Dynamic Linear Models

FISH 507 – Applied Time Series Analysis

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Topics for today

Univariate response

- Stochastic level & growth
- Dynamic Regression
- Dynamic Regression with fixed season
- Forecasting with a DLM
- Model diagnostics

Multivariate response

Simple linear regression

Let's begin with a linear regression model

$$y_i = \alpha + \beta x_i + e_i \text{ with } e_i \sim N(0, \sigma^2)$$

The index i has no explicit meaning in that shuffling (y_i, x_i) pairs has no effect on parameter estimation

Simple linear regression

We can write the model in matrix form

$$y_i = \hat{\ } + \tilde{\ }\, x_i + e_i$$

$$\Downarrow$$

$$y_i = \begin{bmatrix} 1 & x_i \end{bmatrix} \begin{bmatrix} \grave{\ } \\ \tilde{\ } \end{bmatrix} + e_i$$

Simple linear regression

We can write the model in matrix form

$$y_i = \hat{} + \tilde{} \; x_i + e_i$$

$$\Downarrow$$

$$y_i = \begin{bmatrix} 1 & x_i \end{bmatrix} \begin{bmatrix} \grave{} \\ \tilde{} \end{bmatrix} + e_i$$

$$\Downarrow$$

$$y_i = \mathbf{X}_i^\top \; + e_i$$

with

$\mathbf{X}_i^\top = \begin{bmatrix} 1 & x_i \end{bmatrix}$ and $= \begin{bmatrix} \grave{} & \tilde{} \end{bmatrix}^\top$

Dynamic linear model (DLM)

In a *dynamic* linear model, the regression parameters change over time, so we write

$$y_i = \mathbf{X}_i^{\top} \quad + e_i \qquad \text{(static)}$$

as

$$y_t = \mathbf{X}_t^{\top} \;{}_t + e_t \qquad \text{(dynamic)}$$

Dynamic linear model (DLM)

There are 2 important points here:

$$y_{\boxed{t}} = \mathbf{X}_t^{\top} \quad {}_t + e_t$$

1. Subscript t explicitly acknowledges implicit info in the time ordering of the data in $\mathbf{y}$

Dynamic linear model (DLM)

There are 2 important points here:

$$y_t = \mathbf{X}_t^{\top} \boxed{t} + e_t$$

1. Subscript t explicitly acknowledges implicit info in the time ordering of the data in $\mathbf{y}$

2. The relationship between $\mathbf{y}$ and $\mathbf{X}$ is unique for every t

Constraining a DLM

Close examination of the DLM reveals an apparent problem for parameter estimation

$$y_t = \mathbf{X}_t^\top \theta_t + e_t$$

Constraining a DLM

Close examination of the DLM reveals an apparent problem for parameter estimation

$$y_t = \mathbf{X}_t^{\top} \quad {}_t + e_t$$

We only have 1 data point per time step (ie, y_t is a scalar)

Thus, we can only estimate 1 parameter (with no uncertainty)!

Constraining a DLM

To address this issue, we'll constrain the regression parameters to be dependent from t to $t + 1$

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \text{ with } \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

Constraining a DLM

In practice, we often make $\mathbf{G}_t$ time invariant

$$\boldsymbol{\theta}_t = \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{w}_t$$

or assume $\mathbf{G}_t$ is an $m \times m$ identity matrix $\mathbf{I}_m$

$$\begin{aligned} \boldsymbol{\theta}_t &= \mathbf{I}_m\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \\ &= \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \end{aligned}$$

In the latter case, the parameters follow a random walk over time

DLM in state-space form

Observation model relates covariates to data

$$y_t = \mathbf{X}_t^\top \boldsymbol{\theta}_t + e_t$$

State model determines how parameters "evolve" over time

$$\boldsymbol{\theta}_t = \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{w}_t$$

DLM in MARSS notation

Full state-space form

$$
\begin{aligned}
y_t &= \mathbf{X}_t^\top \boldsymbol{\theta}_t + e_t \\
\boldsymbol{\theta}_t &= \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \\
&\Downarrow \\
y_t &= \mathbf{Z}_t \mathbf{x}_t + v_t \\
\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t
\end{aligned}
$$

Contrast in covariate effects

Note: DLMs include covariate effect in the observation eqn much differently than other forms of MARSS models

DLM: $\mathbf{Z}$ is covariates, $\mathbf{x}$ is parameters

$$y_t = \boxed{\mathbf{Z}_t \mathbf{x}_t} + v_t$$

Others: $\mathbf{d}$ is covariates, $\mathbf{D}$ is parameters

$$y_t = \mathbf{Z}_t \mathbf{x}_t + \boxed{\mathbf{D}\mathbf{d}_t} + v_t$$

Other forms of DLMs

The regression model is but one type

Others include:

- stochastic "level" (intercept)
- stochastic "growth" (trend, bias)
- seasonal effects (fixed, harmonic)

The most simple DLM

Stochastic level

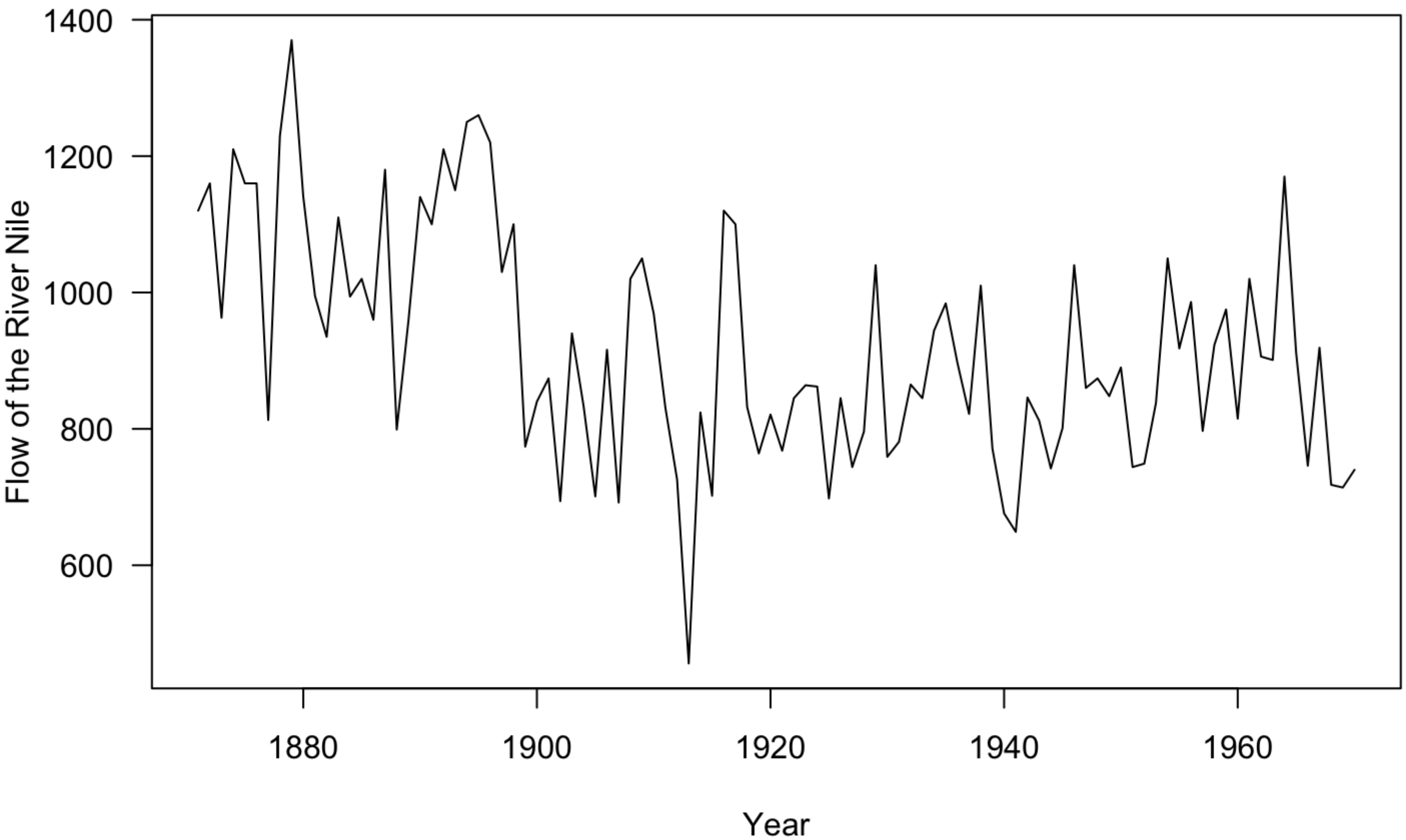
$$
\begin{aligned}
y_t &= \mu_t + e_t \\
\mu_t &= \mu_{t-1} + w_t
\end{aligned}
$$

The most simple DLM

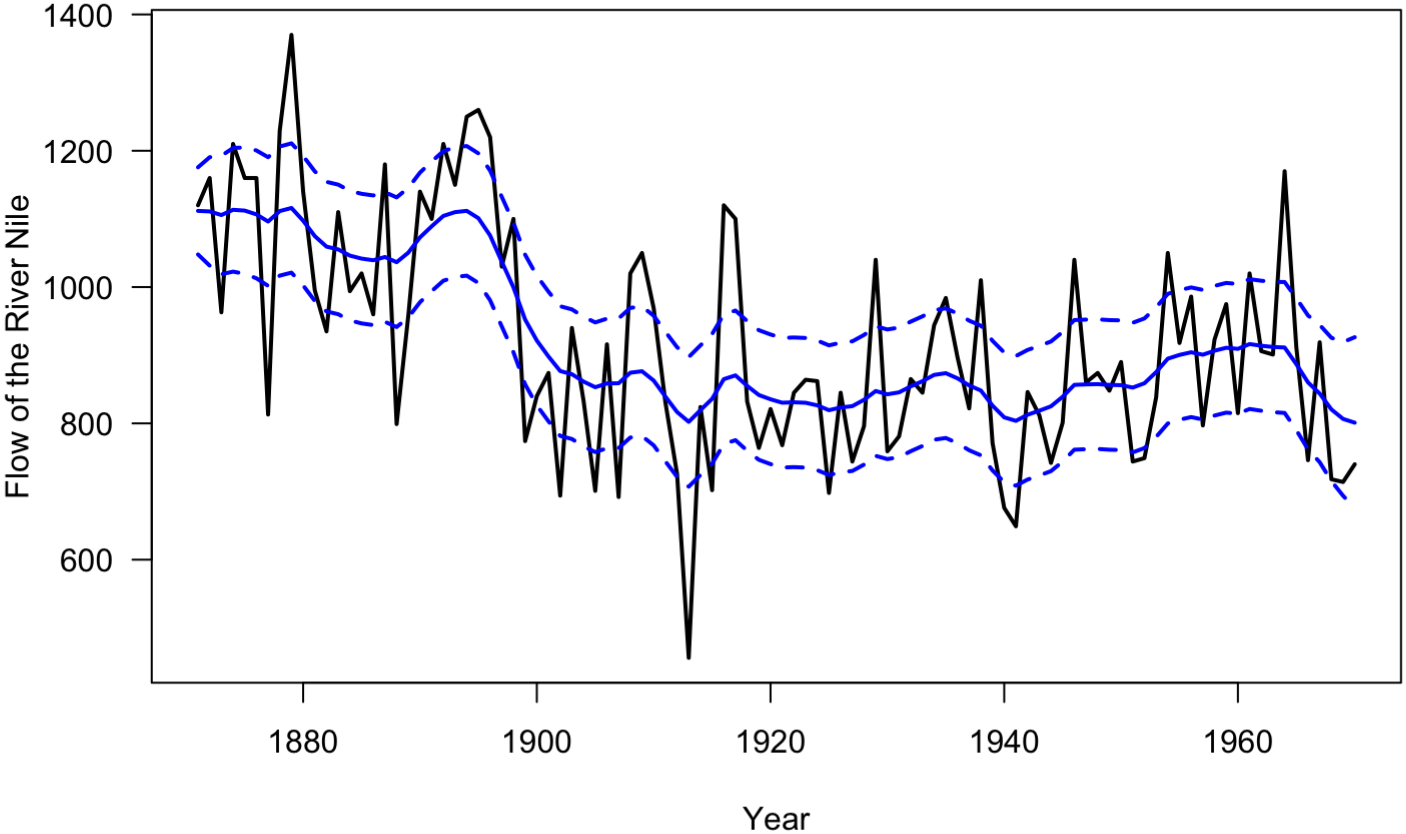
Stochastic level = random walk with obs error

$$
\begin{aligned}
y_t &= \mu_t + e_t \\
\mu_t &= \mu_{t-1} + w_t \\
&\Downarrow \\
y_t &= x_t + v_t \\
x_t &= x_{t-1} + w_t
\end{aligned}
$$

Ex of stochastic level model



Ex of stochastic level model



Univariate DLM for level & growth

Stochastic "level" μ_t with deterministic "growth" η

$$
\begin{aligned}
y_t &= \mu_t + e_t \\
\mu_t &= \mu_{t-1} + \eta + w_t
\end{aligned}
$$

Univariate DLM for level & growth

Stochastic "level" μ_t with deterministic "growth" η

$$
\begin{gathered}
y_t = \mu_t + e_t \\
\mu_t = \mu_{t-1} + \eta + w_t \\
\Downarrow \\
y_t = x_t + v_t \\
x_t = x_{t-1} + u + w_t
\end{gathered}
$$

This is just a random walk with bias u

Univariate DLM for level & growth

Stochastic "level" μ_t with stochastic "growth" β_t

$$
\begin{aligned}
y_t &= \mu_t + e_t \\
\mu_t &= \mu_{t-1} + \beta_{t-1} + w_{\mu,t} \\
\beta_t &= \beta_{t-1} + w_{\beta,t}
\end{aligned}
$$

Now the "growth" term β_t evolves as well

Univariate DLM for level & growth

Evolution of α_t and η_t

$$
\begin{aligned}
\alpha_t &= \alpha_{t-1} + \eta_{t-1} + w_{\alpha,t} \\
\eta_t &= \eta_{t-1} + w_{\eta,t} \\
&\Downarrow \\
\alpha_t &= 1\alpha_{t-1} + 1\eta_{t-1} + w_{\alpha,t} \\
\eta_t &= 0\alpha_{t-1} + 1\eta_{t-1} + w_{\eta,t}
\end{aligned}
$$

Univariate DLM for level & growth

Evolution of $\hat{}_t$ and $\ddot{}_t$

$$
\begin{aligned}
\hat{}_t &= \hat{}_{t-1} + \ddot{}_{t-1} + w_{,t} \\
\ddot{}_t &= \ddot{}_{t-1} + w_{,t} \\
&\Downarrow \\
\hat{}_t &= \underline{1}\,\hat{}_{t-1} + \underline{1}\,\ddot{}_{t-1} + w_{,t} \\
\ddot{}_t &= \underline{0}\,\hat{}_{t-1} + \underline{1}\,\ddot{}_{t-1} + w_{,t} \\
&\Downarrow
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \hat{}_t \\ \ddot{}_t \end{bmatrix}}_{?_t} = \underbrace{\begin{bmatrix} \underline{1} & \underline{1} \\ \underline{0} & \underline{1} \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \hat{}_{t-1} \\ \ddot{}_{t-1} \end{bmatrix}}_{?_{t-1}} + \underbrace{\begin{bmatrix} w_{,t} \\ w_{,t} \end{bmatrix}}_{\mathbf{w}_t}
$$

Univariate DLM for level & growth

Observation model for stochastic "level" & stochastic "growth"

$$
\begin{aligned}
y_t &= \alpha_t + v_t \\
&\Downarrow \\
y_t &= \underline{1}\alpha_t + \underline{0}\beta_t + v_t \\
&\Downarrow \\
y_t &= \underbrace{\begin{bmatrix} \underline{1} & \underline{0} \end{bmatrix}}_{\mathbf{x}_t^\top} \underbrace{\begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix}}_{\theta_t} + v_t
\end{aligned}
$$

Univariate DLM for regression

Stochastic intercept and slope

$$
\begin{aligned}
y_t &= \alpha_t + \beta_t x_t + v_t \\
&\Downarrow \\
y_t &= \underline{1}\,\alpha_t + \underline{x_t}\,\beta_t + v_t \\
&\Downarrow \\
y_t &= \underbrace{\begin{bmatrix} \underline{1} & \underline{x_t} \end{bmatrix}}_{\mathbf{x}_t^\top} \underbrace{\begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix}}_{\theta_t} + v_t
\end{aligned}
$$

Univariate DLM for regression

Parameter evolution follows a random walk

$$
\begin{aligned}
\hat{}\;_t &= \hat{}\;_{t-1} + w_{\;,t} \\
\tilde{}\;_t &= \tilde{}\;_{t-1} + w_{\;,t} \\
&\Downarrow \\
\underbrace{\begin{bmatrix} \hat{}\;_t \\ \tilde{}\;_t \end{bmatrix}}_{\;_t} &= \underbrace{\begin{bmatrix} \grave{}\;_{t-1} \\ \tilde{}\;_{t-1} \end{bmatrix}}_{\;_{t-1}} + \underbrace{\begin{bmatrix} w_{\;,t} \\ w_{\;,t} \end{bmatrix}}_{\mathbf{w}_t}
\end{aligned}
$$

Univariate DLM with seasonal effect

Dynamic linear regression with fixed seasonal effect

$$y_t = \alpha_t + \beta_t x_t + \gamma_{qtr} + e_t$$

$$\gamma_{qtr} = \begin{cases} \gamma_1 & \text{if qtr} = 1 \\ \gamma_2 & \text{if qtr} = 2 \\ \gamma_3 & \text{if qtr} = 3 \\ \gamma_4 & \text{if qtr} = 4 \end{cases}$$

Univariate DLM with seasonal effect

$$y_t = \alpha_t + \beta_t x_t + \gamma_{qtr} + e_t$$

$$\Downarrow$$

$$y_t = \begin{bmatrix} 1 & x_t & 1 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_{qtr} \end{bmatrix} + e_t$$

Univariate DLM with seasonal effect

$$y_t = \begin{bmatrix} 1 & x_t & 1 \end{bmatrix} \begin{bmatrix} \hat{\ }_t \\ \tilde{\ }_t \\ \bar{\ }_{qtr} \end{bmatrix} + e_t$$

$$\begin{bmatrix} \hat{\ }_t \\ \tilde{\ }_t \\ \bar{\ }_{qtr} \end{bmatrix} = \begin{bmatrix} \hat{\ }_{t-1} \\ \tilde{\ }_{t-1} \\ ? \end{bmatrix} + \begin{bmatrix} w_{\ ,t} \\ w_{\ ,t} \\ ? \end{bmatrix}$$

How should we model the fixed effect of $\bar{\ }_{qtr}$?

Univariate DLM with seasonal effect

$$y_t = \begin{bmatrix} 1 & x_t & 1 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_{qtr} \end{bmatrix} + e_t$$

$$\begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_{qtr} \end{bmatrix} = \begin{bmatrix} \alpha_{t-1} \\ \beta_{t-1} \\ \gamma_{qtr} \end{bmatrix} + \begin{bmatrix} w_{\alpha,t} \\ w_{\beta,t} \\ 0 \end{bmatrix}$$

We don't want the effect of quarter to evolve

Univariate DLM with seasonal effect

$$y_t = \begin{bmatrix} 1 & x_t & 1 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_{qtr} \end{bmatrix} + e_t$$

$$\begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_{qtr} \end{bmatrix} = \begin{bmatrix} \alpha_{t-1} \\ \beta_{t-1} \\ \gamma_{qtr} \end{bmatrix} + \begin{bmatrix} w_{\alpha,t} \\ w_{\beta,t} \\ 0 \end{bmatrix}$$

OK, but how do we select the right quarterly effect?

Univariate DLM with seasonal effect

Let's separate out the quarterly effects

$$y_t = \alpha_t + \beta_t x_t + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + e_t$$

$$\Downarrow$$

$$y_t = \begin{bmatrix} 1 & x_t & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

But how do we select only the current quarter?

Univariate DLM with seasonal effect

We can set some values in $\mathbf{x}_t$ to 0 ($qtr = 1$)

$$
y_t = \begin{bmatrix} 1 & x_t & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}
$$

$$
\Downarrow
$$

$$
y_t = \alpha_t + \beta_t x_t + \gamma_1 + e_t
$$

Univariate DLM with seasonal effect

We can set some values in $\mathbf{x}_t$ to 0 (qtr = 2)

$$y_t = \begin{bmatrix} 1 & x_t & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

$$\Downarrow$$

$$y_t = \alpha_t + \beta_t x_t + \gamma_2 + e_t$$

Univariate DLM with seasonal effect

But *how* would we set the correct 0/1 values?

$$\mathbf{X}_t^\top = \begin{bmatrix} 1 & x_t & ? & ? & ? & ? \end{bmatrix}$$

Univariate DLM with seasonal effect

We could instead reorder the γ_i within θ_t ($qtr = 1$)

$$y_t = \begin{bmatrix} 1 & x_t & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

$$\Downarrow$$

$$y_t = \alpha_t + \beta_t x_t + \gamma_1 + e_t$$

Univariate DLM with seasonal effect

We could instead reorder the γ_i within θ_t $(\text{qtr} = 2)$

$$y_t = \begin{bmatrix} 1 & x_t & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_1 \end{bmatrix}$$

$$\Downarrow$$

$$y_t = \alpha_t + \beta_t x_t + \gamma_2 + e_t$$

Univariate DLM with seasonal effect

But *how* would we shift the $\bar{}_i$ within ˀ $_t$?

$$
ˀ_t = \begin{bmatrix} \hat{}_t \\ \tilde{}_t \\ ? \\ ? \\ ? \\ ? \end{bmatrix}
$$

Example of non-diagonal $\mathbf{G}$

We can use a non-diagonal $\mathbf{G}$ to get the correct quarter effect

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Evolving parameters

$$\underbrace{\begin{bmatrix} \hat{\ }_{t} \\ \tilde{\ }_{t} \\ -_{2} \\ -_{3} \\ -_{4} \\ -_{1} \end{bmatrix}}_{?_{t}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \hat{\ }_{t-1} \\ \tilde{\ }_{t-1} \\ -_{1} \\ -_{2} \\ -_{3} \\ -_{4} \end{bmatrix}}_{?_{t-1}} + \underbrace{\begin{bmatrix} \mathrm{w}_{\ ,t} \\ \mathrm{w}_{\ ,t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{w}_t}$$

Evolving parameters

$$\underbrace{\begin{bmatrix} {}_{t} \\ \sim_{t} \\ -_{3} \\ -_{4} \\ -_{1} \\ -_{2} \end{bmatrix}}_{{}_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} {}_{t-1} \\ \sim_{t-1} \\ -_{2} \\ -_{3} \\ -_{4} \\ -_{1} \end{bmatrix}}_{{}_{t-1}} + \underbrace{\begin{bmatrix} \mathrm{w}_{,t} \\ \mathrm{w}_{,t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{w}_t}$$

Forecasting with a DLM

DLMs are often used in a forecasting context where we want a prediction for time t based on the data up through time $t - 1$

Forecasting with a DLM

Pseudo-code

1. get estimate of today's parameters from yesterday's
2. make prediction based on today's parameters & covariates
3. get observation for today
4. update parameters and repeat

Forecasting with a DLM

1. Define the parameters at time $t = 0$

$$\boldsymbol{\theta}_0 | y_0 = \hat{\boldsymbol{\pi}} + \mathbf{w}_1 \text{ with } \mathbf{w}_1 \sim \text{MVN}(\mathbf{0}, \boldsymbol{\Lambda})$$

Forecasting with a DLM

1. Define the parameters at time $t = 0$

$$\boldsymbol{\theta}_0 | y_0 = \boldsymbol{\pi} + \mathbf{w}_1 \text{ with } \mathbf{w}_1 \sim \text{MVN}(\mathbf{0}, \boldsymbol{\Lambda})$$
$$\Downarrow$$
$$\text{E}(\boldsymbol{\theta}_0) = \boldsymbol{\pi}$$
$$\text{Var}(\boldsymbol{\theta}_0) = \boldsymbol{\Lambda}$$
$$\Downarrow$$
$$\boldsymbol{\theta}_0 | y_0 \sim \text{MVN}(\boldsymbol{\pi}, \boldsymbol{\Lambda})$$

Forecasting with a DLM

1. Define the parameters at time $t = 1$

$$\boldsymbol{\theta}_1 | y_0 = \mathbf{G}\boldsymbol{\theta}_0 + \mathbf{w}_1 \text{ with } \mathbf{w}_1 \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

Forecasting with a DLM

1. Define the parameters at time $t = 1$

$$
\begin{gathered}
\boldsymbol{\theta}_1 | y_0 = \mathbf{G} \boldsymbol{\theta}_0 + \mathbf{w}_1 \text{ with } \mathbf{w}_1 \sim \text{MVN}(\mathbf{0}, \mathbf{Q}) \\
\Downarrow \\
\text{E}(\boldsymbol{\theta}_1) = \mathbf{G} \boldsymbol{\theta}_0 \\
\text{E}(\boldsymbol{\theta}_1) = \mathbf{G} \boldsymbol{\pi} \\
\text{and} \\
\text{Var}(\boldsymbol{\theta}_1) = \mathbf{G} \text{Var}(\boldsymbol{\theta}_0) \mathbf{G}^\top + \mathbf{Q} \\
\text{Var}(\boldsymbol{\theta}_1) = \mathbf{G} \boldsymbol{\Lambda} \mathbf{G}^\top + \mathbf{Q} \\
\Downarrow \\
\boldsymbol{\theta}_1 | y_0 \sim \text{MVN}(\mathbf{G} \boldsymbol{\pi}, \mathbf{G} \boldsymbol{\Lambda} \mathbf{G}^\top + \mathbf{Q})
\end{gathered}
$$

Forecasting with a DLM

1. Make a forecast of y_t at time $t = 1$

$$
\begin{gathered}
y_1 | y_0 = \mathbf{X}_1^\top \;\; {}_1 + e_1 \text{ with } e_1 \sim \text{N}(0, \text{R}) \\
\Downarrow \\
\text{E}(y_1) = \mathbf{X}_1^\top \;\; {}_1 \\
\text{E}(y_1) = \mathbf{X}_1^\top [\mathbf{G} \quad] \\
\text{and} \\
\text{Var}(y_1) = \mathbf{X}_1^\top \text{Var}(\;\; {}_1) \mathbf{X}_1 + \text{R} \\
\text{Var}(y_1) = \mathbf{X}_1^\top [\mathbf{G} \Lambda \mathbf{G}^\top + \mathbf{Q}] \mathbf{X}_1 + \text{R} \\
\Downarrow \\
y_1 | y_0 \sim \text{N}(\mathbf{X}_1^\top [\mathbf{G} \quad], \mathbf{X}_1^\top [\mathbf{G} \Lambda \mathbf{G}^\top + \mathbf{Q}] \mathbf{X}_1 + \text{R})
\end{gathered}
$$

Forecasting with a DLM

Putting it all together

$$
\begin{aligned}
\boldsymbol{\theta}_0 | y_0 &\sim \text{MVN}(\boldsymbol{\pi}, \boldsymbol{\Lambda}) \\
\boldsymbol{\theta}_t | y_{t-1} &\sim \text{MVN}(\mathbf{G}\boldsymbol{\pi}, \mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\top + \mathbf{Q}) \\
y_t | y_{t-1} &\sim \text{N}(\mathbf{X}_t^\top [\mathbf{G}\boldsymbol{\pi}], \mathbf{X}_t^\top [\mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\top + \mathbf{Q}] \mathbf{X}_t + R)
\end{aligned}
$$

Forecasting with a DLM

Putting it all together

$$
\begin{aligned}
\boldsymbol{\theta}_0 | y_0 &\sim \text{MVN}(\boldsymbol{\pi}, \boldsymbol{\Lambda}) \\
\boldsymbol{\theta}_t | y_{t-1} &\sim \text{MVN}(\mathbf{G}\boldsymbol{\pi}, \mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\top + \mathbf{Q}) \\
y_t | y_{t-1} &\sim \text{N}(\mathbf{X}_t^\top [\mathbf{G}\boldsymbol{\pi}], \mathbf{X}_t^\top [\mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\top + \mathbf{Q}]\mathbf{X}_t + R)
\end{aligned}
$$

Using `MARSS()` will make this easy to do

Diagnostics for DLMs

Just as with other models, we'd like to know if our fitted DLM meets its underlying assumptions

We can calcuate the forecast error e_t as

$$e_t = y_t - \hat{y}_t$$

and check if

$$\begin{aligned} &(1) \;\; e_t \sim N(0, \quad) \\ &(2) \;\; \text{Cov}(e_t, e_{t-1}) = 0 \end{aligned}$$

with a QQ-plot (1) and an ACF (2)

MULTIVARIATE DLMs

The most simple multivariate DLM

Multiple observations of a stochastic level

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{Z}\alpha_t + \mathbf{v}_t & \mathbf{y}_t \text{ is } n \times 1 \\
\alpha_t &= \alpha_{t-1} + w_t & \alpha_t \text{ is } 1 \times 1
\end{aligned}
$$

with

$$
\mathbf{Z} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
$$

Another simple multivariate DLM

Multiple observations of multiple levels

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{Z} \;\; {}_t + \mathbf{v}_t \qquad & \mathbf{y}_t \text{ is } n \times 1 \\
\hat{\;} \;\; {}_t &= \hat{\;} \;\; {}_{t-1} + \mathbf{w}_t \qquad & \hat{\;} \;\; {}_t \text{ is } n \times 1
\end{aligned}
$$

with

$$
\mathbf{Z} = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}
$$

Multivariate DLMs

Regression model

Our univariate model

$$y_t = \mathbf{X}_t^\top \boldsymbol{\beta}_t + e_t \text{ with } e_t \sim \text{N}(0, \text{R})$$

becomes

$$\mathbf{y}_t = (\mathbf{X}_t^\top \otimes \mathbf{I}_n) \boldsymbol{\beta}_t + \mathbf{e}_t \text{ with } \mathbf{e}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$$

Kronecker products

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix

then $\mathbf{A} \otimes \mathbf{B}$ will be an $mp \times nq$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

Kronecker products

For example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

so

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1 \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} & 2 \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \\ 3 \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} & 4 \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 & 8 \\ 6 & 8 & 12 & 16 \\ 6 & 12 & 8 & 16 \\ 18 & 24 & 24 & 32 \end{bmatrix}$$

Multivariate DLMs

Regression model with $n = 2$

$$
\mathbf{y}_t = (\mathbf{X}_t^\top \otimes \mathbf{I}_n) {}_t + \mathbf{e}_t
$$

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \left(\begin{bmatrix} 1 & x_t \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} {}_{1,t} \\ {}_{2,t} \\ {}_{1,t} \\ {}_{2,t} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix}
$$

Multivariate DLMs

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \left(\begin{bmatrix} 1 & x_t \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \alpha_{1,t} \\ \alpha_{2,t} \\ \beta_{1,t} \\ \beta_{2,t} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix}
$$

$$
\Downarrow
$$

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_t & 0 \\ 0 & 1 & 0 & x_t \end{bmatrix} \begin{bmatrix} \alpha_{1,t} \\ \alpha_{2,t} \\ \beta_{1,t} \\ \beta_{2,t} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix}
$$

Multivariate DLMs

Covariance of observation errors

$$\mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \text{ or } \mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{bmatrix}$$

$$\mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma & \gamma & \gamma & \gamma \\ \gamma & \sigma & \gamma & \gamma \\ \gamma & \gamma & \sigma & \gamma \\ \gamma & \gamma & \gamma & \sigma \end{bmatrix} \text{ or } \mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & \gamma_{2,4} \\ 0 & 0 & \sigma_3 & 0 \\ 0 & \gamma_{2,4} & 0 & \sigma_4 \end{bmatrix}$$

Multivariate DLMs

Parameter evolution

$$\boldsymbol{\theta}_t = \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \text{ with } \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

becomes

$$\boldsymbol{\theta}_t = (\mathbf{G} \otimes \mathbf{I}_n)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \text{ with } \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

Multivariate DLMs

Parameter evolution

If we have 2 regression parameters and $n = 2$, then

$$\boldsymbol{\theta}_t = \begin{bmatrix} \alpha_{1,t} \\ \alpha_{2,t} \\ \beta_{1,t} \\ \beta_{2,t} \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

Multivariate DLMs

Parameter evolution

$$
\begin{gathered}
\boldsymbol{\theta}_t = (\mathbf{G} \otimes \mathbf{I}_n)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \\
\Downarrow \\
\boldsymbol{\theta}_t = (\mathbf{I}_2 \otimes \mathbf{I}_2)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t
\end{gathered}
$$

Multivariate DLMs

$\mathbf{I}_{m} \otimes \mathbf{I}_{n} = \mathbf{I}_{mn}$

$$\mathbf{I}_2 \otimes \mathbf{I}_2 = \begin{bmatrix} 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multivariate DLMs

Parameter evolution

$$
\begin{aligned}
\boldsymbol{\theta}_t &= (\mathbf{G} \otimes \mathbf{I}_n)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \\
\boldsymbol{\theta}_t &= (\mathbf{I}_2 \otimes \mathbf{I}_2)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t
\end{aligned}
$$

$$
\begin{bmatrix} \alpha_{1,t} \\ \alpha_{2,t} \\ \beta_{1,t} \\ \beta_{2,t} \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \alpha_{1,t-1} \\ \alpha_{2,t-1} \\ \beta_{1,t-1} \\ \beta_{2,t-1} \end{bmatrix}
+
\begin{bmatrix} w_{1,t} \\ w_{2,t} \\ w_{1,t} \\ w_{2,t} \end{bmatrix}
$$

$$
\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t
$$

Multivariate DLMs

Evolution variance

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \text{ with } \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \underline{\mathbf{Q}})$$

What form should we choose for $\mathbf{Q}$?

Multivariate DLMs

Evolution variance

$$\begin{bmatrix} \hat{\ }_{t} \\ \tilde{\ }_{t} \end{bmatrix} \sim \text{MVN}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \right)$$

$$\mathbf{Q}_{(\cdot)} = \begin{bmatrix} \text{q}_{(\cdot)} & 0 & \dots & 0 \\ 0 & \text{q}_{(\cdot)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \text{q}_{(\cdot)} \end{bmatrix}$$

Diagonal and equal (IID)

Multivariate DLMs

Evolution variance

$$
\begin{bmatrix} \hat{\ }_{t} \\ \tilde{\ }_{t} \end{bmatrix} \sim \text{MVN}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \right)
$$

$$
\mathbf{Q}_{(\cdot)} = \begin{bmatrix} q_{(\cdot)1} & 0 & \dots & 0 \\ 0 & q_{(\cdot)2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_{(\cdot)n} \end{bmatrix}
$$

Diagonal and unequal

Multivariate DLMs

Evolution variance

$$\begin{bmatrix} \hat{}\ _t \\ \tilde{}\ _t \end{bmatrix} \sim \text{MVN}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \right)$$

$$\mathbf{Q}_{(\cdot)} = \begin{bmatrix} q_{(\cdot)1,1} & q_{(\cdot)1,2} & \cdots & q_{(\cdot)1,n} \\ q_{(\cdot)2,1} & q_{(\cdot)2,2} & \cdots & q_{(\cdot)2,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{(\cdot)n,1} & q_{(\cdot)n,2} & \cdots & q_{(\cdot)n,n} \end{bmatrix}$$

Unconstrained

Multivariate DLMs

Evolution variance

$$\begin{bmatrix} \hat{\ } \quad {}_t \\ \tilde{\ } \quad {}_t \end{bmatrix} \sim \text{MVN}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}\end{bmatrix}\right)$$

Topics for today

Univariate response

- Stochastic level & growth
- Dynamic Regression
- Dynamic Regression with fixed season
- Forecasting with a DLM
- Model diagnostics

Multivariate response

In practice, keep $\mathbf{Q}$ as simple as possible