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Assessing the association between two spatial or temporal sequences

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This paper deals with the codispersion coefficient for spatial and temporal series. We present some results and simulations concerning the codispersion coefficient in the context of spatial models. The results obtained are immediate consequences of the asymptotic normality of the sample codispersion coefficient and show certain limitations of the coefficient. New simulation studies provide information about the performance of the coefficient with respect to other coefficients of spatial association. The behavior of the codispersion coefficient under additively contaminated processes is also studied via Monte Carlo simulations. In the context of time series, explicit expressions for the asymptotic variance of the sample version of the coefficient are given for autoregressive and moving average processes. Resampling methods are used to compute the variance of the coefficient. A real data example is presented to explore how well the codispersion coefficient captures the comovement between two time series in practice.

Keywords: spatial association; autoregressive models; correlation coefficient; codispersion coefficient; time series

1. Introduction

This paper addresses the study of the association between two spatial/time sequences. This topic arises frequently in various fields such as geography and regional sciences [7], hydrology [24], soil sciences [25], epidemiology [10] and geology [29].

Several coefficients of association have been proposed to measure the quantitative assessment between two spatial or temporal series. Most of the proposals have been oriented from a non-parametric perspective. For example, Tjøstheim [43] suggested a measure of association between spatial variables. Subsequently, that coefficient was generalized by Hubert and Golledge [22].

The correlation coefficient has also been used as a measure of spatial association. The standard covariance statistic $\sum(X_i - \bar{X})(Y_i - \bar{Y})$ could be interpreted as a crude similarity index for two

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time/space sequences. One centers the series at zero, codes the movement of X and Y in the direction with respect to zero as a positive contribution to the statistic, and movement in the opposite direction as a negative contribution. Interpreted in this fashion, the normalized statistic is an indicator of codispersion, although it may be inadequate as a detailed characteristic. The statistic, for example, does not take into account the direction of movement from point to point.

Clifford *et al.* [8] approached this problem by introducing a modified test of association based on the correlation coefficient. Later, Richardson and Clifford [35] gave some complementary results on the performance of the tests for small domains and on their power.

Alternatively, the correlation between two or more sequences has also been tackled using spectral domain approaches. While univariate spectral methods allow the detection of movements inside each series, the bivariate spectral analysis provides a way to describe pairs of sequences in a frequency domain, by decomposing their covariance in frequency components. Similar to the one-dimensional spectral analysis, the cross spectrum is obtained by substituting the cross-covariance function for the autocovariance function. In general, the cross spectrum is a complex function; the real part of the cross spectrum is called cospectrum and the imaginary part is called the quadrature spectrum. Two functions that have a time-domain equivalent (indicated in parentheses) are defined in terms of the cross spectrum; they are the coherency function (correlation function) and the phase spectrum (time lag). The coherency spectrum measures the degree to which one series can be represented as a linear function of the other. The phase spectrum measures the phase difference between the frequency components of the two time series. A description of these functions and examples can be found in Shumway and Stoffer [39]. An application of spectral methods to urban traffic flows was studied by Stathopoulos and Karlaftis [41]. Extensions of spectral approaches to spatial sequences have also been studied in the literature, see [47, pp. 151–153].

Matheron [30] introduced a measure of spatial association between two spatial variables. This coefficient is called the codispersion coefficient and has been used in several applications [6,16,17,18,45]. The codispersion coefficient is useful to study the association between variables at different spatial scales. Rukhin and Vallejos [37] studied the codispersion coefficient from both the theoretical and applied perspectives. The limiting distribution of the sample coefficient was obtained for arbitrary lags, and the performance of the estimators was examined via Monte Carlo simulation. The codispersion coefficient was compared with the known coefficients of association. The hypothesis testing problem of independence between two spatial processes was explored and the coverage probability of a confidence interval for the population coefficient was also addressed.

The codispersion coefficient in a time series is of interest to study how well two time sequences move together. Non-parametric tests for the comovement of time series have appeared in several pieces of the literature [48]. Such tests have typically been oriented to providing the independence of two sequences. Other independence tests for two time series make use of statistics derived from the cross-correlation function [19]. Leigh, Pearlman and Rukhin [28] studied the codispersion coefficient in this context.

In this work, we provide results and examples that enrich the discussion about the codispersion coefficients for spatial and temporal models. We extend the results of Rukhin and Vallejos [37] in the following ways. First, the theoretical results in this paper, which are straightforward consequences of the asymptotic normality of the sample codispersion coefficient, show some limitations of this coefficient. Secondly, new simulations experiments are carried out to observe the sensitivity of the codispersion coefficient with respect to the correlation coefficient when there is additive contamination in one of the processes. Thirdly, a particularization of the codispersion coefficient to a time series is motivated and discussed. Examples are provided to illustrate the computation of the asymptotic variance of the coefficient. Very explicit expressions for the asymptotic variance are given for AR(1) and MA(1) processes. Finally, resampling techniques are introduced to compute the variance of the coefficient in practice.

2. Codispersion coefficient for spatial sequences

2.1 *Preliminaries and notation*

Consider two spatial processes, X and Y, defined on a part of a rectangular lattice $D \subset \mathbb{Z}^d$. The cross-variogram is defined as follows [9, p. 67]:

$$\gamma(h) = \mathbb{E}[X(s+h) - X(s)][Y(s+h) - Y(s)],$$

such that $s, s+h \in D$. The codispersion coefficient is defined as

$$\rho(h) = \frac{\gamma(h)}{\sqrt{V_X(h)V_Y(h)}}, \tag{1}$$

where $V_X(h) = \mathbb{E}[X(s+h) - X(s)]^2$.

Let us consider processes of the form

$$X(s+h) - X(s) = \sum_t c_t \epsilon_{s-t}, \tag{2}$$

with a squarely summable coefficient c_t and ϵ_i being zero mean uncorrelated random variables with common variance. For $Z(s) = (X(s), Y(s))^{\mathrm{T}}$, we assume that

$$Z(s+h) - Z(s) = \sum C_t \epsilon(s-t), \tag{3}$$

where $C_t = C_t(h)$ are 2×2 matrices such that $\sum_t ||C_t|| < \infty$ and ϵ_t are uncorrelated random errors with mean 0 and covariance matrix Σ. We denote $\mathbb{S} = \{t : C_t \neq 0\}$. There are several processes that satisfy the condition given in Equation (2), for example the first-order spatial autoregressive models [9,43] and the spatial moving average models [13]. Explicit expressions for $\rho(h)$ have been derived for first-order spatial AR models [36], however, for high-order spatial ARMA models there is no closed form for $\rho(h)$. Numerical approximations have been implemented to inspect the behavior of $\rho(h)$ for models containing a large number of parameters [46].

A natural estimator of $\rho(h)$ is the sample codispersion coefficient for two processes satisfying Equation (3),

$$\widehat{\rho}(h) = \frac{\sum_{s \in N(h)}(X(s+h) - X(s))(Y(s+h) - Y(s))}{\sqrt{\sum_{s \in N(h)}(X(s+h) - X(s))^2 \sum_{s \in N(h)}(Y(s+h) - Y(s))^2}}, \tag{4}$$

where $N(h) = N_M(h) = \{s : s+h \in D_M\}$. $N = |N(h)|$ is the cardinality of $N(h)$. The following result was established by Rukhin and Vallejos [37] in the context of spatial models.

THEOREM 1 *If the processes X and Y can be represented by Equation* (3) *with zero mean error* $\epsilon(t)$ *possessing a fourth finite moment, then*

$$N^{1/2}[\widehat{\rho}(h) - \rho(h)] \xrightarrow{\mathcal{L}} \mathcal{N}(0, v_h^2),$$

where

$$v_h^2 = \frac{\varphi_{1122}}{\kappa_{11}\kappa_{22}} + \frac{\kappa_{12}^2\varphi_{1111}}{4\kappa_{11}^3\kappa_{22}} + \frac{\kappa_{12}^2\varphi_{2222}}{4\kappa_{11}\kappa_{22}^3} - \frac{\kappa_{12}\varphi_{1112}}{\kappa_{11}^2\kappa_{22}} - \frac{\kappa_{12}\varphi_{1222}}{\kappa_{11}\kappa_{22}^2} + \frac{\kappa_{12}^2\varphi_{1212}}{2\kappa_{11}^2\kappa_{22}^2}, \tag{5}$$

$\kappa_{i,j}$ *are the elements of the matrix K defined as*

$$K = E[Z(s+h) - Z(s)][Z(s+h) - Z(s)]^{\mathrm{T}},$$

and φ_{ijkl} are the entries of 4×4 *matrix* Φ *defined as*

$$\Phi = \mathbb{E}[(Z(s+h) - Z(s))(Z(s+h) - Z(s))^{\mathrm{T}} - K] \\ \otimes [(Z(s+h) - Z(s))(Z(s+h) - Z(s))^{\mathrm{T}} - K].$$

The spatial ARMA models are defined as

$$\Phi(B_1, B_2)X(i, j) = \Theta(B_1, B_2)\epsilon(i, j),$$

where

$$\Phi(B_1, B_2) = \sum_k \sum_l \phi(k, l) B_1^k B_2^l,$$

$$\Theta(B_1, B_2) = \sum_k \sum_l \theta(k, l) B_1^k B_2^l,$$

with $B_1 X(i, j) = X(i-1, j)$, $B_2 X(i, j) = X(i, j-1)$, and $\epsilon(i, j)$ are a collection of independent random variables with mean zero and common variance σ^2.

A finite version of the spatial ARMA models are the unilateral ARMA models where the value at the site (i, j) is a finite autoregression on the values at the sites which are in the lower quadrant of (i, j) [43]. Thus

$$X(i, j) = \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \phi(k, l) X(i-k, j-l) + \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \theta(k, l)\epsilon(i-k, j-l), \tag{6}$$

with $\phi(0, 0) = \theta(0, 0) = 0$.

2.2 *Some Results*

RESULT 1 *Consider the moving average processes X and Y described by the equations*

$$X(i, j) = \epsilon_1(i, j) - \theta_1 \epsilon_1(i-1, j) - \theta_2 \epsilon_1(i, j-1), \tag{7}$$

$$Y(i, j) = \epsilon_2(i, j) - \eta_1 \epsilon_2(i-1, j) - \eta_2 \epsilon_2(i, j-1), \tag{8}$$

where $\epsilon(t) = (\epsilon_1(t), \epsilon_2(t))^{\mathrm{T}}$ form independent and identically distributed (i.i.d.) random vectors with mean 0 and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix}. \tag{9}$$

Then

$$\rho(h_1, h_2) = \begin{cases} \dfrac{\rho[2(\theta_1\eta_1 + \theta_2\eta_2) - \theta_1 - \eta_1]}{2\sqrt{(1+\theta_1^2+\theta_2^2+\theta_1)(1+\eta_1^2+\eta_2^2+\eta_1)}}, & h_1 = 1, \ h_2 = 0, \\ \dfrac{\rho[2(\theta_1\eta_1 + \theta_2\eta_2) - \theta_2 - \eta_2]}{2\sqrt{(1+\theta_1^2+\theta_2^2+\theta_2)(1+\eta_1^2+\eta_2^2+\eta_2)}}, & h_1 = 0, \ h_2 = 1, \\ \dfrac{\rho(\theta_1\eta_1 + \theta_2\eta_2)}{\sqrt{(1+\theta_1^2+\theta_2^2)(1+\eta_1^2+\eta_2^2)}}, & h_1 \geq 1, \ h_2 \geq 1. \end{cases}$$

Proof The proof follows from Equation (1). ■

The following result establishes the asymptotic normality of the sample similarity coefficient for two three-dimensional models. This illustrates that condition 3 is not restrictive only for models defined on $D \subset \mathbb{Z}^2$.

Consider a separable three-dimensional ARMA model. This class of models is an extension of the spatial ARMA models and can be described by

$$\Phi(B_1, B_2, B_3)X(i, j, k) = \epsilon(i, j, k),$$

where $B_3 X(i, j, k) = X(i, j, k-1)$, $\epsilon(i, j, k)$ is a white noise, $\phi_3 = \phi_1\phi_2$, $\lambda_3 = \lambda_1\lambda_2$, and the parameters satisfy the conditions necessary to have a stationary process. Similar to the two-dimensional case, we assume that the polynomial $\Phi(z_1, z_2, z_3) \neq 0$ for $|z_1| < 1, |z_2| < 1$ and $|z_3| < 1$.

Result 2 *If X and Y are described by the equations*

$$\begin{aligned}X(i, j, k) &= \phi_1 X(i-1, j, k) + \phi_2 X(i, j-1, k) + \phi_3 X(i, j, k-1) - \phi_1\phi_2 X(i-1, j-1, k)\\ &\quad - \phi_2\phi_3 X(i, j-1, k-1) - \phi_1\phi_3 X(i-1, j, k-1)\\ &\quad + \phi_1\phi_2\phi_3 X(i-1, j-1, k-1) + \epsilon_1(i, j, k), \qquad (10)\\ Y(i, j, k) &= \xi_1 Y(i-1, j, k) + \xi_2 Y(i, j-1, k) + \xi_3 Y(i, j, k-1) - \xi_1\xi_2 Y(i-1, j-1, k)\\ &\quad - \xi_2\xi_3 Y(i, j-1, k-1) - \xi_1\xi_3 Y(i-1, j, k-1)\\ &\quad + \xi_1\xi_2\xi_3 Y(i-1, j-1, k-1) + \epsilon_2(i, j, k), \qquad (11)\end{aligned}$$

where $\epsilon(t) = (\epsilon_1(t), \epsilon_2(t))^{\mathrm{T}}$ *are random vectors with a covariance structure given by Equation* (9), *then*

$$\rho(h) = \frac{\rho(2 - \phi_1^{h_1}\phi_2^{h_2}\phi_3^{h_3} - \xi_1^{h_1}\xi_2^{h_2}\xi_3^{h_3})}{2(1-\phi_1\xi_1)(1-\phi_2\xi_2)(1-\phi_3\xi_3)\sqrt{\frac{(1-\phi^{h_1}\phi_2^{h_2}\phi_3^{h_3})}{(1-\phi_1^2)(1-\phi_2^2)(1-\phi_3^2)(1-\xi_1^2)(1-\xi_2^2)(1-\xi_3^2)}}}. \qquad (12)$$

Proof Because of the factorization of the polynomial $\Phi(z_1, z_2, z_3)$, we have a representation for X and Y of the form

$$X(i, j, k) = \sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty} \phi_1^{t_1}\phi_2^{t_2}\phi_3^{t_3}\epsilon_1(i-t_1, j-t_2, k-t_3),$$

$$Y(i, j, k) = \sum_{t_4=0}^{\infty}\sum_{t_5=0}^{\infty}\sum_{t_6=0}^{\infty} \xi_1^{t_4}\xi_2^{t_5}\xi_3^{t_6}\epsilon_1(i-t_4, j-t_5, k-t_6).$$

Then, with $h = (h_1, h_2, h_3), h_1, h_2, h_3 \geq 0$,

$$\begin{aligned}
\mathbb{E}[X(i,j,k)Y(i,j,k)] &= \mathbb{E}[X(i+h_1, j+h_2, k+h_3) \\
&\quad \times Y(i+h_1, j+h_2, k+h_3)] \\
&= \frac{\rho\sigma\tau}{(1-\phi_1\xi_1)(1-\phi_2\xi_2)(1-\phi_3\xi_3)}, \\
\mathbb{E}[X(i+h_1, j+h_2, k+h_3)Y(i,j,k)] &= \frac{\rho\sigma\tau\phi_1^{h_1}\phi_2^{h_2}\phi_3^{h_3}}{(1-\phi_1\xi_1)(1-\phi_2\xi_2)(1-\phi_3\xi_3)}, \\
\mathbb{E}[X(i,j,k)Y(i+h_1, j+h_2, k+h_3)] &= \frac{\rho\sigma\tau\xi_1^{h_1}\xi_2^{h_2}\xi_3^{h_3}}{(1-\phi_1\xi_1)(1-\phi_2\xi_2)(1-\phi_3\xi_3)}, \\
\mathbb{E}[X(i+h_1, j+h_2, k+h_3) - X(i,j,k)]^2 &= \frac{2\sigma^2(1-\phi^{h_1}\phi_2^{h_2}\phi_3^{h_3})}{(1-\phi_1^2)(1-\phi_2^2)(1-\phi_3^2)}.
\end{aligned}$$

Similar formulae for Y can be derived. The result follows from Equation (1). ■

Corollary 1 *$\widehat{\rho}(h)$ is asymptotically normal with mean zero and the variance given by Equation* (5)

Proof Notice that $Z(i,j,k) = (X(i,j,k), Y(i,j,k))^{\mathrm{T}}$ has the convergent representation

$$Z(s) = \sum_{t\in\mathbb{T}} A_t \epsilon(s-t),$$

with $\mathbb{T} = \mathbb{Z}_+^3$, and $A_t = A_{t_1t_2t_3} = A_1^{t_1}A_2^{t_2}A_3^{t_3}$, where $A_l = \mathrm{diag}(\phi_l, \xi_l), l = 1, 2, 3$. From [36, Example 6.1 and Equation (17)], we have that $Z(s)$ satisfies Equation (3) with the corresponding matrices C_t also diagonal. By Theorem 1, $\widehat{\rho}(h)$ is asymptotically normal with the mean zero and the variance given by Equation (5). ■

Remark 1 If we relax the separability condition on the polynomial $\Phi(z_1, z_2, z_3)$, Corollary 1 still holds, with

$$A_t = A_{t_1t_2t_3} = \frac{(t_1+t_2+t_3)!}{t_1!t_2!t_3!}\begin{pmatrix}\phi_1^{t_1}\phi_2^{t_2}\phi_3^{t_3} & 0 \\ 0 & \xi_1^{t_1}\xi_2^{t_2}\xi_3^{t_3}\end{pmatrix}.$$

There are some useful models for which the condition given in Equation (3) is not satisfied. As an example, consider a particular case of model (6) (Pickard's model) described by

$$X(i,j) = \phi_1 X(i-1,j) + \phi_2 X(i,j-1) + \phi_3 X(i-1,j-1) + \epsilon(i,j). \tag{13}$$

Stationarity requires $|\phi_1 + \phi_2| < 1 - \phi_3$ and $|\phi_1 - \phi_2| < 1 + \phi_3$ (see [38]). Then, using a multinomial expansion for $(1 - \phi_1 B_1 - \phi_2 B_2 - \phi_3 B_1 B_2)^{-1}$, we have the convergent representation [4]

$$X(i,j) = \sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty}\frac{(t_1+t_2+t_3)!}{t_1!t_2!t_3!}\phi_1^{t_1}\phi_2^{t_2}\phi_3^{t_3}\epsilon(i-t_3-t_2, j-t_2-t_3).$$

Notice that X cannot be represented in the form (2) and thus, representation (3) does not exist. Moreover, in this case, $\rho(h)$ does not have a closed form. Hence the asymptotic normality of the sample codispersion coefficient for models like Equation (13) needs further research.

2.3 *Simulations*

Consider X and Y following

$$X(i,j) = \phi_1 X(i-1,j) + \phi_2 X(i,j-1) + \epsilon_1(i,j), \tag{14}$$

$$Y(i,j) = \psi_1 Y(i-1,j) + \psi_2 Y(i,j-1) + \epsilon_2(i,j), \tag{15}$$

and the correlation structure (9) for the errors. Here, we developed a Monte Carlo simulation study to explore the capability of the codispersion coefficient to capture the correlation between X and Y. In the Monte Carlo simulation, two sets of parameters and six values for ρ were used: $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\psi_1 = 0.3$, $\psi_2 = 0.2$; $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\psi_1 = 0.6$, $\psi_2 = 0.1$, $\rho = 0, 0.1, 0.3, 0.5, 0.7$ and 0.9. In each case, the parameters satisfy the necessary conditions to have stationary processes. For instance, for model (14), these conditions are $|\phi_1|, |\phi_2| < 1$, $(1+\phi_1^2-\phi_2^2)^2 - 4\phi_1^2 > 0$ and $1-\phi_2^2 > |\phi_1|$ (see [32]). One thousand replicates were obtained for each case of the two sets of parameters. The average values and the standard deviations of the estimates over these replicates can be found in Tables 1 and 2. The simulations were performed for the grid size 30×30.

Table 1. The means and standard deviations of $\hat{\rho}(h)$, r and $\hat{\rho}_T$ for $\rho = 0, 0.1$ and 0.3.

ρ	ϕ_1	ϕ_2	ψ_1	ψ_2	h_1	h_2	$\hat{\rho}(h)$ $s.d.(\hat{\rho}(h))$	r $s.d.(r)$	$\hat{\rho}_T$ $s.d.(\hat{\rho}_T)$
0	0.3	0.2	0.3	0.2	1	0	−0.0006 (0.0384)	0.0002 (0.0389)	0.0028 (0.0391)
0	0.3	0.2	0.3	0.2	0	1	0.0008 (0.0416)	0.0004 (0.0388)	0.0027 (0.0391)
0	0.3	0.2	0.3	0.2	1	1	0.0012 (0.0405)	0.0006 (0.0394)	0.0028 (0.0391)
0	0.3	0.2	0.6	0.1	1	0	−0.0008 (0.0365)	0.0003 (0.0421)	0.0054 (0.0397)
0	0.3	0.2	0.6	0.1	0	1	0.0006 (0.0454)	0.0002 (0.0421)	0.0054 (0.0397)
0	0.3	0.2	0.6	0.1	1	1	0.0012 (0.0434)	0.0005 (0.0426)	0.0054 (0.0397)
0.1	0.3	0.2	0.3	0.2	1	0	0.0990 (0.0411)	0.1008 (0.0423)	0.0466 (0.0406)
0.1	0.3	0.2	0.3	0.2	0	1	0.1009 (0.0424)	0.1006 (0.0417)	0.0466 (0.0406)
0.1	0.3	0.2	0.3	0.2	1	1	0.0992 (0.0443)	0.1010 (0.0427)	0.0466 (0.0406)
0.1	0.3	0.2	0.6	0.1	1	0	0.0955 (0.0382)	0.0941 (0.0449)	0.0453 (0.0421)
0.1	0.3	0.2	0.6	0.1	0	1	0.0941 (0.0462)	0.0941 (0.0446)	0.0453 (0.0421)
0.1	0.3	0.2	0.6	0.1	1	1	0.0929 (0.0469)	0.0944 (0.0455)	0.0453 (0.0421)
0.3	0.3	0.2	0.3	0.2	1	0	0.2989 (0.0378)	0.3006 (0.0389)	0.1343 (0.0405)
0.3	0.3	0.2	0.3	0.2	0	1	0.3006 (0.0390)	0.3004 (0.0383)	0.1343 (0.0405)
0.3	0.3	0.2	0.3	0.2	1	1	0.2990 (0.0408)	0.3007 (0.0393)	0.1343 (0.0405)
0.3	0.3	0.2	0.6	0.1	1	0	0.2878 (0.0352)	0.2793 (0.0416)	0.1246 (0.0401)
0.3	0.3	0.2	0.6	0.1	0	1	0.2806 (0.0428)	0.2792 (0.0413)	0.1246 (0.0401)
0.3	0.3	0.2	0.6	0.1	1	1	0.2796 (0.0434)	0.2795 (0.0421)	0.1246 (0.0401)

Table 2. The means and standard deviations of $\hat{\rho}(h)$, r and $\hat{\rho}_T$ for $\rho = 0.5$, 0.7 and 0.9.

ρ	ϕ_1	ϕ_2	ψ_1	ψ_2	h_1	h_2	$\hat{\rho}(h)$ $s.d.(\hat{\rho}(h))$	r $s.d.(r)$	$\hat{\rho}_T$ $s.d.(\hat{\rho}_T)$
0.5	0.3	0.2	0.3	0.2	1	0	0.4990 (0.0312)	0.5003 (0.0320)	0.2257 (0.0398)
0.5	0.3	0.2	0.3	0.2	0	1	0.5004 (0.0322)	0.5002 (0.0316)	0.2257 (0.0398)
0.5	0.3	0.2	0.3	0.2	1	1	0.4990 (0.0337)	0.5004 (0.0324)	0.2257 (0.0398)
0.5	0.3	0.2	0.6	0.1	1	0	0.4803 (0.0292)	0.4644 (0.0349)	0.2092 (0.0385)
0.5	0.3	0.2	0.6	0.1	0	1	0.4670 (0.0359)	0.4643 (0.0346)	0.2092 (0.0385)
0.5	0.3	0.2	0.6	0.1	1	1	0.4665 (0.0364)	0.4646 (0.0354)	0.2092 (0.0385)
0.7	0.3	0.2	0.3	0.2	1	0	0.6992 (0.0212)	0.7001 (0.0218)	0.3232 (0.0373)
0.7	0.3	0.2	0.3	0.2	0	1	0.7002 (0.0219)	0.7000 (0.0215)	0.3232 (0.0373)
0.7	0.3	0.2	0.3	0.2	1	1	0.6992 (0.0230)	0.7002 (0.0220)	0.3232 (0.0373)
0.7	0.3	0.2	0.6	0.1	1	0	0.6729 (0.0203)	0.6495 (0.0248)	0.3005 (0.0370)
0.7	0.3	0.2	0.6	0.1	0	1	0.6535 (0.0254)	0.6494 (0.0246)	0.3005 (0.0370)
0.7	0.3	0.2	0.6	0.1	1	1	0.6535 (0.0257)	0.6497 (0.0251)	0.3005 (0.0370)
0.9	0.3	0.2	0.3	0.2	1	0	0.8996 (0.0079)	0.9001 (0.0081)	0.4316 (0.0353)
0.9	0.3	0.2	0.3	0.2	0	1	0.9000 (0.0082)	0.8999 (0.0080)	0.4316 (0.0353)
0.9	0.3	0.2	0.3	0.2	1	1	0.8996 (0.0086)	0.9000 (0.0082)	0.4316 (0.0353)
0.9	0.3	0.2	0.6	0.1	1	0	0.8657 (0.0084)	0.8348 (0.0111)	0.3931 (0.0365)
0.9	0.3	0.2	0.6	0.1	0	1	0.8401 (0.0113)	0.8346 (0.0111)	0.3931 (0.0365)
0.9	0.3	0.2	0.6	0.1	1	1	0.8407 (0.0113)	0.8349 (0.0113)	0.3931 (0.0365)

For models (14) and (15), we get

$$\rho(h) = \frac{\rho[2D(\phi_1, \phi_2, \psi_1, \psi_2, 0, 0) - D(\phi_1, \phi_2, \psi_1, \psi_2, h_1, h_2)(\psi_1^{h_1}\psi_2^{h_2} + \phi_1^{h_1}\phi_2^{h_2})]}{2\sqrt{R}}, \qquad (16)$$

where

$$\begin{aligned} R &= [D(\phi_1, \phi_2, \phi_1, \phi_2, 0, 0) - \phi_1^{h_1}\phi_2^{h_2} D(\phi_1, \phi_2, \phi_1, \phi_2, h_1, h_2)] \\ &\quad \times [D(\psi_1, \psi_2, \psi_1, \psi_2, 0, 0) - \psi_1^{h_1}\psi_2^{h_2} D(\psi_1, \psi_2, \psi_1, \psi_2, h_1, h_2)], \end{aligned}$$

$$D(\phi_1, \phi_2, \psi_1, \psi_2, h_1, h_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(k+l)!(l+h_2+k+h_1)!}{k!l!(l+h_2)!(k+h_1)!}(\phi_1\psi_1)^k(\phi_2\psi_2)^l.$$

Due to Equation (16)

$$\mathbb{E}[\widehat{\rho}(h)] = c\rho(h),$$

where

$$c = \frac{2D(\phi_1, \phi_2, \psi_1, \psi_2, 0, 0) - D(\phi_1, \phi_2, \psi_1, \psi_2, h_1, h_2)(\psi_1^{h_1}\psi_2^{h_2} + \phi_1^{h_1}\phi_2^{h_2})}{2\sqrt{R}}.$$

Hence all simulation averages reported in Tables 1 and 2 were corrected by the factor $1/c$.

From Tables 1 and 2 we see that in all cases the codispersion coefficient and the correlation coefficient (r) are comparable and less biased than Tjøstheim's coefficient ($\hat{\rho}_T$). The bias of Tjøstheim's coefficient increases as ρ increases. Also, notice that the standard deviation of the codispersion coefficient is smaller than the standard deviation of Tjøstheim's coefficient when $\rho \geq 0.5$. The standard deviations of the codispersion coefficient and the correlation coefficient decrease as ρ increases. All estimators underestimate the true value. In general, the performance of the codispersion and correlation coefficients is better than that of Tjøstheim's coefficient for $\rho \neq 0$.

2.4 *Codispersion coefficient for contaminated models*

The importance of contaminated models has been demonstrated by numerous applied publications in the area of image processing and image analysis (see, for instance, [5]). Most of the proposals are oriented to provide robust estimators in parametric models, which represent the image intensity of a given picture by a small number of parameters. There are many estimation algorithms for different image models, but most of these methods are based on the assumption of Gaussian image intensity. However, the actual distribution of image intensity clearly deviates from the Gaussian distribution, and traditional estimation methods can be very sensitive to deviations from the Gaussian assumption. A more realistic assumption for the image models is a contaminated Gaussian noise. Robust estimators assuming a mixture of normal distributions have been widely studied [21,27]. In the context of spatial autoregressive models Allende, Galbiati and Vallejos [2] have developed generalized M- estimators for the parameters of a contaminated autoregressive model.

In time series there are two different types of outliers [12]: outliers of type I, known as innovation outliers (IOs), which affect all subsequent observations; and outliers of type II, called additive outliers (AOs), which have no effect on subsequent observations. Since IOs agree with the structure of (properly chosen) time series models, ordinary M-estimates or even LS are quite capable of mastering this benign situation. For this reason, the main interest in the literature is aimed at the development of robust estimation procedures for AO situations [1]. These concepts are formally described as follows.

Innovation outliers: In this case, the $\epsilon_2(s)$ have a G contaminated normal distribution

$$G = (1 - \zeta)N(0, \tau^2) + \zeta H, \tag{17}$$

where H is an arbitrary distribution with variance $\lambda^2 > \tau^2$. Hence, the innovations $\epsilon_2(s)$ come from a $N(0, \tau^2)$ with probability $1 - \zeta$ and from an arbitrary distribution H with probability ζ. The variables $\epsilon_2(s)$ coming from H are considered outliers. In practice, ζ varies between 0 and 0.15.

Additive outliers: Here, it is assumed that the spatial autoregressive model is not perfectly observable due to a small fraction ζ of observations which are generated by the outlier process $\{\delta(s)\nu(s)\}$, where $\delta(s)$ is such that $P[\delta(s) = 1] = \zeta$, $P[\delta(s) = 0] = 1 - \zeta$ and the variables $\nu(s)$ have an arbitrary distribution H; ζ is similar to the IO case. Thus, the observational model is

$$W(s) = Y(s) + \delta(s)\nu(s). \tag{18}$$

Therefore the spatial autoregressive model Y itself is observed with a probability $1-\zeta$, and with a probability ζ the observations $Y(s)$ are corrupted by an error with a distribution H, $s \in \mathbb{R}^2$.

It is of interest to observe the sensitivity of the codispersion and correlation coefficient under additive contamination. Intuitively, both estimators must be strongly affected by the presence of outliers. However, the performance of the codispersion coefficient in comparison with the correlation coefficient is unknown. To study the sensitivity of these coefficients, a numerical experiment was carried out considering X as in Equation (14) and Y as in Equation (18); that is, X is not contaminated (model (14)), and Y is subjected to additive contamination. This in not the only way to introduce contamination for spatial models. In an extension of the work of Fox [12], four types of outliers have been proposed for univariate time series analysis. They are the AOS, IOS, level shifts and temporary changes. These four types of outliers affect an observed time series and its residual process differently. The definitions have been extended to a multivariate framework, and the effects of multivariate outliers on the joint and marginal models have been examined by Tsay, Peña and Pankratz [44].

In our study, H is zero mean normal with variance $\lambda^2 = 10$. Two values of ζ were used, $\zeta = 0.05$ and $\zeta = 0.1$. The number of runs in each case is 1000. The results of the experiment are presented in Table 3 for each set of parameters. Table 3 shows that all estimators are strongly biased. We cannot conclude that one estimator performs better than the others in terms of the bias. In all cases, the simulation standard deviation of the correlation coefficient is slightly smaller than the simulation standard deviation of the codispersion coefficient. Also, note that since we have positive and negative outliers, the means of X and W are not seriously affected.

Table 3. The means and standard deviations of the codispersion and correlation coefficients under additive contamination.

ρ	ϕ_1	ϕ_2	ψ_1	ψ_2	$\widehat{\rho}(1,0)$ $(st.dev.)$	$\widehat{\rho}(0,1)$ $(st.dev.)$	$\widehat{\rho}(1,1)$ $(st.dev.)$	$\widehat{\rho}$ $(st.dev.)$
$\sigma^2=\tau^2=1, \zeta=0.05, \lambda^2=10$								
0.9	0.3	0.2	0.3	0.2	0.5795	0.5928	0.5847	0.5912
					(0.0451)	(0.0417)	(0.0399)	(0.0361)
0.9	0.3	0.2	0.6	0.1	0.4725	0.5045	0.4815	0.4809
					(0.0457)	(0.0343)	(0.0330)	(0.0291)
0.5	0.3	0.2	0.3	0.2	0.3237	0.3309	0.3261	0.3301
					(0.0427)	(0.0376)	(0.0395)	(0.0326)
0.5	0.3	0.2	0.6	0.1	0.2651	0.2815	0.2682	0.2674
					(0.0429)	(0.0355)	(0.0360)	(0.0313)
0.1	0.3	0.2	0.3	0.2	0.0658	0.0662	0.0660	0.0673
					(0.0440)	(0.0386)	(0.0406)	(0.0331)
0.1	0.3	0.2	0.6	0.1	0.0516	0.0575	0.0525	0.0525
					(0.0423)	(0.0368)	(0.0364)	(0.0313)
$\sigma^2=\tau^2=1, \zeta=0.1, \lambda^2=10$								
0.9	0.3	0.2	0.3	0.2	0.4664	0.4822	0.4831	0.4926
					(0.0435)	(0.0436)	(0.0401)	(0.0371)
0.9	0.3	0.2	0.6	0.1	0.3717	0.4279	0.4087	0.4128
					(0.0456)	(0.0362)	(0.0378)	(0.0319)
0.5	0.3	0.2	0.3	0.2	0.2605	0.2688	0.2691	0.2741
					(0.0439)	(0.0388)	(0.0409)	(0.0346)
0.5	0.3	0.2	0.6	0.1	0.2071	0.2353	0.2260	0.2289
					(0.0383)	(0.0383)	(0.0373)	(0.0322)
0.1	0.3	0.2	0.3	0.2	0.0530	0.0546	0.0532	0.0554
					(0.0437)	(0.0392)	(0.0405)	(0.0334)
0.1	0.3	0.2	0.6	0.1	0.0404	0.0465	0.0457	0.0455
					(0.0422)	(0.0364)	(0.0385)	(0.0319)

2.5 *Applications*

As immediate consequences of Theorem 1, confidence intervals for $\rho(h)$ and hypothesis testing for $H_0 : \rho(h) = 0$ can be derived in practical situations. Several Monte Carlo simulations were developed to explore both, the coverage probability of an approximate confidence interval for $\rho(h)$ and the power of the test $H_0 : \rho(h) = 0$ (see [46]).

To illustrate a direct application of codispersion in spatial statistics, Rukhin and Vallejos [37] discussed an example concerning properties of polymers. There is interest in the study of flammability properties of polymers. However, one weak aspect of polymers is that they are very combustible under certain conditions. One technique to improve its performance is the application of some resins that act as flame retardants. A data set produced at the National Institute of Standards and Technology (NIST) was analyzed to show that the codispersion coefficient is able to capture the similarity of images taken at the same distance and the dissimilarity of images taken from a different distance than the previous ones.

There are other potential applications of the codispersion coefficient in the context of spatial statistics. For example, in forensic sciences it is relevant to perform ballistic identification of bullets that have been retrieved from crime scenes. Gel images obtained from test-fires can be compared to identify the nature and features of the bullets [34].

3. The codispersion coefficient for temporal series

3.1 *The codispersion coefficient as a measure of the comovement*

In the context of temporal series, the codispersion coefficient defined in Equation (1) corresponds to a normalized inner product of first differences for the time sequences X and Y

$$\mathrm{cm}(X, Y) = \rho(1, 0) = \frac{\sum \Delta X \Delta Y}{\left(\sum(\Delta X)^2 \sum(\Delta Y)^2\right)^{1/2}}.$$

The coefficient $\mathrm{cm}(X, Y)$ is a geometrically natural comovement coefficient in that it compares proportional slopes at matched pairs of points across the sequences. Here, we quantitatively examine how well the two time series move together. Intuitively, it is clear what one means by comovement. Two curves or sequences comove if their respective sets of slopes are proportional, or are nearly so. Here, we give a motivation of the codispersion coefficient in time series.

For wide-sense stationarity differentiable processes, $X(t)$ and $Y(t)$, such that $X'(t) = X(t+1) - X(t)$ and $Y'(t) = Y(t+1) - Y(t)$, the comovement coefficient is defined by the formula

$$\mathrm{cm}(X(t), Y(t)) = \mathbb{E}[X'(t)Y'(t)]/[\mathrm{Var}(X'(t))\mathrm{Var}(Y'(t))]^{1/2}.$$

It is assumed that $\mathbb{E}[X'_t] < \infty$ and $\mathbb{E}[Y'_t] < \infty$.

For the two sequences $X(t)$ and $Y(t)$, the estimator of this coefficient is the following:

$$\widehat{\mathrm{cm}}(X(t), Y(t)) = \frac{\sum(X(t+1) - X(t))(Y(t+1) - Y(t))}{\left(\sum(X(t+1) - X(t))^2 \sum(Y(t+1) - Y(t))^2\right)^{1/2}}.$$

In the integral form for the smooth curves $X(t)$ and $Y(t)$, the sample comovement statistic is defined as

$$\widehat{\mathrm{cm}}(X_t, Y_t) = \frac{\int X'(t) \int Y'(t)\,\mathrm{d}t}{\left(\int (X'(t))^2\,\mathrm{d}t \int (Y'(t))^2\,\mathrm{d}t\right)^{1/2}}.$$

As defined, the comovement coefficient is not the correlation coefficient between the first derivatives. Because of the local nature of the comovement, in both the numerator and denominator of the coefficient, $\mathbb{E}[X(t)]$ and $\mathbb{E}[Y(t)]$ are not subtracted off. In fact, it would be undesirable to subtract these expected values (as is done, for instance, in the case of the ordinary covariance and correlation). Indeed, consider the example of two straight lines with positive slopes, where the value of the comovement coefficient equals 1, but where subtraction of the means leads to an indeterminate expression of the form 0/0. An alternative method is to think that the first differencing has already achieved the substraction of the mean(s).

Comparison of correlation between first-order differences of sequences, known in the literature as the variate difference correlation method (see [42]) is a technique used for examining the correlation between higher-frequency components of two sequences, the differencing being employed to filter the low-frequency excursions of the time series sequences being compared, so as to enhance the comparison of finer higher-frequency features.

The comovement coefficient shares a number of the standard properties of the correlation coefficient. It is straightforward to show that the comovement coefficient and its sampling forms are traslation invariant, positive homogeneous, symmetric in its arguments, positive definite for a sequence and lagged versions of itself, and interpretable as the cosine of the angle between the vectors formed by the first difference of the sampled series. The only slightly non-trivial point is the positive definitness, and that can be proved by imitating a standard proof for the autocovariance function [31]. Note that there is nothing in the definition of the comovement statistic that requires that the two sequences to be assessed be equispace sampled, as is usually the case with time series statistics. The first differencing can be done regardless of the uniformity or continuity of the spacings between observations. What is required by the definition is that the observations of the two sequences be matched in time.

If one compares a phase-lagged sine wave, $f(t) = f(t+h)$ with the unlagged $g(t) = \sin(t)$, is is easy to show that the comovement coefficient will be the cosine of the lag angle $\cos(h)$. Thus, in particular, for a lag of $\pi/2$ radians, the comovement is zero, as one would expect. As in the case of the classic correlation, a comovement coefficient of $+1$ indicates that the sample functions/processes being compared are rescaled/retranslated versions of one another. Similarly, a profile matched with its reflection across the time axis gives a comovement of -1.

A statistic related to the codispersion coefficient is the mean square successive difference $\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2/(n-1)$. This statistic was studied by Von Neumann in the 1940s and its exact distribution has been derived by Kamat [26] and Geisser [14,15], who showed that its variance in the normal case is $4(3n-4)\sigma^4/(n-1)^2$. The ratio of the mean square successive difference statistic to the sample variance $\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2/\sum_{i=1}^{n}(x_i-\overline{x})^2$ is known variously as the Von Neumann ratio or the Durbin-Watson statistic in regression analysis. Although there is no closed-form expression for the sampling distribution of the Von Neumann ratio, numerical approximations to the distribution was explored by Durbin and Watson [11].

In the simulations and examples developed in the next sections, we compute $\rho(h)$ for $h \geq 1$.

COROLLARY 2 *For the one-dimensional AR(1) processes of the form*

$$X(t) = \phi_1 X(t-1) + \epsilon_1(t), \tag{19}$$

$$Y(t) = \phi_2 Y(t-1) + \epsilon_2(t). \tag{20}$$

where the error vectors $\epsilon(t) = (\epsilon_1(t), \epsilon_2(t))^{\mathrm{T}}$, $t = 1, \ldots, n$, are independent normal random vectors with mean 0 and the covariance matrix Σ given by (9), the asymptotic variance of

$$n^{1/2}(\widehat{\mathrm{cm}}(X(t), Y(t)) - \mathrm{cm}(X(t), Y(t)))$$

is

$$v_1^2 = \left(1 - \frac{\rho^2(2-\phi_1-\phi_2)^2(1+\phi_1)(1+\phi_2)}{4(1-\phi_1\phi_2)^2}\right)^2. \tag{21}$$

Proof See the Appendix. ■

According to this formula, $v_1^2 = (1-\rho^2)^2$, when $\phi_1 = \phi_2 = \phi$, i.e. when each of the sequences $X(t)$ and $Y(t)$ are formed by i.i.d. random variables. When errors are uncorrelated, the asymptotic distribution of $\widehat{\rho}(1)$ is normal with mean zero and variance $1/n$.

Similar calculations show that for any lag h,

$$v_h^2 = \left(1 - \frac{\rho^2(1-(\phi_1^h+\phi_2^h)/2)^2(1-\phi_1^2)(1-\phi_2^2)}{(1-\phi_1\phi_2)^2(1-\phi_1^h)(1-\phi_2^h)}\right). \tag{22}$$

In Figure 1, we plotted $\rho(1)$ versus ϕ_1, ϕ_2 for $\rho = 0.9$. If $\phi_1 = \phi_2$, then $\rho(1) = \rho$, and the maximum value of the surface is attained. This pattern is graphically confirmed from the generated surface.

Corollary 3 *Consider the two* MA(1) *models described by the equations*

$$X(t) = \nu_1(t) + \theta_1\nu_1(t-1), \tag{23}$$

$$Y(t) = \nu_2(t) + \theta_2\nu_2(t-1). \tag{24}$$

where $|\theta_1|, |\theta_2| < 1$, *and the errors* $\epsilon(t) = (\nu_1(t), \nu_1(t))^{\mathrm{T}}$, $t = 1, 2, \ldots, n$ *form independent and identically distributed random vectors with mean zero and the covariance matrix* Σ. *Then*

$$\rho(1) = \frac{\rho(2-\theta_1-\theta_2+2\theta_1\theta_2)}{2\sqrt{(1-\theta_1+\theta_1^2)(1-\theta_2+\theta_2^2)}},$$

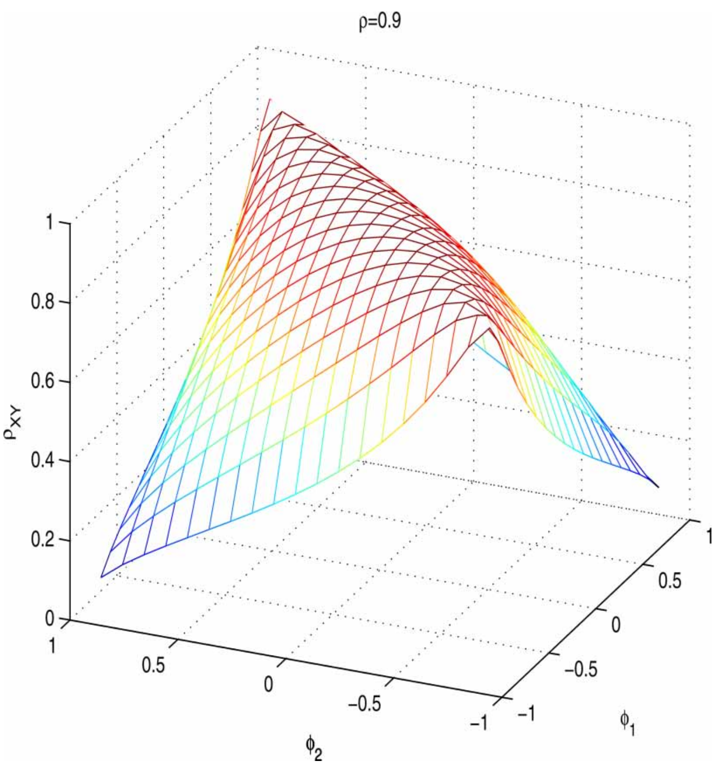

Figure 1. $\rho(1)$ versus ϕ_1, ϕ_2 for $\rho = 0.9$.

$$v_1^2 = \left(1 - \frac{\rho^2(2-\theta_1-\theta_2+2\theta_1\theta_2)^2}{4(1-\theta_1+\theta_1^2)(1-\theta_2+\theta_2^2)}\right)^2.$$

When $h \geq 2$,

$$v_h^2 = \left(1 - \frac{\rho^2(1+\theta_1\theta_2)^2}{(1+\theta_1^2)(1+\theta_2^2)}\right)^2.$$

3.2 *Computation of the variance*

In the previous subsection, the codispersion coefficient was discussed in the context of time series. One drawback of this coefficient is that for models with a large number of parameters, there is no explicit form for the variance v_h^2. This motivates the introduction of computational techniques to compute the variance of the coefficient in practice. One way to accomplish this is using a resampling procedure. Several bootstrap methods for stationary time series have been developed in the last decade. Some of them are based on blocking arguments, that is, the data are divided into blocks of consecutive observations, the blocks of length l are resampled with replacement and the selected blocks are concatenated to obtain resampled time series.

In this work, we consider the stationary bootstrap first introduced by Politis and Romano [33]. In this method, the block length is not fixed but random. Indeed let $L_1, L_2, \ldots$ be a sequence of i.i.d. random variables having a geometric distribution such that $\mathbb{P}(L_i = m) = (1-p)^{m-1}p, m = 1, 2, \ldots$. Independent of the L_i let $I_1, I_2, \ldots$ be a sequence of i.i.d. random variables from a discrete uniform distribution on $\{1, 2, \ldots, \}$. The blocks are $B(I_i, L_i) = \{X_{I_1}, X_{I_1+1}, \ldots, X_{I_i+L_i-1}\}$ and the parameter p is related to the block length of the moving block by $p = 1/l$. A discussion of the optimal choice of the block size can be found in Hall *et al.* [20]. In the context of the estimation of bias and variance of an estimator $l \sim n^{1/3}$.

The statistic T is a function of the $2n$ observations of the processes X_t and Y_t, such that

$$T = T(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n).$$

To explore the performance of this procedure we developed a Monte Carlo simulation experiment. The autoregressive models described in Equations (19) and (20) were considered under the correlation structure for the errors given in Equation (9). In each case, 300 observations from processes given in Equations (19) and (20) were generated. For several values of ρ, three sets of parameters are considered: (i) $\phi_1 = \phi_2 = 0.5$, (ii) $\phi_1 = 0.5$, $\phi_2 = 0.3$ and (iii) $\phi_1 = 0.1$, $\phi_2 = 0.2$. For each run, the codispersion coefficient was computed for $h = 1, 2, 3$. One thousand replicates were considered. All estimation results are summarized in Table 4, which report the mean and the bootstrap standard error of $\widehat{\rho}(h)$, $\widehat{se}_B = \widehat{se}_B(\widehat{\rho}(h))$. The value σ_ρ is given by $\sigma_\rho = v_h/\sqrt{n}$.

The simulation experiment was repeated for $X(t)$ and $Y(t)$ following Equations (23) and (24), respectively, with the correlation structure (9) for the errors. The combinations for θ_1 and θ_2 were the same as in the autoregressive case. In both experiments, 300 observations were considered for each series. The block length used was $l = (2n)^{1/3} \approx 8$.

From Table 1, we observed that the bootstrap estimate of the standard error has a better performance for the AR(1) models than the MA(1) models. The largest values of $|\widehat{se}_B - \sigma_\rho|$ for the MA(1) models are attained for the third combination of the parameters $\theta_1 = 0.1$, $\theta_2 = 0.2$ and $\rho = 0.8$. In all cases the performance of the bootstrap estimates are better for small h and small ρ.

3.3 *Real data example*

To illustrate the proposed coefficient, we make use of three series corresponding to the cardiovascular mortality, temperature and pollution from a study by Shumway *et al.* [40] of the possible

Table 4. The mean and bootstrap standard error of the codispersion coefficient.

AR(1)							MA(1)						
$\sigma^2 = \tau^2 = 1$							$\sigma^2 = \tau^2 = 1$						
ϕ_1	ϕ_2	h	$\hat{\rho}(h)$	$\widehat{se}_B$	σ_ρ	ρ	θ_1	θ_2	h	$\hat{\rho}(h)$	$\widehat{se}_B$	σ_ρ	ρ
0.5	0.5	1	0.0016	0.0604	0.0577	0	0.5	0.5	1	0.0018	0.0644	0.0577	0
0.5	0.5	1	0.3002	0.0546	0.0525	0.3	0.5	0.5	1	0.2991	0.0567	0.0478	0.3
0.5	0.5	2	0.3008	0.0561	0.0525	0.3	0.5	0.5	2	0.2984	0.0651	0.0478	0.3
0.5	0.5	3	0.2987	0.0658	0.0525	0.3	0.5	0.5	3	0.2982	0.0696	0.0478	0.3
0.5	0.5	1	0.5011	0.0446	0.0433	0.5	0.5	0.5	1	0.4988	0.0475	0.0324	0.5
0.5	0.5	2	0.4982	0.0445	0.0433	0.5	0.5	0.5	2	0.4977	0.0564	0.0324	0.5
0.5	0.5	3	0.4993	0.0569	0.0433	0.5	0.5	0.5	2	0.5000	0.0591	0.0324	0.5
0.5	0.5	1	0.7983	0.0221	0.0207	0.8	0.5	0.5	1	0.7991	0.0232	0.0074	0.8
0.5	0.5	2	0.7996	0.0209	0.0207	0.8	0.5	0.5	2	0.7993	0.0254	0.0074	0.8
0.5	0.5	3	0.7989	0.0273	0.0207	0.8	0.5	0.5	3	0.7986	0.0279	0.0074	0.8
0.5	0.3	1	0.0009	0.0621	0.0577	0	0.5	0.3	1	-0.0031	0.0631	0.0577	0
0.5	0.3	1	0.2961	0.0573	0.0526	0.3	0.5	0.3	1	0.2900	0.0588	0.0482	0.3
0.5	0.3	2	0.2892	0.0539	0.0526	0.3	0.5	0.3	2	0.2978	0.0624	0.0480	0.3
0.5	0.3	3	0.2883	0.0667	0.0526	0.3	0.5	0.3	3	0.2940	0.0681	0.0480	0.3
0.5	0.3	1	0.4912	0.0446	0.0437	0.5	0.5	0.3	1	0.4880	0.0457	0.0335	0.5
0.5	0.3	2	0.4889	0.0456	0.0437	0.5	0.5	0.3	2	0.4893	0.0562	0.0331	0.5
0.5	0.3	3	0.4847	0.0539	0.0437	0.5	0.5	0.3	3	0.4923	0.0571	0.0331	0.5
0.5	0.3	1	0.7865	0.0224	0.0218	0.8	0.5	0.3	1	0.7800	0.0247	0.0088	0.8
0.5	0.3	2	0.7847	0.0229	0.0218	0.8	0.5	0.3	2	0.7878	0.0268	0.0082	0.8
0.5	0.3	3	0.7763	0.0286	0.0218	0.8	0.5	0.3	3	0.7875	0.0286	0.0082	0.8
0.1	0.2	1	0.0000	0.0641	0.0577	0	0.1	0.2	1	0.0001	0.0638	0.0577	0
0.1	0.2	1	0.2956	0.0617	0.0525	0.3	0.1	0.2	1	0.2962	0.0602	0.0479	0.3
0.1	0.2	2	0.2956	0.0595	0.0525	0.3	0.1	0.2	2	0.2956	0.0607	0.0479	0.3
0.1	0.2	3	0.2959	0.0626	0.0525	0.3	0.1	0.2	3	0.2980	0.0641	0.0479	0.3
0.1	0.2	1	0.4934	0.0486	0.0434	0.5	0.1	0.2	1	0.4978	0.0491	0.0326	0.5
0.1	0.2	2	0.4987	0.0479	0.0434	0.5	0.1	0.2	2	0.4980	0.0511	0.0326	0.5
0.1	0.2	3	0.4953	0.0516	0.0434	0.5	0.1	0.2	3	0.4951	0.0511	0.0326	0.5
0.1	0.2	1	0.7968	0.0245	0.0210	0.8	0.1	0.2	1	0.7953	0.0245	0.0077	0.8
0.1	0.2	2	0.7958	0.0235	0.0210	0.8	0.1	0.2	2	0.7950	0.0249	0.0077	0.8
0.1	0.2	3	0.7937	0.0271	0.0210	0.8	0.1	0.2	3	0.7961	0.0242	0.0077	0.8

effects of pollution and temperature on daily mortality in Los Angeles, CA. This data set has been extensively studied in Shumway and Stoffer [39]. In Figure 2 we show the plots of these series. There are 508 observations for each series over the 10 year period 1970–1979.

Using the codispersion coefficient, we investigate the level of comovement between all possible pairs of series. Before computing the codispersion coefficient, a suitable ARMA model was fitted to each series, considering the classical Box–Jenkins approach. The matrix $\widehat{\Gamma} = \{\hat{\rho}(h)_{ij}\}_{i,j=1,\dots,3}$, was computed for $h = 1, \dots, 40$. The results are shown in Figure 3.

The sample codispersion coefficient between mortality and temperature series reveals an anti comovement between the series at lags beyond 6. The sample codispersion coefficient reaches the estimated correlation coefficient $\widehat{\rho} = -0.438$ at a lag greater than 13. The minimum value of the codispersion function is attained at $h = 27$.

The codispersion function between the mortality and pollution series have positive values for all lags indicating a comovement between these series. The maximum value is attained at $h = 22$ showing a long-term correlation. The estimated correlation coefficient is $\hat{\rho} = 0.443$.

For the temperature and pollution series, the codispersion function exhibits positive and negative correlation. Notice that $\widehat{\rho}(h)_{23} < 0$, for $h \geq 13$. The estimated correlation coefficient is $\widehat{\rho} = -0.017$ and the minimum codispersion is attained at $h = 27$.

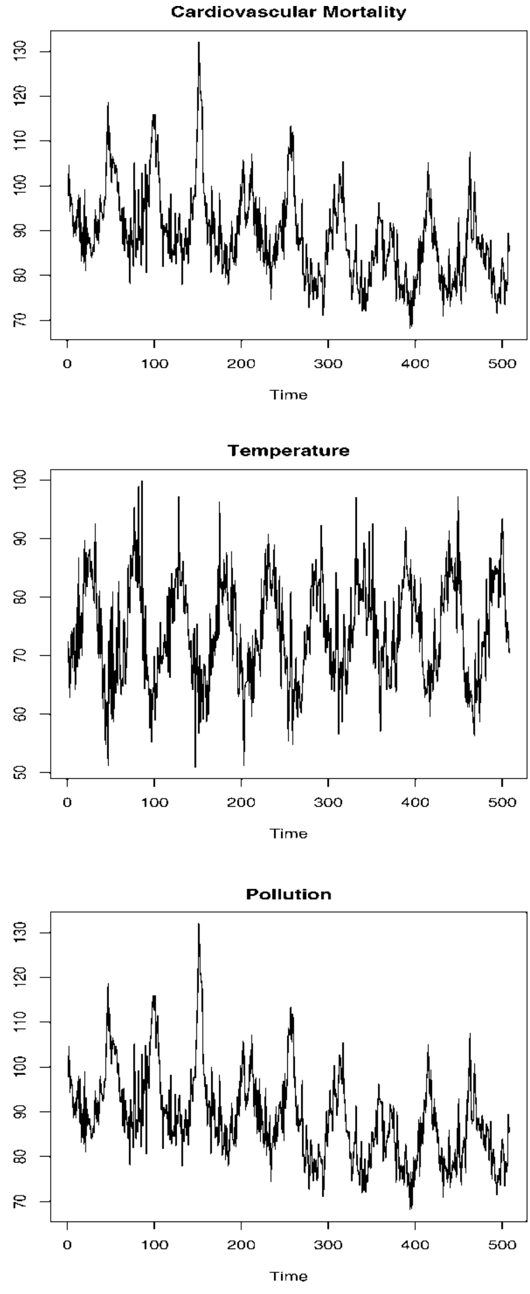

Figure 2. Cardiovascular mortality, temperature and pollution series.

Figure 3 show 95% bootstrap percentile confidence intervals for $\rho(h)$,

$$(\widehat{\rho}^{(\alpha/2)}(h), \widehat{\rho}^{(\alpha/2)}(h)),$$

where $\widehat{\rho}^{(\alpha/2)}(h)$ is the $\alpha/2$ percentile of the bootstrap estimators of $\rho(h)$. Notice that the bootstrap percentile confidence intervals are slightly wider for higher values of $|\widehat{\rho}(h)|$; this is in agreement with the results obtained by Vallejos [46], who reported that the codispersion coefficient is more biased and less precise for large values of $|\widehat{\rho}(h)|$.

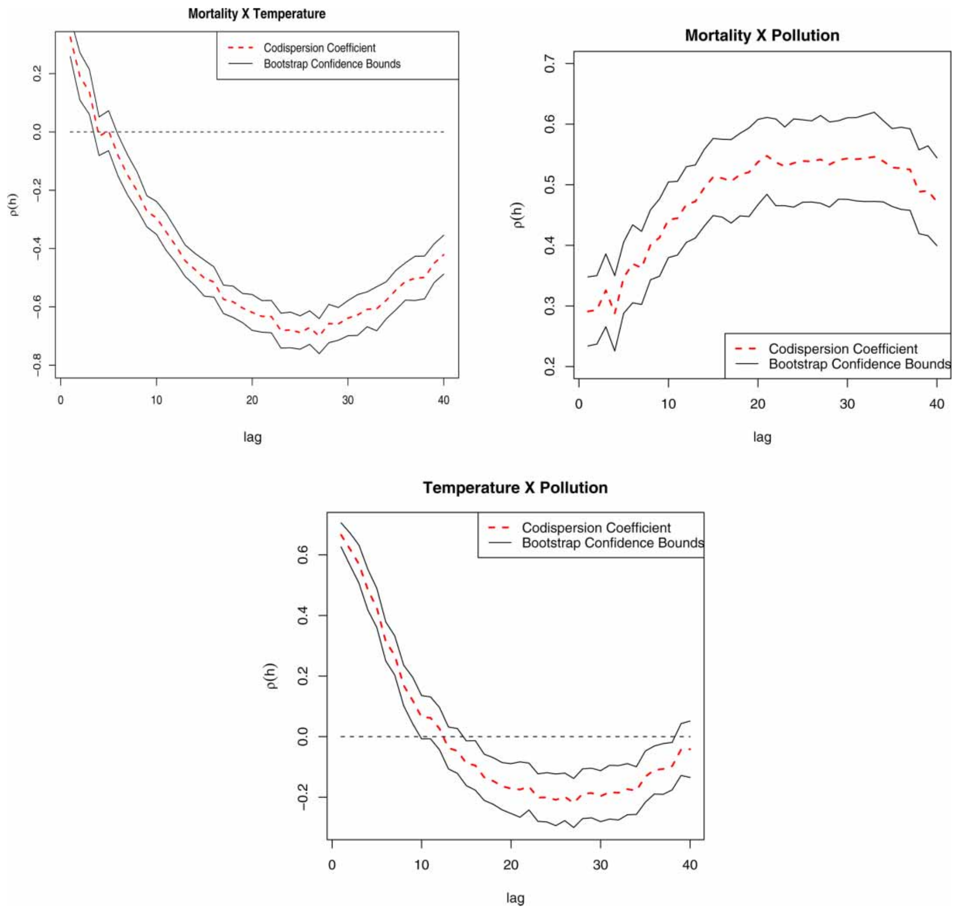

Figure 3. $\widehat{\rho}(h)$ and 95% bootstrap percentile confidence intervals for $\rho(h)$.

4. Conclusions

In this paper we studied the codispersion coefficient for spatial and temporal sequences. From the results and simulations presented in the first part of the paper we see that the codispersion coefficient is useful to describe the association between two spatial processes. From the Monte Carlo study developed in Section 2.3 we observe that the codispersion coefficient is better than Tjøstheim's coefficient and slightly better than the correlation coefficient. When ρ is large, the differences among these estimators are noticed. The performance of the coefficient under contamination needs further research. Indeed, the definition of a robustified version of the codispersion coefficient is still an open problem.

In the context of time series, resampling techniques are needed since there is no formula for the variance when the model contains a large number of parameters. In the simulation experiment developed in Section 3.2, the confidence intervals provided by the stationary bootstrap method appeared to perform well. The real data example developed in Section 3.3 shows that the codispersion coefficient can be used as a measure of comovement between two time series. This coefficient was able to detect a positive comovement between the mortality and pollution series, and a mixture of positive and negative comovements between the mortality and temperature, and the temperature and pollution series.

One interesting aspect to investigate in further research is the performance of the codispersion coefficient as a competitive alternative to spectral approaches in time series.

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Appendix

Proof of Corollary 2 A straightforward calculation shows that

$$\rho(1) = \frac{\rho(2-\phi_1-\phi_2)\sqrt{(1+\phi_1)(1+\phi_2)}}{2(1-\phi_1\phi_2)}.$$

Assuming independence of $Z(t-1)$ and $\epsilon(t)$ (see [3, Theorem 5.5.1]), we have that the vector $Z(t) = (X(t), Y(t))^{\mathrm{T}}$ satisfies

$$Z(t+1) - Z(t) = \sum_s A_s \epsilon(t-s),$$

where

$$A_s = \begin{cases} I, & s = -1 \\ -[F+I](-F)^s, & s \geq 0 \end{cases},$$

$$F = \begin{pmatrix} -\phi_1 & 0 \\ 0 & -\phi_2 \end{pmatrix}.$$

In this notation,

$$K = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \sum_{s_1, s_2 \geq -1} A_{s_1} E\epsilon_{t-s_1}\epsilon_{t-s_2}^{\mathrm{T}} A_{s_2} = \sum_{s=-1}^{\infty} A_s \Sigma A_s.$$

Hence

$$K = \begin{pmatrix} \dfrac{2\sigma_1^2}{1+\phi_1} & \dfrac{\rho\sigma_1\sigma_2(2-\phi_1-\phi_2)}{1-\phi_1\phi_2} \\ \dfrac{\rho\sigma_1\sigma_2(2-\phi_1-\phi_2)}{1-\phi_1\phi_2} & \dfrac{2\sigma_2^2}{1+\phi_2} \end{pmatrix}.$$

Taking into account the explicit form of matrix coefficients A_s, we have

$$A_s \otimes A_s = \mathrm{diag}(a^2\phi_1^{2s}, ab(\phi_1\phi_2)^s, ab(\phi_1\phi_2)^s, b^2\phi_2^{2s}),$$

where $a = \phi_1 - 1$, $b = \phi_2 - 1$. Notice that

$$\Phi = \sum_{s,u}[A_s \otimes A_s]U[A_u^{\mathrm{T}} \otimes A_u^{\mathrm{T}}] + \sum_{s,u}[A_s \otimes A_u]V[A_u^{\mathrm{T}} \otimes A_s^{\mathrm{T}}].$$

Defining

$$\Phi_1 = \sum_{s,u\geq -1}[A_s \otimes A_s]U[A_u \otimes A_u], \;\; \Phi_2 = \sum_{s,u\geq -1}[A_s \otimes A_u]V[A_u \otimes A_s],$$

where

$$U = \mathbb{E}[\epsilon(t)\epsilon(s)^{\mathrm{T}} \otimes \epsilon(t)\epsilon(s)^{\mathrm{T}}], \quad V = \mathbb{E}[\epsilon(t)\epsilon(s)^{\mathrm{T}} \otimes \epsilon(s)\epsilon(t)^{\mathrm{T}}],$$

one obtains

$$\Phi_1 = U + \sum_{u\geq 0} U[A_u \otimes A_u] + \sum_{s\geq 0}[A_s \otimes A_s]U + \sum_{s,u\geq 0}[A_s \otimes A_s]U[A_u \otimes A_u].$$

The relevant elements of Φ_1 are

$$\varphi^1_{1111} = \frac{4\sigma_{11}^2}{(1+\phi_1)^2},$$

$$\varphi^1_{1112} = \sigma_{11}\sigma_{12}\left[1 + \frac{(\phi_1-1)^2}{(1-\phi_1^2)} + \frac{(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_1-1)^3(\phi_2-1)}{(1-\phi_1^2)(1-\phi_1\phi_2)}\right]$$

$$= \frac{2\sigma_{11}\sigma_{12}(2-\phi_1-\phi_2)}{(1+\phi_1)(1-\phi_1\phi_2)},$$

$$\varphi^1_{1122} = \sigma_{11}\sigma_{22}\left[1 + \frac{(\phi_1-1)^2}{(1-\phi_1^2)} + \frac{(\phi_2-1)^2}{(1-\phi_2^2)} + \frac{(\phi_1-1)^2(\phi_2-1)^2}{(1-\phi_1^2)(1-\phi_2^2)}\right] = \frac{4\sigma_{11}\sigma_{22}}{(1+\phi_1)(1+\phi_2)},$$

$$\varphi^1_{1212} = \sigma_{12}^2\left[1 + \frac{2(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_1-1)^2(\phi_2-1)^2}{(1-\phi_1\phi_2)^2}\right] = \sigma_{12}^2\frac{(2-\phi_1-\phi_2)^2}{(1-\phi_1\phi_2)},$$

$$\varphi^1_{1222} = \sigma_{12}\sigma_{22}\left[1 + \frac{(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_2-1)^2}{(1-\phi_2^2)} + \frac{(\phi_1-1)(\phi_2-1)^3}{(1-\phi_1\phi_2)(1-\phi_2^2)}\right]$$

$$= \frac{2\sigma_{12}\sigma_{22}(2-\phi_1-\phi_2)}{(1+\phi_2)(1-\phi_1\phi_2)},$$

$$\varphi^1_{2222} = \frac{4\sigma_{22}^2}{(1+\phi_2)^2}.$$

Similarly

$$\Phi_2 = V + \sum_{u\geq 0}[I \otimes A_u]V[A_u \otimes I] + \sum_{s\geq 0}[A_s \otimes I]V[I \otimes A_s] + \sum_{s,u\geq 0}[A_s \otimes A_u]V[A_u \otimes A_s].$$

For $u \geq 0$, the matrix $I \otimes A_u$ is diagonal, with diagonal elements $(\phi_1-1)\phi_1^s, (\phi_2-1)\phi_2^s,$ $(\phi_1-1)\phi_1^s$ and $(\phi_2-1)\phi_2^s$. Also, for $u \geq 0$, $A_u \otimes I$ is diagonal with diagonal elements

$(\phi_1 - 1)\phi_1^u$, $(\phi_1 - 1)\phi_1^u$, $(\phi_2 - 1)\phi_2^u$ and $(\phi_2 - 1)\phi_2^u$. The relevant elements of Φ_2 are

$$\varphi_{1111}^2 = \frac{4\sigma_{11}^2}{(1+\phi_1)^2},$$

$$\varphi_{1112}^2 = \sigma_{11}\sigma_{12}\left[1 + \frac{(\phi_1-1)^2}{(1-\phi_1^2)} + \frac{(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_1-1)^3(\phi_2-1)}{(1-\phi_1^2)(1-\phi_1\phi_2)}\right] = \varphi_{1112}^1,$$

$$\varphi_{1122}^2 = \sigma_{12}^2\left[1 + \frac{2(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_1-1)^2(\phi_2-1)^2}{(1-\phi_1\phi_2)^2}\right] = \sigma_{12}^2\frac{(2-\phi_1-\phi_2)^2}{(1-\phi_1\phi_2)},$$

$$\varphi_{1212}^2 = \varphi_{1122}^2,$$

$$\varphi_{1222}^2 = \sigma_{12}\sigma_{22}\left[1 + \frac{(\phi_1-1)(\phi_2-1)}{(1-\phi_1\phi_2)} + \frac{(\phi_2-1)^2}{(1-\phi_2^2)} + \frac{(\phi_1-1)(\phi_2-1)^3}{(1-\phi_1\phi_2)(1-\phi_2^2)}\right],$$

$$\varphi_{2222}^2 = \frac{4\sigma_{22}^2}{(1+\phi_2)^2}.$$

By combining these facts, one obtains explicit formulas for the coefficients

$$\varphi_{ijkl} = \varphi_{ijkl}^1 + \varphi_{ijkl}^2;$$

$$\frac{\kappa_{12}^2\varphi_{1111}}{4\kappa_{11}^3\kappa_{22}} = \frac{\kappa_{12}^2\varphi_{2222}}{4\kappa_{22}^3\kappa_{11}} = \frac{\rho^2(2-\phi_1-\phi_2)^2(1+\phi_1)(1+\phi_2)}{8(1-\phi_1\phi_2)^2},$$

$$\frac{\kappa_{12}\varphi_{1112}}{\kappa_{11}^2\kappa_{22}} = \frac{\kappa_{12}\varphi_{1222}}{\kappa_{11}\kappa_{22}^2} = \frac{\rho^2(2-\phi_1-\phi_2)^2(1+\phi_1)(1+\phi_2)}{2(1-\phi_1\phi_2)^2},$$

$$\frac{\kappa_{12}^2\varphi_{1212}}{2\kappa_{11}^2\kappa_{22}^2} = \frac{\rho^4(2-\phi_1-\phi_2)^4(1+\phi_1)^2(1+\phi_2)^2}{16(1-\phi_1\phi_2)^4},$$

$$\frac{\varphi_{1122}}{\kappa_{11}\kappa_{22}} = 1 + \frac{\rho^2(2-\phi_1-\phi_2)^2(1+\phi_1)(1+\phi_2)}{4(1-\phi_1\phi_2)^2}.$$

The result follows by replacing these expressions in Equation (5). ■