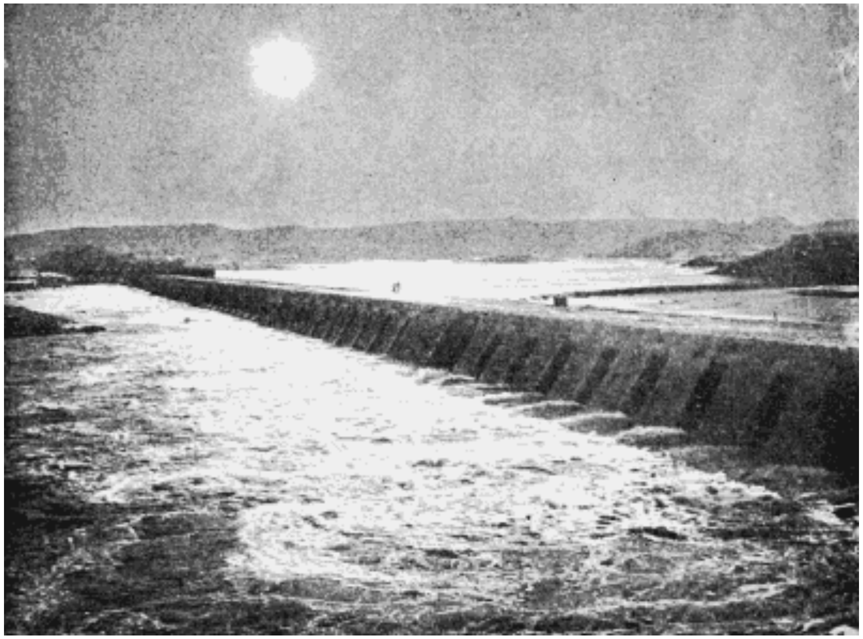
# Basic diagnostics: #2 check acf of residuals

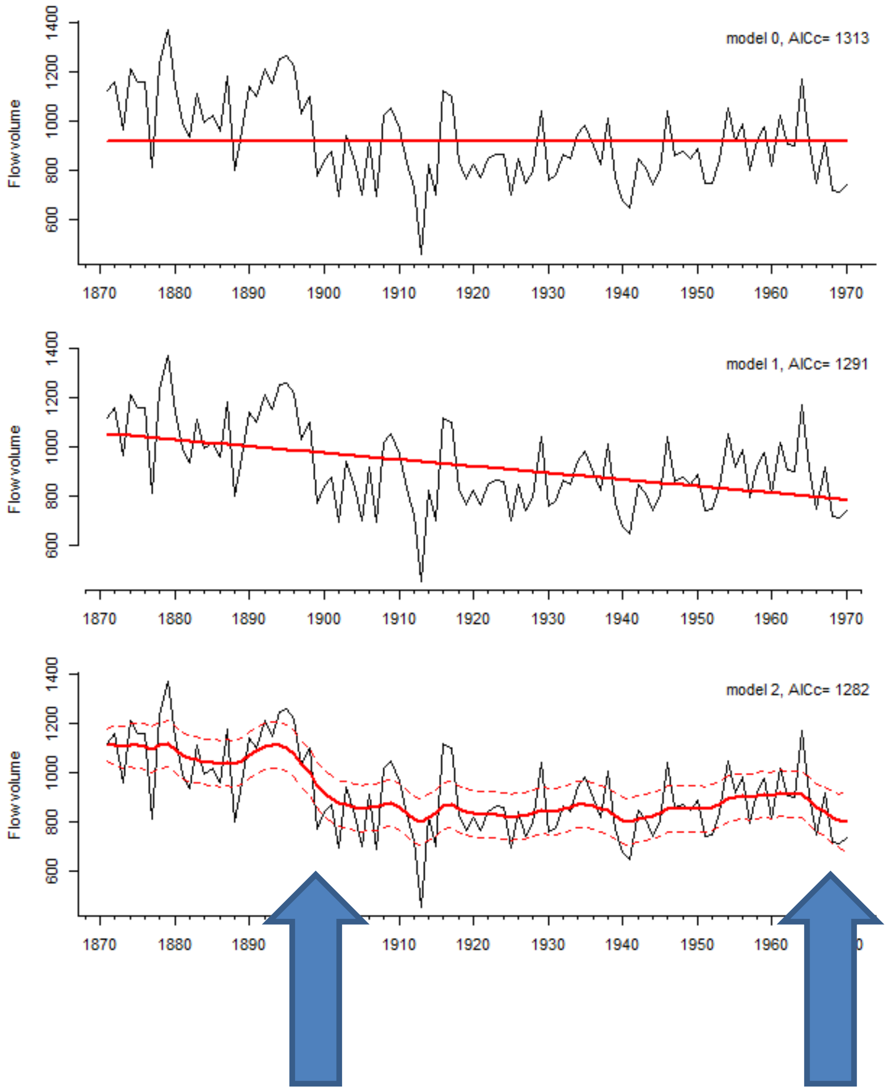
v(t) are model residuals



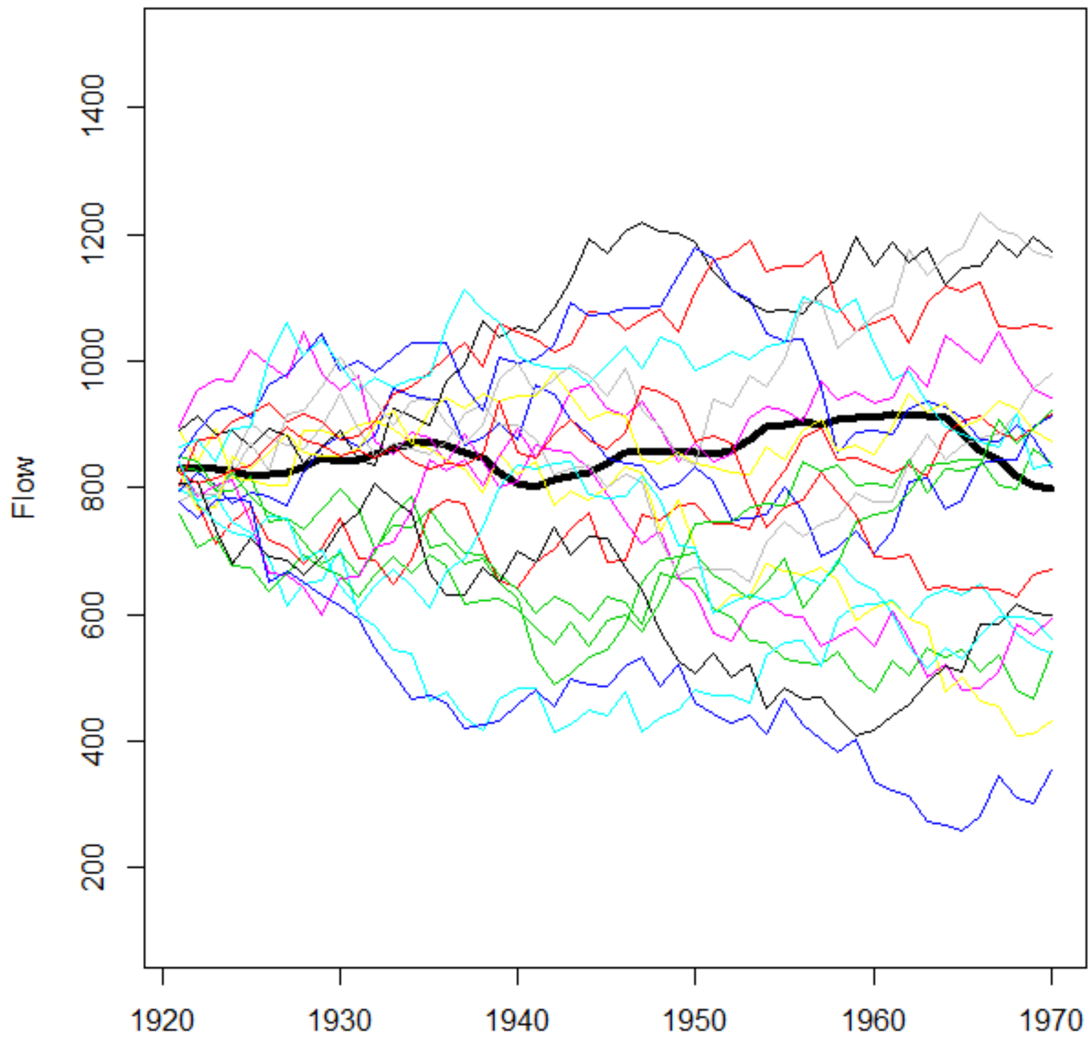

w(t) are state residuals

# Even our ‘best’ model is missing something...

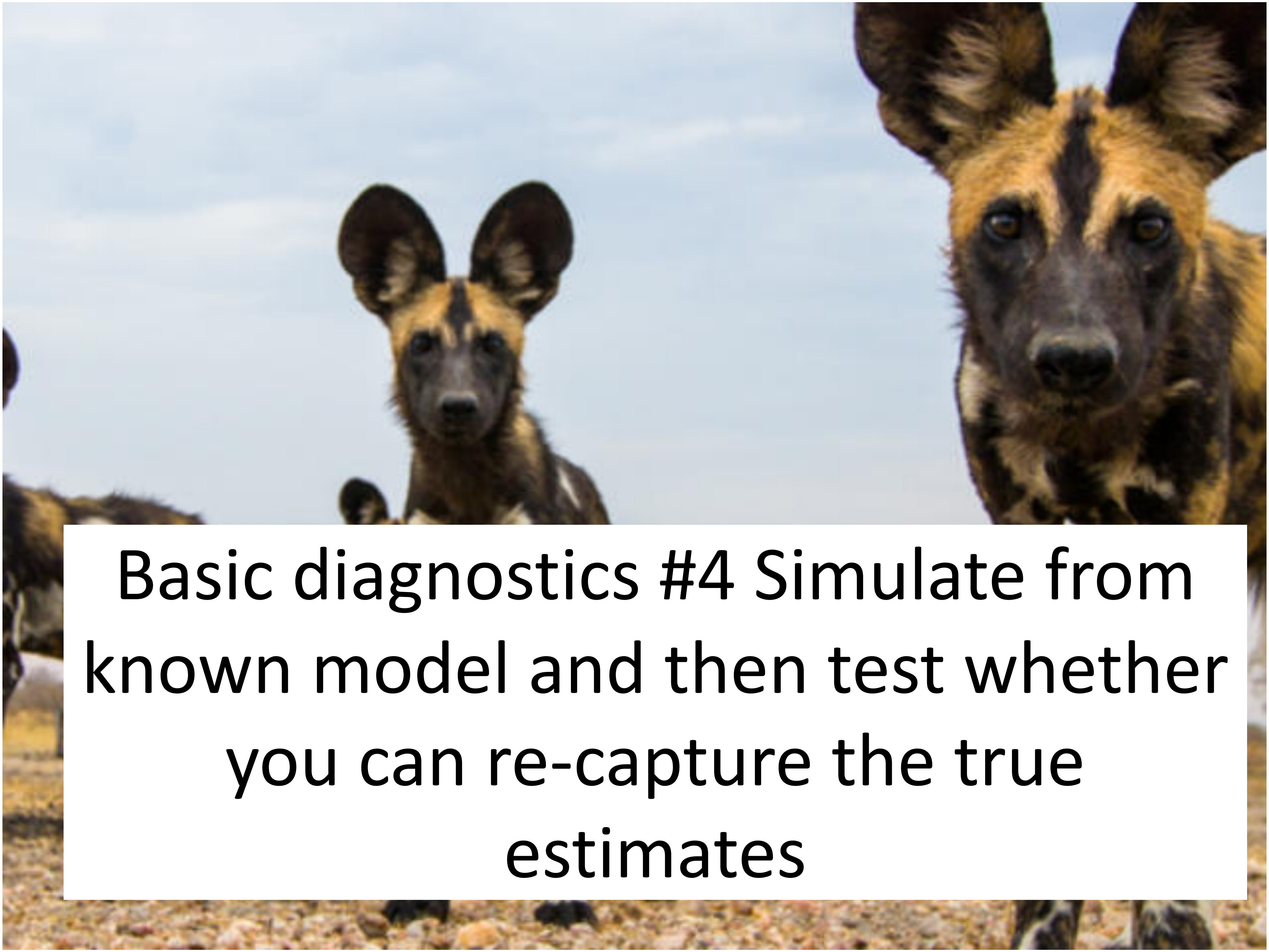




# Basic diagnostics: #3 Simulate from your estimated model and compare to the data.




Black line is the estimated state from model 2



# Basic diagnostics #4 Simulate from known model and then test whether you can re-capture the true estimates

# How do you know when to use a process error or observation error model?

- If your time-series data contain both types, use a model with both types.
- To estimate both variances, you need a) 20+ time steps **OR** b) multi-site data.
- If you don't have enough data, you need to use assumptions about one of the variances.  Meaning a) fix the value or b) incorporate a prior.
- Diagnostics: Observation error induces autocorrelation in the noise of an autoregressive process. Fit a process-error only model (R=0) and check for autocorrelation of residuals

# Other types of “non-process” error

- Fluctuations that don’t have “feedback” (variance doesn’t explode)

- Lots of biological processes also create noise that looks like that
  - age-structure cycles
  - density-dependence
    - cyclic variability in fecundity
    - predator-prey interactions

- If your model cannot accommodate that cycling,
  - it tends to get ‘soaked’ up in the ‘non-process’ error component

- If your model can accommodate that cycling,
  - estimation of ‘observation error’ variance can be confounded, unless you have long, long datasets or replicates

# Thursday lecture: multivariate state-space

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \\ \eta_{4,t} \\ \eta_{5,t} \end{bmatrix}$$

Thursday lab: fitting univariate and multivariate state-space models