Stochastic Hybrid Systems: Application to Communication Networks⋆

João P. Hespanha

Dept. of Electrical and Computer Eng.,
Univ. of California, Santa Barbara, CA 92106-9560

Abstract. We propose a model for Stochastic Hybrid Systems (SHSs) where transitions between discrete modes are triggered by stochastic events much like transitions between states of a continuous-time Markov chains. However, the rate at which transitions occur is allowed to depend both on the continuous and the discrete states of the SHS. Based on results available for Piecewise-Deterministic Markov Process (PDPs), we provide a formula for the extended generator of the SHS, which can be used to compute expectations and the overall distribution of the state.

As an application, we construct a stochastic model for on-off TCP flows that considers both the congestion-avoidance and slow-start modes and takes directly into account the distribution of the number of bytes transmitted. Using the tools derived for SHSs, we model the dynamics of the moments of the sending rate by an infinite system of ODEs, which can be truncated to obtain an approximate finite-dimensional model.

1 Introduction

One of the more general formal models for stochastic hybrid systems was proposed by Hu et al. [15], where the deterministic differential equations for the continuous flows are replaced by their stochastic counterparts, and the reset maps are generalized to (state-dependent) distributions that define the probability density of the state after a discrete transition. However, in this model transitions are always triggered by deterministic conditions (guards) on the state.

In the model proposed in this paper for SHSs, transitions between modes are triggered by stochastic events, much like the transitions between states of a continuous-time Markov chains. However, the probability that a transition occurs in a given interval of time depends both on the continuous and discrete components of the current SHS's state. The state of the SHS so defined is a Piecewise-Deterministic Markov Process (PDP) in the sense of [9]. Based on this observation, we provide a formula for the extended generator of the SHS, which can be used to compute expectations and probability distributions/densities of its discrete/continuous states.

The basic model proposed is (i) limited to a deterministic evolution of the continuous state inside each discrete mode and (ii) it does not allow for transitions triggered by deterministic conditions on the state (e.g., guards being crossed). The first restriction is only introduced for simplicity of presentation and it is shown how the model can be generalized to allow for continuous evolutions driven by stochastic differential equations. The rationale for disallowing transitions triggered by guard-crossings is that this allows us to exclude Zeno phenomena [18]. However, we show how one can approximate systems with guards (and potentially prone to Zeno phenomena) by a sequence of Zeno-free SHSs.

The model proposed is inspired by piecewise deterministic jump systems (PDJSs), where the evolution of the continuous state in each mode is modeled by a deterministic differential equation and transitions between modes are governed by a continuous-time Markov process [3, 8, 10, 11, 31]. In general, the transitions rates in PDJSs are assumed independent of the continuous state, which is too restrictive for our applications. The work of Filar et al. [11] is a notable exception but requires a time-scale separation between the (purely deterministic) continuous dynamics and the discrete jump dynamics. In switched diffusion processes (SDPs), as defined by Ghosh et al. [12], the evolution of the continuous state in each mode is modeled by a stochastic differential equation and transitions between modes are controlled by a continuous-time Markov process. The transition rates of the Markov process can depend on the state but transitions do not generate jumps on the continuous state (i.e., no resets). The reader is referred to [28] for a comparison of the models in [9, 12, 15]. The SHSs considered here can be viewed as special cases of general jump-diffusion processes [17]. In fact, Theorem 1 can be viewed as an Itô rule for SHSs. However, in the application of interest, we are faced with the technical difficulty that jump-intensities are not bounded and moreover the jump-distributions do not have compact support.

Our SHS model was inspired by the need to obtain accurate models for TCP congestion control in communication networks. The use of hybrid models to characterize the behavior of congestion control was proposed by Hespanha et al. [14] and further pursued in [5, 6]. In these models, packet drops trigger transitions between different "modes" for the evolution of TCP's congestion window size. The drop models in [5, 14] were completely deterministic but, as noted in [6], their use is limited to simple network topologies.

There is an extensive literature on models that describe the behavior of TCP congestion control for long-lived flows, i.e., flows that have an infinite amount of data to transmit. A great deal of effort has been placed in characterizing the steady-state behavior of these flows [23, 24, 26, 27, 30]. In particular, in studying the relationship between the average transmission rate μ, the average round-trip time RTT, and the per-packet drop rate p_{drop} for a single TCP flow. In most of this work, μ and p_{drop} should be understood as *time-averages for a single TCP flow*. This type of approach was also pursued in [19–22, 27, 29] to derive dynamic models for the congestion avoidance stage of long-lived TCP flows. However, these single-flow models are only valid over time scales much longer

than the round-trip time for one packet. To avoid averaging over long time intervals, Misra et al. [25] utilized *ensemble averages* to construct models for the dynamics of long-lived flows. Shakkottai and Srikant [29] also used stochastic aggregation to reduce the time-scales over which a single-flow model is valid.

We pursue here a stochastic analysis of the hybrid models proposed in [6]. As in [25] time averaging is done over intervals of roughly one round-trip time to obtain continuously varying sending rates, and we then investigate the dynamics of ensemble averages. However, here we consider ensembles of *on-off* TCP flows. The off-periods are assumed exponentially distributed whereas the on-periods are determined by the amount of data being transfered. We take as given the probability distribution of the transfer-sizes, which implicitly determines the distribution for the on-periods. The model takes into account the correlation between the (variable) sending rate and the duration of the (also variable) on-periods. Not surprisingly, the ensemble behavior of on-off TCP flows varies very significantly with the probability distribution of the transfer sizes. By feeding our models with realistic distributions reported in the literature, we conclude that the dynamics of the sending rate is dominated by high-order statistics, exhibiting much larger standard deviations than the average value. Moreover, the packet drop rate seems to have a surprisingly small effect on the average drop rate but provides a strong control on its standard deviation. This has significant implications for the design of congestion control mechanisms.

Notation. By a *piecewise continuous signal* it is meant a function $x : [0, \infty) \to \mathbb{R}^n$ that is right-continuous and has left-limit at every point. We denote by $x^-(t)$ the left-limit of $x(\tau)$ as $\tau \uparrow t$. A signal x is called *piecewise constant* if it is piece-continuous and it is constant on every interval where it is continuous. Given a *measurable space* $(\Omega, \mathfrak{F})$ and *probability measure* $P : \mathfrak{F} \to [0, 1]$, vector-valued random variables $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^n$ and stochastic processes $\mathbf{x} : \Omega \times [0, \infty) \to \mathcal{X} \subset \mathbb{R}^n$ are denoted in **boldface**. A stochastic process with piecewise constant sample paths is called a *jump process*. When a jump process takes values on the set $\mathbb{N}$ of nonnegative integers it is called a *stochastic counter*.

2 Stochastic Hybrid Systems

A *stochastic hybrid system (SHS)* is defined by a differential equation

$$\dot{\mathbf{x}} = f(\mathbf{q}, \mathbf{x}, t), \qquad f : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n, \tag{1}$$

a family of m *discrete transition/reset maps*

$$(\mathbf{q}, \mathbf{x}) = \phi_\ell(\mathbf{q}^-, \mathbf{x}^-, t), \quad \phi_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \to \mathcal{Q} \times \mathbb{R}^n, \ \ell \in \{1, \ldots, m\}, \tag{2}$$

and a family of m *transition intensities*

$$\lambda_\ell(\mathbf{q}, \mathbf{x}, t), \qquad \lambda_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \to [0, \infty), \ \ell \in \{1, \ldots, m\}, \tag{3}$$

where $\mathcal{Q}$ denotes a (typically finite) set with no particular topological structure. A SHS characterizes a jump process $\mathbf{q} : \Omega \times [0, \infty) \to \mathcal{Q}$ called the *discrete state*; a stochastic process $\mathbf{x} : \Omega \times [0, \infty) \to \mathbb{R}^n$ with piecewise continuous sample paths called the *continuous state*; and m stochastic counters $\mathbf{N}_\ell : \Omega \times [0, \infty) \to \mathbb{N}$ called the *transition counters*.

In essence, between transition counter increments the discrete state remains constant whereas the continuous state flows according to (1); and at transition times the continuous and discrete states are reset according to (2). Each transition counter $\mathbf{N}_\ell$ counts the number of times that the corresponding discrete transition/reset map ϕ_ℓ is "activated." The frequency at which this occurs is determined by the transition intensities (3). In particular, the probability that the counter $\mathbf{N}_\ell$ will increment and therefore that the corresponding transition takes place in an "elementary interval" $(t, t + dt]$ is given by $\lambda_\ell(\mathbf{q}(t), \mathbf{x}(t), t)dt$. In practice, one can think of the intensity of a transition as the instantaneous rate at which that transition occurs. We will shortly make these statements mathematically precise.

It is often convenient to represent SHSs by a directed graph as in Figure 1, where each vertex corresponds to a discrete mode and each edge to a transition between discrete modes. The vertices are labeled with the corresponding discrete mode and the vector fields that determines the evolution of the continuous state in that particular mode. The source of each edge is labeled with the probability that the transition will take place in an elementary interval $(t, t + dt]$ and the destination is labeled with the corresponding reset-map.
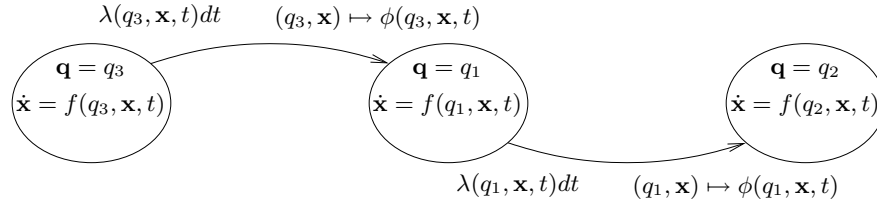

Fig. 1. Graphical representation of a stochastic hybrid system

2.1 Construction of the stochastic processes

We now provide the formal procedure to construct the sample paths of the discrete state, continuous state, and transition counters and show that it is indeed consistent with the intuition given above. Aside from its theoretical interest, this construction is useful to run Monte Carlo simulations of SHSs. It is inspired by the one used in [9, Chapter 2] to define Piecewise-Deterministic Markov Processes (PDPs). The following regularity assumption on the vector field f is required for the SHS to be well defined:

Assumption 1. *For every $(q_0, x_0, t_0) \in \mathcal{Q} \times \mathbb{R}^n \times [0, \infty)$ there exists a unique global solution $\varphi(\cdot; t_0, q_0, x_0) : [t_0, \infty) \to \mathbb{R}^n$ to (1) with initial condition $\mathbf{x}(t_0) = x_0$ and $\mathbf{q}(t_0) = q_0$.* $\square$

In what follows, the $\boldsymbol{\mu}_k^\ell$, $\ell \in \{1, \ldots, m\}, k \in \mathbb{N}$ denote independent random variables all uniformly distributed in the interval $[0, 1]$. We will call these *transition triggers*. Consider an initial condition $(q_0, x_0, t_0) \in \mathcal{Q} \times \mathbb{R}^n \times [0, \infty)$. For a given $\omega \in \Omega$, the sample paths of $\mathbf{q}(\omega, \cdot) : [t_0, \infty) \to \mathcal{Q}$, $\mathbf{x}(\omega, \cdot) : [t_0, \infty) \to \mathbb{R}^n$, and all the $\mathbf{N}_\ell(\omega, \cdot) : [t_0, \infty) \to \mathbb{N}$ can be constructed as follows:

1. Set $\mathbf{t}_0(\omega) = t_0$, $\mathbf{q}(\omega, 0) = q_0$, $\mathbf{x}(\omega, 0) = x_0$, $\mathbf{N}_\ell(\omega, 0) = 0$, $\forall \ell$.
2. Let $\mathbf{t}_1(\omega)$ be the largest time on $(\mathbf{t}_0(\omega), \infty]$ for which

$$e^{-\int_{\mathbf{t}_0(\omega)}^t \lambda_\ell(\mathbf{q}(\omega, \mathbf{t}_0(\omega)), \varphi(s; \mathbf{t}_0(\omega), \mathbf{q}(\omega, \mathbf{t}_0(\omega)), \mathbf{x}(\omega, \mathbf{t}_0(\omega))), s)ds} > \boldsymbol{\mu}_0^\ell(\omega), \qquad (4)$$

$\forall t \in [\mathbf{t}_0(\omega), \mathbf{t}_1(\omega)), \ \ell \in \{1, \ldots, m\}$.
3. On the interval $[\mathbf{t}_0(\omega), \mathbf{t}_1(\omega))$, the sample paths of $\mathbf{q}(\omega, \cdot)$ and all the counters $\mathbf{N}_\ell(\omega, \cdot)$ remain constant, whereas the sample path of $\mathbf{x}(\omega, \cdot)$ equals $\varphi(\cdot; \mathbf{t}_0(\omega), \mathbf{q}(\omega, \mathbf{t}_0(\omega)), \mathbf{x}(\omega, \mathbf{t}_0(\omega)))$.
4. Denoting by $\boldsymbol{\ell}_1(\omega) \in \{1, 2, \ldots, m\}$ the index for which (4) is violated at time $t = \mathbf{t}_1(\omega)$, the counter $\mathbf{N}_{\boldsymbol{\ell}_1(\omega)}(\omega)$ is incremented by one and

$$\big(\mathbf{q}(\omega, \mathbf{t}_1(\omega)), \mathbf{x}(\omega, \mathbf{t}_1(\omega))\big) = \phi_{\boldsymbol{\ell}_1(\omega)}\big(\mathbf{q}^-(\omega, \mathbf{t}_1(\omega)), \mathbf{x}^-(\omega, \mathbf{t}_1), \mathbf{t}_1(\omega)\big).$$

5. In case $\mathbf{t}_1(\omega) < \infty$, repeat the construction from the step 2 above with $\mathbf{t}_0(\omega)$, $\boldsymbol{\mu}_0^\ell(\omega)$, $\mathbf{t}_1(\omega)$, $\boldsymbol{\ell}_1(\omega)$ replaced by $\mathbf{t}_k(\omega)$, $\boldsymbol{\mu}_k^\ell(\omega)$, $\mathbf{t}_{k+1}(\omega)$, $\boldsymbol{\ell}_{k+1}(\omega)$, respectively, for $k = 1, 2, \ldots$

   The random variables $\mathbf{t}_k$ defined by (4) are called *transition times*.

The stochastic processes so defined depend on the initial condition for the SHS. To emphasize this dependence we sometimes use P_{z_0} and E_{z_0} to denote the probability measure and expected value corresponding to the initial condition $z_0 := (q_0, x_0, t_0) \in \mathcal{Q} \times \mathbb{R}^n \times [0, \infty)$.

Step 1 provides the initialization for all the stochastic processes. Step 3 guarantees that the discrete state remains constant and the continuous state flows according to (1) between transitions. Step 4 enforces that the continuous and discrete states are reset according to (2) at transition times. The frequency at which these occur is determined by Step 2. In fact, one can derive from (4) [13] that

$$\lim_{dt \downarrow 0} \frac{\mathrm{P}\left(\mathbf{N}_\ell(t + dt) > \mathbf{N}_\ell(t)\right)}{dt} = \lim_{dt \downarrow 0} \mathrm{E}\left[\frac{1}{dt} \int_t^{t+dt} \lambda_\ell(\mathbf{q}(t), \varphi(s; t, \mathbf{q}(t), \mathbf{x}(t)), s)ds\right],$$

$$(5)$$

which shows that the probability $\mathrm{P}\left(\mathbf{N}_\ell(t + dt) > \mathbf{N}_\ell(t)\right)$ that the transition ϕ_ℓ will occur in an (arbitrarily) small interval $(t, t+dt]$ is proportional to the length of the interval with the proportionality constant given by the right-hand-side of (5). This equation specifies the precise meaning of the observation made above

to the extent that "the probability that the counter $\mathbf{N}_\ell$ will increment in an elementary interval $(t, t + dt]$ is given by $\lambda_\ell(\mathbf{q}(t), \mathbf{x}(t), t)dt$." Note that when λ_ℓ is continuous, the right-hand-side of (5) is precisely equal to $\mathrm{E}[\lambda_\ell(\mathbf{q}(t), \mathbf{x}(t), t)]$.

The above construction guarantees that the sample-paths are indeed right-continuous and have left-limits at every point with probability one. However, without further assumptions there is no guarantee that the sample path are defined globally on $[0, \infty)$. We will return to this issue later.

2.2 Generalizations

The model for stochastic hybrid systems presented above is more general than it may appear at first. We discuss next some of the generalizations possible and refer the reader to [13] for a discussion of their implication on the results in Sect. 3.

The model allows for transitions where the *next state is chosen according to a given distribution.* For example, suppose one would like the intensity $\lambda(\mathbf{q}, \mathbf{x}, t)$ to trigger transitions to the discrete-states q_1 or q_2 with probabilities p_1 or $1 - p_1$, respectively. This could be achieved by considering two transitions with intensities $p_1\lambda(\mathbf{q}, \mathbf{x}, t)$ and $(1 - p_1)\lambda(\mathbf{q}, \mathbf{x}, t)$, respectively, and reset maps $\phi_1(q, x, t) = (q_1, \varphi_1(x, t), t)$ and $\phi_2(q, x, t) = (q_2, \varphi_2(x, t), t)$, respectively, where the φ_i denote (possibly distinct) continuous-state resets.

The above model does not directly consider differential equations *driven by stochastic processes.* However, many important classes of stochastic processes can be obtained as the limit of jump processes that can be modeled by SHSs. For example, the stochastic differential equation

$$\dot{\mathbf{x}} = a\mathbf{x} + b\dot{\mathbf{w}} \tag{6}$$

where $\mathbf{w}$ denotes Brownian motion, can be regarded as the limit as $\epsilon \downarrow 0$ of the jump system with continuous dynamics $\dot{\mathbf{x}} = a\mathbf{x}$ and resets $\mathbf{x} \mapsto \mathbf{x} + b\sqrt{\epsilon}$ and $\mathbf{x} \mapsto \mathbf{x} - b\sqrt{\epsilon}$ both triggered with fixed intensity $\frac{1}{2\epsilon}$. We can therefore model continuous evolutions of the form (6) as limits to a sequences of SHSs.

The basic model also does not directly consider *discrete transitions triggered by deterministic conditions of the state*, e.g., a guard being crossed. However, this behavior can also be obtained as the limiting solution to a sequence of SHSs. Consider for example the well known bouncing-ball single-mode deterministic hybrid system with dynamics $\ddot{x} = -g$, $g > 0$ and state reset $(x, \dot{x}) \mapsto (0, -c\dot{x})$, $c \in (0, 1)$ triggered by the condition $\dot{x} < 0$ and $x \leq 0$. We could approximate this system by a sequence of SHSs for which the resets are triggered with intensities given by a "barrier function" of the form

$$\lambda_\epsilon(x, \dot{x}) := \begin{cases} \epsilon\, e^{-x/\epsilon} & \dot{x} < 0 \\ 0 & \dot{x} > 0, \end{cases}, \qquad \epsilon > 0. \tag{7}$$

As $\epsilon \downarrow 0$, transitions will occur in a small neighborhood of $x = 0$ with increasingly higher probability. Figure 2 shows confidence intervals for the sample-paths of three SHSs that approximate with increased accuracy the deterministic

bouncing-ball. It is important to emphasize that, for any $\epsilon > 0$, the sample paths of the SHSs are globally defined with probability one. This approach may in fact be a promising technique to overcome difficulties posed by the Zeno phenomena that occur for the deterministic bouncing-ball system [18].
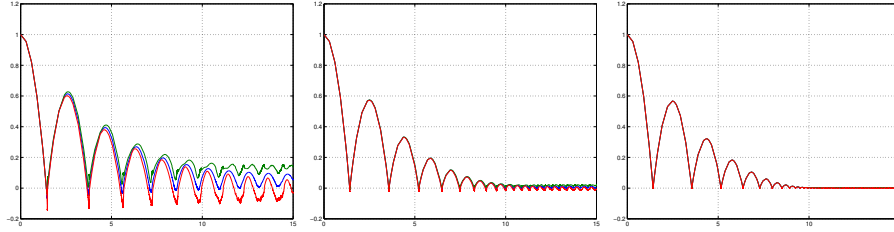

Fig. 2. 95% confidence intervals for the **x** sample paths of three SHS that approximate the bouncing-ball deterministic system with Zeno time equal to 10. The transition intensities of the SHSs are given by (7) with $\epsilon = 10^{-2}$, 10^{-3}, 10^{-4} from left to right. These results were obtained via Monte Carlo simulations.

3 Generator for SHSs

In this section we provide a result to compute expectations on the state of a SHS. The following assumptions are needed:

Assumption 2. *(i) The transition intensities $\lambda_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \to [0, \infty)$, $\ell \in \{1, \ldots, m\}$ are measurable functions (e.g., continuous).*
(ii) For every initial condition $z_0 := (q_0, x_0, t_0) \in \mathcal{Q} \times \mathbb{R}^n$ there exists a continuous functions $\alpha_{z_0} : [0, \infty) \to [0, \infty)$ such that the sample-paths are defined globally and $\|\mathbf{x}(t)\| \leq \alpha_{z_0}(t)$, $\forall t \geq t_0$ with probability one with respect to P_{z_0}. □

Assumption 2(ii) may be difficult to check, but we will shortly provide conditions that are more friendly to verify. We are now ready to state the main result of this section. Due to space limitations, we refer the reader to [13] for its proof.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. For every initial condition $z_0 := (q_0, x_0, t_0) \in \mathcal{Q} \times \mathbb{R}^n \times [0, \infty)$ and every function $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that is continuously differentiable with respect to its second and third arguments, we have that*

$$\mathrm{E}_{z_0}[\psi(\mathbf{q}(t), \mathbf{x}(t), t)] = \psi(q_0, x_0, t_0) + \mathrm{E}_{z_0}\left[\int_{t_0}^{t}(L\psi)(\mathbf{q}(s), \mathbf{x}(s), s)ds\right], \quad (8)$$

where $\forall(q, x, t) \in \mathcal{Q} \times \mathbb{R}^n \times [0, \infty)$

$$(L\psi)(q,x,t) := \frac{\partial\psi(q,x,t)}{\partial x}f(q,x,t) + \frac{\partial\psi(q,x,t)}{\partial t} +$$
$$+ \sum_{\ell=1}^{m}\Big(\psi\big(\phi_\ell(q,x,t),t\big) - \psi(q,x,t)\Big)\lambda_\ell(q,x,t), \quad (9)$$

and $\frac{\partial\psi(q,x,t)}{\partial x}$ and $\frac{\partial\psi(q,x,t)}{\partial t}$ denote the gradient of $\psi(q,x,t)$ with respect to x and the partial derivative of $\psi(q,x,t)$ with respect to t, respectively. □

Following [9], we call the operator $\psi \mapsto L\psi$ defined by (9) the *extended generator* of the SHS. It is often convenient to write (8) in the following differential form[1]

$$\frac{\mathrm{d}\,\mathrm{E}[\psi(\mathbf{q}(t),\mathbf{x}(t),t)]}{\mathrm{d}t} = \mathrm{E}[(L\psi)(\mathbf{q}(t),\mathbf{x}(t),t)].$$

Assumption 2(ii) rules out finite escape time almost surely. Although this is a mild requirement, it may be difficult to verify directly. The following lemma provides a condition that is more restrictive but easily checkable:

Lemma 1. *[13] Let* $\phi_\ell^x : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, $\ell \in \{1,\ldots,m\}$ *denote the projection of* ϕ_ℓ *into* $\mathbb{R}^n$ *[i.e.,* $\phi_\ell^x(q,x,t) = \bar{x}$ *where* $(\bar{q},\bar{x}) = \phi_\ell(q,x,t)$*]. Assumptions 2(ii) holds, if there exists a continuous function* $\gamma_f : [0,\infty) \to [0,\infty)$ *and constants* c_f, c_ϕ *such that*

$$\|f(q,x,t)\| \leq \max\{\gamma_f(t)\|x\|, c_f\}, \qquad \|\phi_\ell^x(q,x,t)\| \leq \max\{\|x\|, c_\phi\},$$

$\forall q \in \mathcal{Q}, x \in \mathbb{R}^n, t \geq 0, \ell \in \{1,\ldots,m\}$. □

This lemma essentially requires f and the ϕ_ℓ to have linear growth in x over $\mathbb{R}^n$. Moreover, the growth constant of the ϕ_ℓ must not be larger than one. This is a strong requirement and one might be tempted to think that it could be replaced by a local condition if one would restrict one's attention to a finite time interval $[0,T]$, $T < \infty$. It turns out that in general this is not true, as shown in [13] by counter example.

4 A stochastic model for TCP flows

In this section we present a SHSs model for a single-user on-off TCP flow based on the hybrid modeling framework proposed by Bohacek et al. [6]. The model is represented graphically in Figure 3. It has two continuous states—TCP's congestion window size $\mathbf{w}$ and the cumulative number of packets sent in a particular connection $\mathbf{s}$—and three discrete states $\{off, ss, ca\}$. During the *off* mode the flow is inactive and no packets are transmitted. The *ss* mode corresponds to TCP's slow-start and during this mode the window size $\mathbf{w}$ increases exponentially. The *ca* mode corresponds to TCP's congestion-avoidance and during this mode $\mathbf{w}$

---

[1] Recall that all signals are right-continuous with probability one and derivatives should be understood as right-limits.
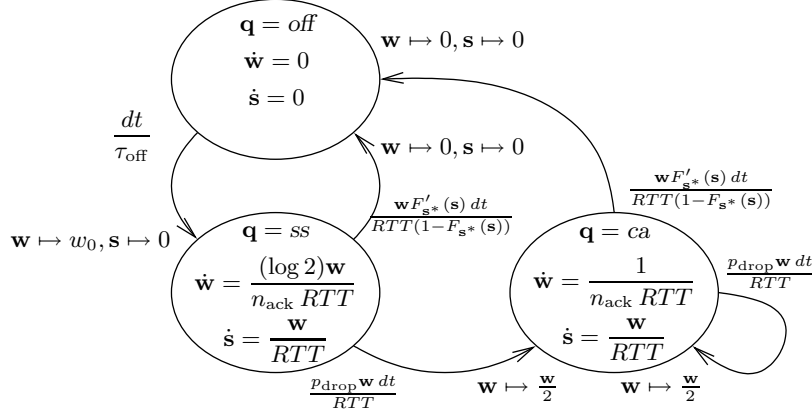
Fig. 3. Stochastic hybrid model for a TCP flow, where n_{ack} denotes the number of data packets acknowledged per each ACK packet received and $w_0 := .693$ when $n_{\mathrm{ack}} = 1$ or $w_0 := 1.428$ when $n_{\mathrm{ack}} = 2$.

increases linearly. In any of the active modes, $\mathbf{w}$ packets are sent each round-trip time RTT, leading to a packet transmission rate $\dot{\mathbf{s}} = \mathbf{r} := \frac{\mathbf{w}}{RTT}$. The reader is referred to [13] for further details. The transitions between modes occur as follows:

1. Drops occurrences—which correspond to transitions from the *ss* or the *ca* modes to the *ca* mode—occur at a rate $p_{\mathrm{drop}}\mathbf{r}$, where p_{drop} denotes the per-packet drop probability and $\mathbf{r} := \frac{\mathbf{w}}{RTT}$ the packet sending rate.
2. The start of new flows—which correspond to the transitions from the *off* to the *ss* mode—occur at a rate $\frac{1}{\tau_{\mathrm{off}}}$. This is consistent with an exponentially distributed duration of the *off* periods with average τ_{off}.
3. The termination of flows—which correspond to transitions from the *ss* and *ca* modes to the *off* mode—occur at a rate $\frac{\mathbf{r} F'_{\mathbf{s}^*}(\mathbf{s})}{1 - F_{\mathbf{s}^*}(\mathbf{s})}$. This is consistent with a distribution $F_{\mathbf{s}^*} : [0, \infty) \to [0, 1]$ for the number $\mathbf{s}^*$ of packets sent in each TCP session [13].

We focus our attention on two specific instances of the general model in Figure 3:

Exponential sizes This model is obtained by assuming that the number of packets to transmit is exponentially distributed with mean k, i.e., $F_{\mathbf{s}^*}(s) = 1 - e^{-\frac{s}{k}}$, $\forall s \geq 0$.

Mixed-exponential sizes It has been observed that modeling the distribution of transfer-sizes as an exponential is an over-simplification. For example, it has been observed that heavy-tail models are more fitting to experimental data (cf., e.g., [1, 2]). An alternative that turns out to be computationally attractive and still fits reasonably well with experimental data is a mixture of exponentials. According to this model, transfer-sizes are sampled from a family of M

exponential random variables $\mathbf{s}_i$, $i \in \{1, 2, \ldots, M\}$ by selecting a sample from the ith random variable $\mathbf{s}_i$ with probability p_i. Each $\mathbf{s}_i$ corresponds to a distinct mean transfer-size k_i. To model this as a SHS, we consider M alternative $\{ss_i, ca_i : i = 1, 2, \ldots, M\}$ modes, each corresponding to a specific exponential distribution for the transfer-sizes. The transition from the inactive mode *off* to the slow-start mode ss_i corresponding to a mean transfer-size of k_i occurs with probability p_i.

4.1   Analysis of the TCP SHS models

To investigate the dynamics of the moments of the sending rate $\mathbf{r}(t) = \frac{\mathbf{w}(t)}{RTT(t)}$ for the mixed-exponentials TCP model, $\forall n \geq 0$, $q_0 \in \mathcal{Q}$ let

$$\mu_{q_0,n}(t) := \mathrm{E}\left[\psi_{q_0,n}(\mathbf{q}(t), \mathbf{w}(t), t)\right], \quad \psi_{q_0,n}(q, w, t) := \begin{cases} \frac{w^n}{RTT(t)^n} & q = q_0 \\ 0 & \text{otherwise.} \end{cases}$$

The following result can be obtained by directly applying Theorem 1 to the SHS TCP model. Details of the computations can be found in [13].

Theorem 2 (Full-order models). *For the mixed-exponentials model in Sect. 4 we have*[2]

$$\dot{\mu}_{off,0} = -\frac{\mu_{off,0}}{\tau_{off}} + \sum_{j=1}^{M} k_j^{-1}(\mu_{ss_j,1} + \mu_{ca_j,1}) \tag{10}$$

$$\dot{\mu}_{ss_i,n} = \frac{p_i w_0^n \mu_{off,0}}{\tau_{off} RTT^n} + n \frac{(\log 2) - n_{ack} \dot{RTT}}{n_{ack} RTT} \mu_{ss_i,n} - (p_{drop} + k_i^{-1})\mu_{ss_i,n+1} \tag{11}$$

$$\dot{\mu}_{ca_i,n} = \frac{n \mu_{ca_i,n-1}}{n_{ack} RTT^2} - \frac{n \dot{RTT} \mu_{ca_i,n}}{RTT} - (p_{drop} + k_i^{-1})\mu_{ca_i,n+1}$$
$$+ \frac{p_{drop}}{2^n}(\mu_{ss_i,n+1} + \mu_{ca_i,n+1}). \tag{12}$$

4.2   Reduced-order model

The system of infinitely many differential equations that appear in Theorem 2 describes exactly the evolution of the moments of the sending rate $\mathbf{r}$ but finding the exact solution to these equations does not appear to be simple. However, as noted by Bohacek [4], Monte Carlo simulations reveal that the steady-state distribution of the sending rate is often well approximated by a Log-Normal distribution. Assuming that on each mode the sending rate $\mathbf{r}$ approximately obeys a Log-Normal distribution even during transients, we can truncate the systems of infinitely many differential equations that appear in Theorem 2. We recall that, if the random variable $\mathbf{x}$ has a Log-Normal distribution then $\mathrm{E}[\mathbf{x}^3] = \frac{\mathrm{E}[\mathbf{x}^2]^3}{\mathrm{E}[\mathbf{x}]^3}$. Therefore if $\mathbf{r}$ is approximately Log-Normal distributed in the mode $q \in \mathcal{Q}$, we have that

$$\mu_{q,3} = \mu_{q,0} \,\mathrm{E}[\mathbf{r}^3 \mid \mathbf{q} = q] \approx \mu_{q,0} \frac{\mathrm{E}[\mathbf{r}^2 \mid \mathbf{q} = q]^3}{\mathrm{E}[\mathbf{r} \mid \mathbf{q} = q]^3} = \frac{\mu_{q,0} \,\mu_{q,2}^3}{\mu_{q,1}^3}, \tag{13}$$

---

[2] To simplify the notation, we omit the time-dependence of RTT and p_{drop}.

where we used the fact that

$$\mu_{q,n} = P(\mathbf{q} = q) \, E[\mathbf{r}^n \mid \mathbf{q} = q] = \mu_{q,0} \, E[\mathbf{r}^n \mid \mathbf{q} = q].$$

Using (13) in (10)–(12), we can eliminate any terms $\mu_{q_0,n}$, $n \geq 3$ in the equations for $\dot{\mu}_{q_0,n}$, $n \leq 2$, thus can constructing a finite-dimensional model to approximately describe the dynamics of the first two moments of the sending rate. We present next simulations of this reduced model for a few representative parameter values.

Figure 4 corresponds to a transfer-size distribution that results from the mixture of two exponentials ($M = 2$) with parameters

$$p_1 = 88.87\%, \qquad k_1 = 3.5\text{KB}, \qquad p_2 = 11.13\%, \qquad k_2 = 246\text{KB}. \qquad (14)$$

The first exponential corresponds to small "mice" transfers (3.5KB average) and the second to "elephant" mid-size transfers (246KB average). The small transfers are assumed more common (88.87%). These parameters result in a distribution with an average transfer-size of 30.58KB and for which 11.13% of the transfers account for 89.7% of the total volume transfered. This is consistent with the file distribution observed in the UNIX file system [16]. However, it does not accurately capture the tail of the distribution (it lacks the "mammoth" files that will be considered later). The results obtained with the reduced model match reasonably well those obtained from Monte Carlo simulations of the full SHS model, especially taking into account the very large standard deviations. It is worth it to point out that the simulation of the reduced model takes just a few seconds, whereas each Monte Carlo simulation takes several hours of CPU.

Two somewhat surprising conclusions can be drawn from Figure 4 for this distribution of transfer-sizes and off-times:

1. The *average total sending rate varies very little with the drop rate* (at least up to the drop rate of 33% shown in the plots), with most of the packets transmitted belonging to "elephant" mid-size transfers.
2. The *dynamics of TCP are completely dominated by second order moments.* In Figure 4, the standard deviation is 5 to 20 times larger than the average sending rate, which is very accurately predicted by the reduced model.

This behavior is completely different from the one observed for TCP flows that are always on, for which it has been shown that the steady-state average sending rate is approximately given by $\frac{c}{RTT\sqrt{p_{\text{drop}}}}$, where RTT denotes the average round-trip time, p_{drop} the per-packet drop rate, and c a constant ranging from 1.225 to 1.310 depending on the method used to derive the equation [5, 14, 23, 24, 26].

We considered next a transfer-size distribution that results from a mixture of three exponentials ($M = 3$) with parameters

$$p_1 = 98\%, \; k_1 = 6\text{KB}, \; p_2 = 1.7\%, \; k_2 = 400\text{KB}, \; p_3 = .02\%, \; k_3 = 10\text{MB}. \quad (15)$$
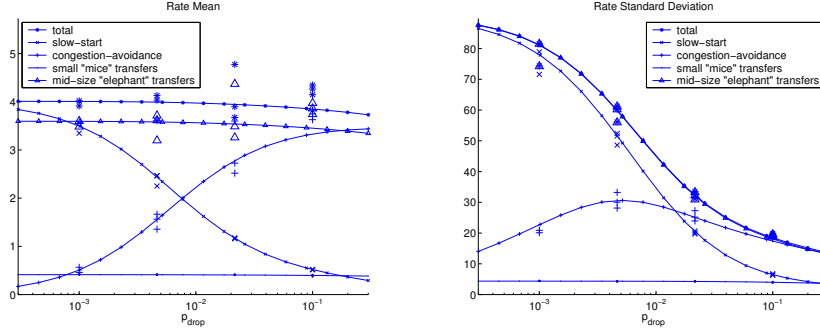
Fig. 4. Steady-state values for the average (left) and standard deviation (right) of the sending rate as a function of the drop probability. The solid lines were obtained from the reduced model whereas the (larger) symbols were obtained from Monte Carlo simulations, with $RTT = 50$ms, $n_{\text{ack}} = 1$, a transfer-size distribution resulting from the mixture of two exponentials with parameters in (14), and mean off-time $\tau_{\text{off}} = 5$sec.

The first exponential corresponds to small "mice" transfers, the second to mid-size "elephant" transfers, and the third to large "mammoth" transfers. The resulting distribution approximates the one reported by Arlitt et al. [1] obtained from monitoring transfers from a world-wide web proxy within an Internet Service Provider. This distribution has a much heavier tail than the one considered before.

Figure 5 contains results obtained from the reduced model. We do not present Monte Carlo results because the simulation times needed to capture the tails of the transfer-size distribution are prohibitively large. It turns out that the main conclusions drawn before still hold: the average sending rate varies relatively little with the drop rate and the dynamics of TCP are completely dominated by second order moments. The mid-size "elephants" still dominate followed by the small "mice." The large "mammoth" transfers occur at a rate that is not sufficiently large to have a significant impact on the average sending rate.

5 Conclusions

This paper presents a new model for SHSs where transitions between discrete modes are triggered by stochastic events, which occur at rates that are allowed to depend on both the continuous and the discrete states of the SHS. Based on results available for Piecewise-Deterministic Markov Process (PDPs), we provide a formula for the extended generator of the SHS.

As an illustration, we presented a SHS model for on-off TCP flows that considers both slow-start and congestion avoidance. One important observation that stems from this work is that for realistic transfer-size distributions high-order statistical moments seem to dominate the dynamics of TCP. Also, the
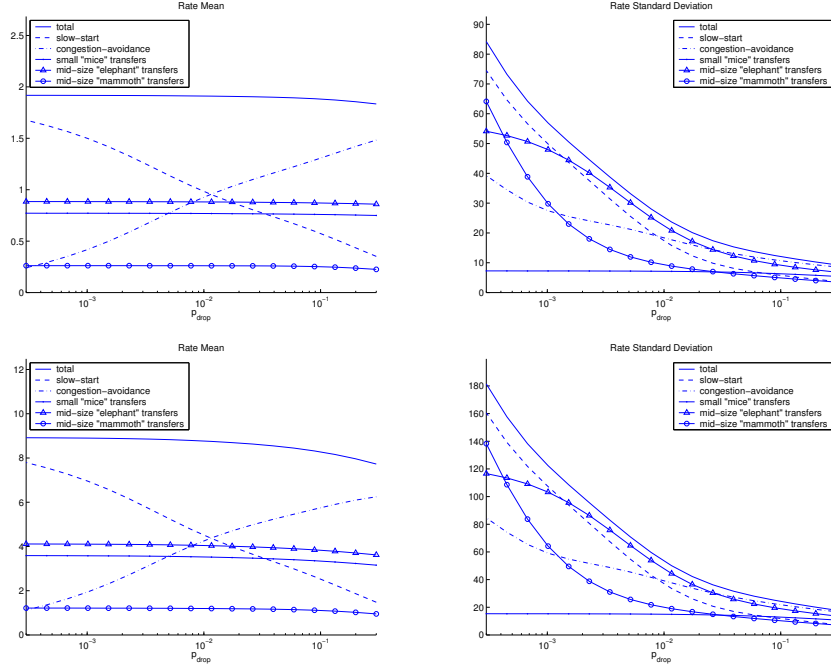
Fig. 5. Steady-state values for the average (left) and standard deviation (right) of the sending rate as a function of the drop probability. These results were obtained from the reduced model, with $RTT = 50$ms, $n_{ack} = 1$, and a transfer-size distribution resulting from the mixture of three exponentials with the parameters in (15). The mean off time was set to $\tau_{off} = 5$sec (top) and 1sec (bottom).

probability of drop appears to have a much larger effect on the standard deviation of the sending rate than on its mean value. We are currently investigating the impact of this observation on the stability and performance of congestion control mechanisms.

This work opens several avenues for future research. We would like to determine conditions under which Assumption 2(ii) holds, milder than those required by Lemma 1. It is also worth to investigate how solutions to deterministic hybrid systems that exist only in a finite interval due to Zeno phenomena can be globally extended using SHSs. Finally, one needs to develop general tools to approximate the dynamics of SHSs by finite-dimensional systems of ODEs, as was done for the TCP example.

Acknowledgments

distribution reported in [1]; Sanjoy Mitter and John Lygeros for provided several relevant references; and Stephan Bohacek, Yonggang Xu, Guillaume Bonnet, and the anonymous reviewers for insightful comments.

Bibliography

[1] M. Arlitt, R. Friedrich, and T. Jin. Workload characterization of a web proxy in a cable modem environment. Technical Report HPL-1999-48, Hwelett-Packard Laroratories, Palo Alto, CA, Apr. 1999.

[2] P. Barford, A. Bestavros, A. Bradley, and M. Crovella. Changes in web client access patterns. *World Wide Web,* Special Issue on Characterization and Performance Evaluation, 2(1–2):15–28, 1999.

[3] K. Benjelloun and E. K. Boukas. Mean square stochastic stability of linear time-delay system with Markovian jumping parameters. *IEEE Trans. on Automat. Contr.*, 43(10):1456–1460, Oct. 1998.

[4] S. Bohacek. A stochastic model of TCP and fair video transmission. In *Proc. of the IEEE INFOCOM*, 2003.

[5] S. Bohacek, J. P. Hespanha, J. Lee, and K. Obraczka. Analysis of a TCP hybrid model. In *Proc. of the 39th Annual Allerton Conf. on Comm., Contr., and Computing*, Oct. 2001.

[6] S. Bohacek, J. P. Hespanha, J. Lee, and K. Obraczka. A hybrid systems modeling framework for fast and accurate simulation of data communication networks. In *Proc. of the ACM Int. Conf. on Measurements and Modeling of Computer Systems (SIGMETRICS)*, June 2003.

[7] R. Brockett. Lecture notes on stochastic control. Provided by the author, 2002.

[8] O. L. V. Costa and M. D. Fragoso. Stability results for discrete-time linear systems with Markovian jumping parameters. *J. Mathematical Anal. and Applications*, 179:154–178, 1993.

[9] M. H. A. Davis. *Markov models and optimization.* Monographs on statistics and applied probability. Chapman & Hall, London, UK, 1993.

[10] Y. Fang and K. A. Loparo. Stabilization of continuous-time jump linear systems. *IEEE Trans. on Automat. Contr.*, 47(10):1590–1603, Oct. 2002.

[11] J. A. Filar, V. Gaitsgory, and A. B. Haurie. Control of singularly perturbed hybrid stochastic systems. *IEEE Trans. on Automat. Contr.*, 46(2):179–190, Feb. 2001.

[12] M. K. Ghosh, A. Arapostathis, and S. Marcus. Ergodic control of switching diffusions. *SIAM J. Contr. Optimization*, 35(6):1952–1988, Nov. 1997.

[13] J. P. Hespanha. Stochastic hybrid systems: Application to communication networks (extended version). Technical report, Dept. of Electrical and Computer Eng., University of California, Santa Barbara, Jan. 2004.

[14] J. P. Hespanha, S. Bohacek, K. Obraczka, and J. Lee. Hybrid modeling of TCP congestion control. In M. D. D. Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control*, number 2034 in Lect. Notes in Comput. Science, pages 291–304. Springer-Verlag, Berlin, Mar. 2001.

[15] J. Hu, J. Lygeros, and S. Sastry. Towards a theory of stochastic hybrid systems. In N. A. Lynch and B. H. Krogh, editors, *Hybrid Systems: Computation and Control*, volume 1790 of *Lect. Notes in Comput. Science*, pages 160–173. Springer, 2000.

[16] G. Irlam. Unix file size survey – 1993. Available at `http://www.base.com/gordoni/ufs93.html`, Nov. 1994.

[17] J. Jacod and A. N. Shiriaev. *Limit Theorems for Stochastic Processes*. Springer Verlag, 2nd edition, 2002.

[18] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. On the regularization of Zeno hybrid automata. *Syst. & Contr. Lett.*, 38:141–150, 1999.

[19] S. Kunniyur and R. Srikant. Analysis and design of an adaptive virtual queue (AVQ) algorithm for active queue management. In *Proc. of the ACM SIGCOMM*, San Diego, California, USA, 8 2001.

[20] A. Lakshmikantha, C. Beck, and R. Srikant. Robustness of real and virtual queue based active queue management schemes. In *Proc. of the 2003 Amer. Contr. Conf.*, pages 266–271, June 2003.

[21] S. H. Low. A duality model of TCP and queue management algorithms. *IEEE/ACM Trans. on Networking*, Oct. 2003. To appear.

[22] S. H. Low, F. Paganini, and J. C. Doyle. Internet congestion control. *IEEE Contr. Syst. Mag.*, 22(1):28–43, Feb. 2002.

[23] J. Mahdavi and S. Floyd. TCP-friendly unicast rate-based flow control. Technical note sent to the end2end-interest mailing list, Jan. 1997.

[24] M. Mathis, J. Semke, J. Mahdavi, and T. Ott. The macroscopic behavior of the TCP congestion avoidance algorithm. *ACM Comput. Comm. Review*, 27(3), July 1997.

[25] V. Misra, W. Gong, and D. Towsley. Stochastic differential equation modeling and analysis of TCP-windowsize behavior. In *Proc. of PERFORMANCE'99*, Istanbul, Turkey, 1999.

[26] T. Ott, J. H. B. Kemperman, and M. Mathis. Window size behavior in TCP/IP with constant loss probability. In *Proc. of the DIMACS Workshop on Performance of Realtime Applications on the Internet*, Nov. 1996.

[27] J. Padhye, V. Firoiu, D. Towsley, and J. Kurose. Modeling TCP Reno performance: A simple model and its empirical validation. *IEEE/ACM Trans. on Networking*, 8(2):133–145, Apr. 2000.

[28] G. Pola, M. L. Bujorianu, J. Lygeros, and M. D. D. Benedetto. Stochastic hybrid models: An overview. In *Proc. of the IFAC Conf .on Anal. and Design of Hybrid Syst.*, June 2003.

[29] S. Shakkottai and R. Srikant. How good are deterministic fluid models of Internet congestion control? In *Proc. of the IEEE INFOCOM*, June 2002.

[30] B. Sikdar, S. Kalyanaraman, and K. Vastola. Analytic models for the latency and steady-state throughput of TCP Tahoe, Reno and SACK. In *Proc. of the IEEE GLOBECOM*, pages 25–29, 2001.

[31] L. Xiao and A. H. J. P. How. Control with random communication delays via a discrete-time jump system approach. In *Proc. of the 2000 Amer. Contr. Conf.*, volume 3, pages 2199–2204, June 2000.