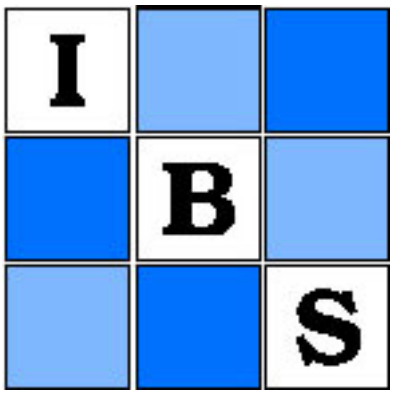

BIOMETRICS 49, 305–314
March 1993

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Modifying the *t* Test for Assessing the Correlation Between Two Spatial Processes

Pierre Dutilleul*

Département de Sciences Biologiques, Université de Montréal, C.P. 6128, Succursale A, Montréal, Québec H3C 3J7, Canada

SUMMARY

Clifford, Richardson, and Hémon (1989, *Biometrics* **45**, 123–134) presented modified tests of association between two spatially autocorrelated processes, for lattice and non-lattice data. These tests are built on the sample covariance and on the sample correlation coefficient; they require the estimation of an effective sample size that takes into account the spatial structure of both processes. Clifford et al. developed their method on the basis of an approximation of the variance of the sample correlation coefficient and assessed it by Monte Carlo simulations for lattice and non-lattice networks of moderate to large size.

In the present paper, the variance of the sample covariance is computed for a finite number of locations, under the multinormality assumption, and the mathematical derivation of the definition of effective sample size is given. The theoretically expected number of degrees of freedom for the modified *t* test with renewed modifications is compared with that computed on the basis of equation (2.9) of Clifford et al. (1989). The largest differences are observed for small numbers of locations and high autocorrelation, in particular when the latter is present with opposite sign in the two processes. Basic references that were missing in Clifford et al. (1989) are given and inherent ambiguities are discussed.

1. Introduction

Spatial autocorrelation in sample data can alter the conclusions of statistical analyses performed without due allowance for it, because autocorrelation does not provide minimum-variance unbiased linear estimators and produces a bias in the estimation of correlation coefficients and variances. This is well known for the analysis of correlation (in space: see, e.g., Bivand, 1980; Cliff and Ord, 1981, §7.3.1; Richardson and Hémon, 1982; Clifford, Richardson, and Hémon, 1989; Haining, 1990, §8.1); other spatial references may be found in the paper by Clifford et al.

Several solutions are open to data analysts. One can attempt to remove the spatial dependency among observations so that the usual statistical tests can be used, by filtering out the spatial structure by a prewhitening transformation, by using trend-surface analysis [as mentioned in Clifford et al. (1989)] or by the method of spatial variate differencing (Cliff and Ord, 1981, §7.4). An alternative, which does not alter the autocorrelation structure of the data, is to modify the statistical method in order to take spatial autocorrelation into account. This can be accomplished in two different ways. The first one is to develop permutation tests that do not destroy the autocorrelation structure of the data; an example is given in Legendre et al. (1990). The second way is to modify the variance estimation (Cliff and Ord, 1981, §7.2; Griffith, 1988, §4.1) or to correct the number of degrees of freedom associated with the distribution of estimated variances. The method of Clifford et al. belongs to this second class of modifications, just as Box (1954a, 1954b) procedures do in the analysis of variance (ANOVA) area (see also Crowder and Hand, 1990, §3.6). We will give special attention below to the properties derived in the pioneering, multipurpose work of Box on the distribution of quadratic forms in the normal random vector.

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* *Current address*: Department of Plant Science, Faculty of Agricultural and Environmental Sciences, Macdonald Campus, McGill University, 21111 Lakeshore Drive, Ste-Anne-de-Bellevue, Québec H9X 3V9, Canada.

Key words: Correlation analysis; Effective sample size; Modified *t* test; Spatial autocorrelation.

2. Computation of the Variance of the Sample Covariance

Richardson and Hémon (1981, 1982) derived an asymptotic expression for the variance of the sample correlation between two stochastically independent and stationary Gaussian lattice processes with zero means and finite variances. In Clifford et al. (1989), there is an ambiguity concerning the conditions under which the variance of the sample covariance is actually derived (see §4). In the present section, we consider the sample covariance S_{XY} between two spatial processes X and Y, from their observation on a network of a finite number N of locations, and we derive the theoretical variance of S_{XY} under the assumption of joint multinormality of the observations on X and Y. Expressions below are valid for lattice and non-lattice networks. New notation only is defined; the remainder is that used in Clifford et al. (1989).

First, let us suppose the $2N \times 1$ random vector $(\mathbf{x}', \mathbf{y}')'$ to be normally distributed as

$$\mathcal{N}_{2N}\left[\begin{pmatrix}\mathbf{m}_x\\ \mathbf{m}_y\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_x & \boldsymbol{\Sigma}_{xy}\\ \boldsymbol{\Sigma}_{xy} & \boldsymbol{\Sigma}_y\end{pmatrix}\right], \tag{2.1}$$

where $\mathbf{x} = (X_1, \ldots, X_N)'$ and $\mathbf{y} = (Y_1, \ldots, Y_N)'$; $\mathbf{m}_x = \mathrm{E}(\mathbf{x})$ and $\mathbf{m}_y = \mathrm{E}(\mathbf{y})$; $\boldsymbol{\Sigma}_x = \mathrm{var}(\mathbf{x})$ and $\boldsymbol{\Sigma}_y = \mathrm{var}(\mathbf{y})$; $\boldsymbol{\Sigma}_{xy} = \mathrm{cov}(\mathbf{x}, \mathbf{y})$. The problem of estimating these matrices will be discussed in Section 5.

Next, the sample covariance S_{XY} may be written in matrix notation as a bilinear form in $\mathbf{x}$ and $\mathbf{y}$, i.e., $S_{XY} = \mathbf{x}'\mathbf{B}\mathbf{y}$ with $\mathbf{B} = N^{-1}(\mathbf{I}_N - N^{-1}\mathbf{J}_N)$, where $\mathbf{I}_N$ and $\mathbf{J}_N$, respectively, denote the $N \times N$ identity matrix and the $N \times N$ matrix of ones only.

By the fourth-order properties of the normal random vector (Graybill, 1983, p. 361), we can then derive that

$$\mathrm{var}(S_{XY}) = \mathrm{tr}(\mathbf{B}\boldsymbol{\Sigma}_x\mathbf{B}\boldsymbol{\Sigma}_y) + \mathrm{tr}\{(\mathbf{B}\boldsymbol{\Sigma}_{xy})\}^2 - 2\mathbf{m}_x'\mathbf{B}\mathbf{m}_y\mathrm{tr}(\mathbf{B}\boldsymbol{\Sigma}_{xy}) - (\mathbf{m}_x'\mathbf{B}\mathbf{m}_y)^2, \tag{2.2}$$

where tr and $'$ denote the trace and the transpose operators, respectively.

Under the assumption of stationarity in mean of processes X and Y, i.e., $\mathrm{E}(X_i) = c_X \ \forall i$ and $\mathrm{E}(Y_j) = c_Y \ \forall j$, where c_X and c_Y denote two constants, it follows that

$$\mathrm{var}(S_{XY}) = \mathrm{tr}(\mathbf{B}\boldsymbol{\Sigma}_x\mathbf{B}\boldsymbol{\Sigma}_y) + \mathrm{tr}\{\mathbf{B}\boldsymbol{\Sigma}_{xy})\}^2, \tag{2.3}$$

and finally, from the supplementary assumption of independence between X and Y, i.e., $\boldsymbol{\Sigma}_{xy} = \mathbf{0}$, it follows from the definition of $\mathbf{B}$ that

$$\mathrm{var}(S_{XY}) = \mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y) - 2\mathrm{tr}(\mathbf{J}_N\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y) + \mathrm{tr}(\mathbf{J}_N\boldsymbol{\Sigma}_x)\mathrm{tr}(\mathbf{J}_N\boldsymbol{\Sigma}_y), \tag{2.4}$$

which is the numerator of the variance of the sample correlation coefficient, given in Clifford et al. (1989, Appendix 1) and in Clifford and Richardson (1985, pp. 157–158). However, the estimator recommended in Clifford et al. [1989, p. 125, expression (2.6)] and its preliminary version introduced in Clifford and Richardson [1985, p. 159, eq. (5)] may be rewritten as $N^{-2}\,\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_x\hat{\boldsymbol{\Sigma}}_y)$ (the constant N^{-2} actually concerns the denominator of $\hat{\sigma}_r^2$; see §3.2). Their estimator thus concerns only the first term in the right-hand side of (2.4) above.

In order to evaluate the quality of that approximation, we have computed values of $\mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y)$ and of the complete expression of $\mathrm{var}(S_{XY})$ in (2.4), assuming processes X and Y to be first-order simultaneous autoregressive on a lattice with parameters a_X and a_Y (cf. Clifford et al., 1989, p. 126; see Bivand, 1980, p. 7, for the row standardization of the contiguity matrix in the computation of $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_y$ for this class of processes, as well as Cliff and Ord, 1981, p. 148). Cases $a_X = a_Y = a$ and $a_X = -a_Y = a$ are considered for these computations; the lattice widths include the 12, 16, and 20 values considered by Clifford et al. (1989, pp. 126–128) in the lattice case. Computations were performed with SAS/IML Release 6.03 and IMSL Version 1.1 matrix algebra procedures. Differences between $\mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y)$ and $\mathrm{var}(S_{XY})$ values are plotted as percentages in Figure 1. The maximum of $\mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y)$ and $\mathrm{var}(S_{XY})$ values is always provided by the $\mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y)$ approximation, so that the corresponding estimator $\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_x\hat{\boldsymbol{\Sigma}}_y)$ will tend to overestimate $\mathrm{var}(S_{XY})$. For small numbers of locations, substantial differences $\mathrm{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y) - \mathrm{var}(S_{XY})$ are observed; these differences are more important for $a_X = a_Y = a$ than for $a_X = -a_Y = a$. Fortunately, differences tend to vanish for large numbers of locations; the recourse to $\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_x\hat{\boldsymbol{\Sigma}}_y)$ "for computational convenience," as argued by Clifford et al. (1989, p. 125), is valid in that case.

3. Modifying the Number of Degrees of Freedom for the *t* Test

In Clifford and Richardson (1985, p. 158) and in Clifford et al. (1989, p. 126), the definition of the effective sample size as 1 plus the inverse of the variance of the sample correlation coefficient r was only proposed; they did not demonstrate that it is so. This definition was essentially based on recovering the standard distribution for r when $\boldsymbol{\Sigma}_x = \boldsymbol{\Sigma}_y = \mathbf{I}_N$ or in a wider class of cases (see Clifford

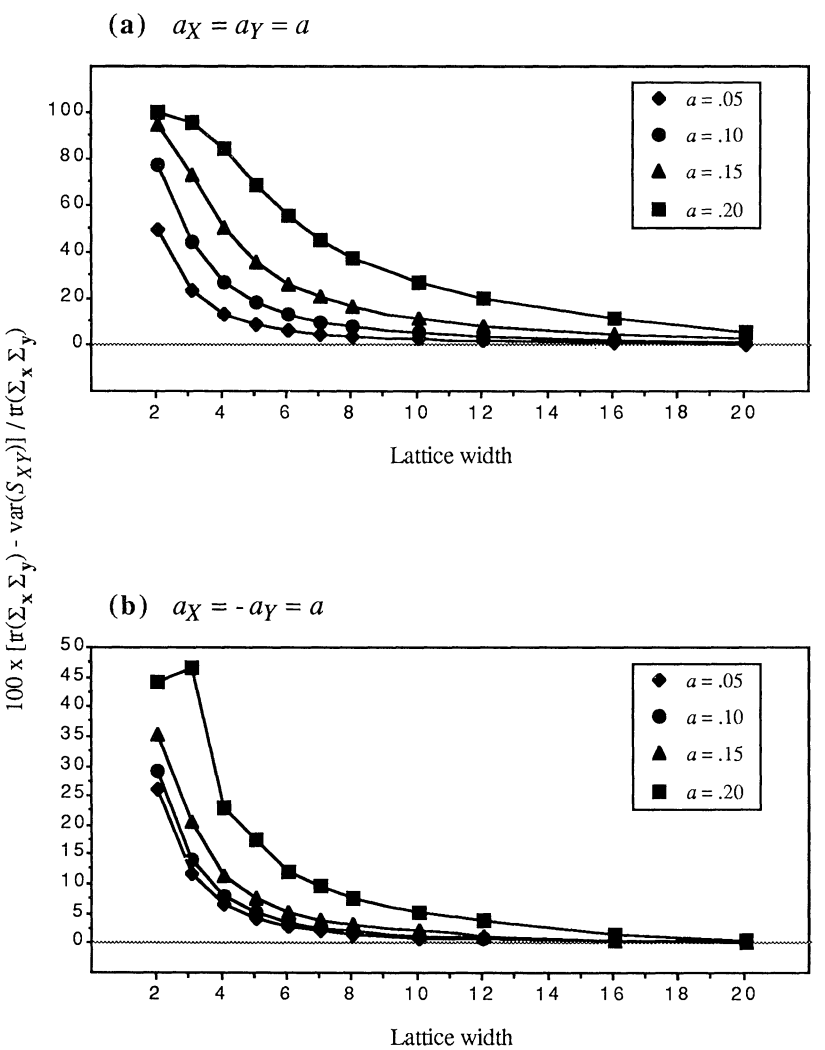

Figure 1. Differences between var(S_{XY}) and its approximation by tr($\mathbf{\Sigma}_x\mathbf{\Sigma}_y$) as a function of a-values and lattice widths, assuming that lattice processes X and Y are first-order simultaneous autoregressive with parameters a_X and a_Y. Differences are expressed in percentages.

et al., 1989, p. 134). Clifford et al. (1989) showed by Monte Carlo simulations that corrections based on their definition work fine, or seem to, for lattice and non-lattice networks of moderate to large size when processes X and Y are both positively autocorrelated. They provided no mathematical justification, possibly because they were not aware of Box's (1954a, 1954b) pioneering results on the distributional properties of quadratic forms in the normal random vector. We will fill this gap in Section 3.1.

The justification of the modified t tests to assess the correlation between two spatial processes follows from the mathematical definition of effective sample size, as empirically argued by Clifford et al. (1989). The theoretically expected numbers of degrees of freedom may thus be computed and compared, with and without complete modification of the t test. Such a comparison is performed in Section 3.2.

3.1 *Mathematical Derivation of the Definition of Effective Sample Size*

The Box (1954a, 1954b) procedure consists of approximating the distribution of a quadratic form in the normal random vector by a chi-square distribution that has the same first two moments. As we will observe below, the effective sample size initially proposed by Clifford and Richardson (1985, p. 158) is based only on the adjustment of the first moment (i.e., the expectation) of $1 - r^2$, which appears under a square root in the denominator of the statistic $(\hat{M} - 2)^{1/2}r/(1 - r^2)^{1/2}$ and whose distribution is approximated by a chi-square distribution with $\hat{M} - 2$ degrees of freedom.

Under the null hypothesis of no correlation between X and Y, $E(r) = 0$ and in particular, $E(1 - r^2) = 1 - \sigma_r^2$. From Clifford et al. (1989, Appendix 1), it follows that

$$\sigma_r^2 \approx \frac{\text{var}(S_{XY})}{E(S_X^2)E(S_Y^2)}. \tag{3.1}$$

From the preliminary results presented in Section 2 (see also Clifford and Richardson, 1985, pp. 157–158), it follows under the assumption $\boldsymbol{\Sigma}_x = \boldsymbol{\Sigma}_y = \mathbf{I}_N$ that to the first order,

$$1 - \sigma_r^2 = \frac{\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_x)\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_y) - \text{tr}(\mathbf{B}\boldsymbol{\Sigma}_x\mathbf{B}\boldsymbol{\Sigma}_y)}{\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_x)\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_y)}$$

so that

$$1 - \sigma_r^2 = \frac{N - 2}{N - 1} \tag{3.2}$$

and finally,

$$N = 1 + \sigma_r^{-2}. \tag{3.3}$$

That is the announced mathematical justification for defining the effective sample size M to be the right-hand side of (3.3). In fact, that value is the solution of (3.2) resulting from the adjustment of the expectation of a chi-square distribution to that of $1 - r^2$.

3.2 *Theoretically Expected Number of Degrees of Freedom of the Modified t Test*

For the range of a-values considered in Figure 1 in assuming that both processes X and Y are first-order simultaneous autoregressive on a lattice, we have computed the numbers of degrees of freedom for the modified t test that are theoretically expected:

(i) on the basis of equation (2.9) in Clifford et al. [1989, p. 126, where NS_X^2 and NS_Y^2 respectively estimate $\text{tr}(\boldsymbol{\Sigma}_x)$ and $\text{tr}(\boldsymbol{\Sigma}_y)$ if X and Y are stationary in variance; this is the origin of the multiplicative constant N^{-2} in estimator (2.6) of Clifford et al. (1989, p. 125)], i.e., with

$$\sigma_r^2 = \frac{\text{tr}(\boldsymbol{\Sigma}_x\boldsymbol{\Sigma}_y)}{\text{tr}(\boldsymbol{\Sigma}_x)\text{tr}(\boldsymbol{\Sigma}_y)}; \tag{3.4}$$

(ii) with the complete expression (2.4) for $\text{var}(S_{XY})$ in the numerator of (3.1) and also with its complete denominator, i.e., with

$$\sigma_r^2 = \frac{\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_x\mathbf{B}\boldsymbol{\Sigma}_y)}{\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_x)\text{tr}(\mathbf{B}\boldsymbol{\Sigma}_y)}. \tag{3.5}$$

If M_1 and M_2 respectively denote the effective sample sizes theoretically expected on the basis of (3.4) and (3.5), the corresponding numbers of degrees of freedom for the modified t test are given by $df_1 = M_1 - 2$ and $df_2 = M_2 - 2$. From the comparison of df_1 and df_2 in Table 1, one can easily see that the largest differences expressed in percentages are observed for the smallest number of locations ($N = 4$) and the highest a-value ($a = .20$), with $a_X = a_Y = a$ (87.66%) or with $a_X = -a_Y = a$ (87.03%). In these extreme cases, df_1 values are equal to about 8 times the corresponding df_2 values, for same-sign autocorrelation, and conversely for opposite-sign autocorrelation. For a given a-value and whatever the combination of a_X and a_Y signs, differences expressed in percentages tend to vanish for large numbers of locations, whereas differences $|df_1 - df_2|$ remain approximately constant. In practice, the difference is of no consequence for a t test with 97.68 degrees of freedom instead of 95.74 for $N = 81$ and $a_X = -a_Y = .10$; for $N = 16$ and $a_X = a_Y = .20$, which is not an extreme case, the use of the incomplete modification has more serious consequences ($df_1 = 2.54$ instead of $df_2 = 4.91$), whereas no modification at all would lead to 14 degrees of freedom.

When $a_X = a_Y = a$ and $a > .10$, df_1 values are always smaller than the corresponding df_2 values; with $a < .10$, the opposite is the rule; and finally, when $a = .10$, the df_1/df_2 ratio is approximately equal to 1. For a given lattice width, the differences, whether expressed in percentages or not, increase on the right-hand side of the "border" a-value of .10 from $a = .15$ to $a = .20$; these differences are more important for the higher autocorrelation value $a = .15$ than for $a = .05$. For the smallest numbers of locations ($N = 4, 9, 16$) and the highest a-values ($a = .15$ and .20), the $100 \times df_1/df_2$ ratio is equal to or higher than 115%.

When $a_X = -a_Y = a$, all df_1 values are greater than the corresponding df_2 values. For a given a-value, the general tendency is that differences decrease in absolute value for lattice widths ranging

Table 1

Theoretically expected numbers of degrees of freedom for the modified t test in assessing the correlation between two first-order simultaneous autoregressive lattice processes X and Y, without (df_1) *and with* (df_2) *complete modification, relative to the number of locations and the values of parameters* a_X *and* a_Y

Number of locations	$a_X = a_Y = a$								$a_X = -a_Y = a$							
	$a = .05$		$a = .10$		$a = .15$		$a = .20$		$a = .05$		$a = .10$		$a = .15$		$a = .20$	
	df_1	df_2	df_1	df_2	df_1	df_2	df_1	df_2	df_1	df_2	df_1	df_2	df_1	df_2	df_1	df_2
4	2.68 (28.19)	1.92	1.80 (.84)	1.79	.81 (50.97)	1.65	.19 (87.74)	1.54	3.35 (36.89)	2.11	4.78 (47.54)	2.51	9.85 (65.06)	3.44	41.78 (87.04)	5.42
9	7.45 (10.45)	6.67	5.79 (.25)	5.78	3.33 (21.17)	4.57	.97 (70.81)	3.33	8.57 (14.05)	7.36	10.69 (10.21)	8.64	16.46 (29.63)	11.58	34.92 (45.41)	19.06
16	14.12 (5.59)	13.33	11.41 (.50)	11.35	7.18 (13.71)	8.32	2.54 (48.32)	4.91	15.90 (7.49)	14.71	19.24 (10.33)	17.25	28.35 (16.89)	23.56	64.43 (34.15)	42.43
25	22.71 (3.48)	21.92	18.67 (.26)	18.62	12.11 (8.52)	13.24	4.68 (31.78)	6.86	25.32 (4.69)	24.13	30.16 (6.51)	28.20	43.08 (10.74)	38.45	89.98 (21.48)	70.65
36	33.22 (2.38)	32.42	27.59 (.10)	27.57	18.20 (6.14)	19.39	7.28 (22.10)	9.34	36.82 (3.22)	35.63	43.46 (4.49)	41.51	61.04 (7.50)	56.46	124.47 (15.77)	104.84
49	45.64 (1.73)	44.85	38.18 (.02)	38.17	25.48 (4.76)	26.75	10.32 (16.68)	12.38	50.40 (2.35)	49.22	59.15 (3.29)	57.21	82.12 (5.53)	77.58	163.41 (11.60)	144.45
64	59.99 (1.31)	59.20	50.43 (.02)	50.44	33.95 (3.85)	35.31	13.84 (13.41)	15.98	66.06 (1.79)	64.88	77.22 (2.51)	75.28	106.34 (4.24)	101.82	208.24 (8.99)	189.52
100	94.44 (.83)	93.65	79.90 (.05)	79.94	54.44 (2.71)	55.96	22.38 (9.70)	24.78	103.65 (1.14)	102.47	120.51 (1.60)	118.58	164.18 (2.72)	159.71	314.21 (5.74)	296.16
144	136.57 (.59)	135.77	116.03 (.04)	115.98	79.74 (1.50)	80.96	32.98 (2.85)	33.95	149.58 (.80)	148.39	173.37 (1.14)	171.40	234.67 (2.12)	229.69	442.60 (5.22)	419.49
256	243.87 (.34)	243.05	208.24 (.01)	208.21	144.50 (.65)	145.45	60.57 (1.38)	61.42	266.44 (.45)	265.25	307.69 (.67)	305.63	413.50 (1.38)	407.79	767.52 (3.54)	740.32
400	381.94 (.21)	381.12	326.98 (<.01)	326.97	228.22 (.37)	229.07	96.19 (1.00)	97.16	416.68 (.29)	415.46	480.47 (.45)	478.30	643.37 (.95)	637.23	1182.25 (2.50)	1152.65

Numbers in parentheses are percentages $100 \times |df_1 - df_2|/\max(df_1, df_2)$.

from 2 ($N = 4$) to 6 ($N = 36$) and stabilize for larger lattices. For a given number of locations, the differences, whether expressed in percentages or not, increase with the value of a. For the smallest numbers of locations ($N = 4, 9, 16$) and the highest a-values ($a = .15$ and $.20$), the $100 \times df_1/df_2$ ratio is equal to or higher than 120%.

4. Discussion

In Section 3.1, we have shown the pioneering results of Box (1954a, 1954b) to be the theoretical foundation of the modified t tests proposed by Clifford et al. (1989). Box's first results concerned the effect of inequality of variances on the distribution adjustment of quadratic forms in the normal random vector in the ANOVA area (Box, 1954a); they were further extended for the effect of serial correlation (Box, 1954b). These results cannot be circumvented in the literature on modified t and F tests (see, for instance, Brownie, Boos, and Hughes-Oliver, 1990). Finally, note the striking resemblance of expression (A.4) of Clifford et al. (1989, p. 133)

$$\sigma_r^{-2} = \frac{\text{tr}(\boldsymbol{\Sigma}_\eta)\text{tr}(\boldsymbol{\Sigma}_\xi)}{\text{tr}(\boldsymbol{\Sigma}_\eta\boldsymbol{\Sigma}_\xi)}$$

when it is evaluated for $\boldsymbol{\Sigma}_\xi = \boldsymbol{\Sigma}_\eta = \boldsymbol{\Sigma}$, with the expression of Box's ε rewritten in matrix notation (see, for instance, Crowder and Hand, 1990, p. 54)

$$\varepsilon = \frac{[\text{tr}(\boldsymbol{\Sigma})]^2}{q\ \text{tr}(\boldsymbol{\Sigma}^2)},$$

where q denotes the number of degrees of freedom of the numerator of the unmodified ANOVA F test, and $\boldsymbol{\Sigma}$, the covariance matrix of q orthonormal contrasts. In view of our Section 3.1, this resemblance is not surprising.

A second basic reference that was missing in Clifford et al. (1989) [and missing also in Richardson and Hémon, 1981, 1982; and Clifford and Richardson, 1985] is Bivand (1980), cited by Cliff and Ord (1981, p. 189), Griffin (1988, p. 86), and Haining (1990, p. 314). That is a major oversight, since Bivand (1980) addressed the problem of estimating the correlation coefficient from spatially autocorrelated observations. In an empirical approach based on Monte Carlo experiments [an approach also used by Clifford et al. (1989)], he explored the risk incurred in inferring from unmodified estimates. His conclusions were that positive spatial autocorrelation induces a bias due to the underestimation of the standard error of the product moment correlation after Fisher's transformation, and that this bias leads to narrower confidence intervals than were in fact justified by the amount of information carried out by the observations. This is nothing but an intuitive predefinition of the concept of effective sample size.

The major remaining ambiguity in the Clifford et al. (1989) paper concerns the use of the "conditional variance of S_{XY}" (p. 125 and top of p. 126). The use of conditional normal distributions for X conditional on $Y = y$ and for Y conditional on $X = x$ (§2.2, p. 124), instead of the multinormal distribution on $(\mathbf{x}, \mathbf{y})$ that we used in Section 2, may at first be appealing because it implies less restrictive hypotheses. But why do the authors suppose thereafter (p. 125, line 1) that "in the general case," X and Y are "independent" and marginally multinormally distributed, and in the next sentence, give the "conditional variance (of) S_{XY}"? I do not know, and I understand all the less since they incorporate in equation (2.8) (p. 126) an expression of $\text{var}(S_{XY})$ that was unconditionally derived in their Appendix 1.

5. Conclusion

For lattice and non-lattice data networks, t tests for assessing the correlation between two spatially autocorrelated processes have to be modified (Clifford et al., 1989). In the case of data networks of small size (Table 1), we recommend the use of modified t tests with a corrected number of degrees of freedom $\hat{M} - 2 = \hat{\sigma}_r^{-2} - 1$, with

$$\hat{\sigma}_r^2 = \frac{\text{tr}(\mathbf{B}\hat{\boldsymbol{\Sigma}}_x\mathbf{B}\hat{\boldsymbol{\Sigma}}_y)}{\text{tr}(\mathbf{B}\hat{\boldsymbol{\Sigma}}_x)\text{tr}(\mathbf{B}\hat{\boldsymbol{\Sigma}}_y)},$$

where the estimated X-autocovariances contained in $\hat{\boldsymbol{\Sigma}}_x$ are the sample autocovariances $\hat{C}_X(k)$ computed among pairs of X-observations for the pairs of locations belonging to stratum S_k ($k = 0, 1, \ldots$); a similar procedure is applied to the estimated Y-autocovariances contained in $\hat{\boldsymbol{\Sigma}}_y$. In the case of large data networks (Table 1), the use of the incomplete modification based on the approximation in the estimation of the variance of the sample correlation coefficient proposed by Clifford et al. (1989) is of no consequence in practice. The computational advantage of that approximation may

be limited, however, even for large samples, depending on the number K of strata. In fact, the difference between the number of elementary operations (sums and products) required by the computation of $\mathrm{tr}(\hat{\mathbf{\Sigma}}_y\hat{\mathbf{\Sigma}}_y)$ and by that of $\mathrm{tr}(\mathbf{B}\hat{\mathbf{\Sigma}}_x\mathbf{B}\hat{\mathbf{\Sigma}}_y)$ is of order K, whereas the number of operations required by the estimation of the sample autocovariances $\hat{C}_X(k)$ and $\hat{C}_Y(k)$ (which has to be performed whether the modification is complete or not) is of order N_k^2 ($k = 0, 1, \ldots$). The complete modification is recommended in computer programs.

Acknowledgements

I would like to thank Drs Robert Sokal, Neal Oden, and Pierre Legendre for fruitful and motivating discussions on the present topic. Neal Oden has kindly made available some of the programs for use in this research. I also acknowledge the insightful comments of Drs Pierre Legendre and Théophile Niyonsenga on a first draft of the manuscript. This research was supported in part by Grant OGP0007738 from the Natural Sciences and Engineering Research Council of Canada to Pierre Legendre.

Résumé

Clifford, Richardson, and Hémon (1989, *Biometrics* **45,** 123–134) ont présenté des tests modifiés d'association entre deux processus spatialement autocorrélés, pour des grilles régulières ou irrégulières de points. Ces tests sont construits sur la covariance et le coefficient de corrélation empiriques; ils requièrent l'estimation d'une taille d'echantillon effective qui tient compte de la structure spatiale des deux processus. Clifford et al. ont développé leur méthode sur la base d'une approximation de la variance du coefficient de corrélation empirique et l'ont éprouvée par simulations de Monte Carlo pour des grilles régulières et irrégulières, de moyenne et de grande taille.

Dans le présent article, la variance de la covariance empirique est calculée pour un nombre fini de points d'observation, sous l'hypothèse de multinormalité, et la démonstration mathématique de la définition de taille d'échantillon effective est donnée. Les nombres de degrés de liberté, théoriquement attendus pour le test t modifié avec les modifications mises à jour, sont comparés avec ceux calculés à partir de l'équation (2.9) de Clifford et al. (1989). Les plus grandes différences sont observées pour de petits nombres de points d'observation et une autocorrélation élevée, en particulier lorsque cette dernière est présente avec des signes opposés dans les deux processus. Des références fondamentales non reprises dans Clifford et al. (1989) sont données et des ambiguïtés inhérentes sont discutées.

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Received December 1991; *revised May* 1992; *accepted September* 1992.

RESPONSE

1. General Points

We are grateful to Professor Dutilleul for his thoughtful comments on our paper. We believe that finding a correct method of assessing apparent correlation between spatial series is an important goal for statistical research. Our own interest is in epidemiological studies of the association between health indicators and the environment. We hope that this discussion will stimulate others to work on the theoretical problems in this area.

First, we should explain that our published paper is an abbreviated version of the original typescript. This was in response to requests from the editors of *Biometrics.* The original paper contained a more substantial historical review, which was shortened. Revisions elsewhere may have made the paper more difficult to follow.

Our paper is based on a simple idea, as follows. The sample correlation, r, between two spatial processes **X** and **Y,** always lies in the interval $(-1, 1)$, regardless of the joint distribution of the variables involved. Under the null hypothesis of independence between the two spatial processes, the mean of r is zero. The simple idea is to approximate the distribution of r by a member of the class

$$f_M(r) = \frac{(1 - r^2)^{(M-4)/2}}{B[1/2, (M - 2)/2]}, \quad \text{for } -1 < r < 1,$$

that is, the standard null distribution of the correlation coefficient with sample size M. We do not insist that M be an integer. The problem is to find a suitable value for M. Our approach is to equate σ_r^2, the variance of the sample correlation between the two spatial processes, to the variance of $f_M(r)$, i.e., $1/(M - 1)$. This reduces to the equation

$$M = 1 + \sigma_r^{-2}.$$

We then replace σ_r^2 by an estimate and obtain an estimate of M,

$$\hat{M} = 1 + \hat{\sigma}_r^{-2},$$

as the effective sample size. The significance of the observed value of r is then assessed as if it had come from the density $f_{\hat{M}}(r)$. This is how we actually arrived at the approximation, as opposed to the more elaborate explanation suggested in the Reader Reaction. The ultimate check on the effectiveness of this chain of assumptions is to compute the Type I error rate of the test, e.g., to determine whether under the null hypothesis

$$\Pr[|r| > K(\hat{M})] = .05, \tag{1}$$

where $K(\hat{M})$ is given by

$$\int_{-K(\hat{M})}^{K(\hat{M})} f_{\hat{M}}(r)dr = .95.$$

The steps along the way are therefore less important than the final check on the accuracy of (1).

Professor Dutilleul takes us to task for not referring to the papers of Box, which deal with, as he says, "approximating the distribution of a quadratic form in the normal random vector by a chi-square distribution that has the same first two moments." However, we feel bound to point out that the tradition of approximating distributions by equating moments goes back much further than that, having been widely used by K. Pearson and his coworkers. Our stated approach is to approximate the distribution of r under the null hypothesis by equating the mean and variance of r to the corresponding moments of members of the f_M family. To complete the approximation we must estimate M. We have suggested one estimate and Professor Dutilleul has suggested another. The choice between the estimates will depend on the accuracy of the associated Type I error rates.

It is possible to investigate the bias of our estimate of the variance of S_{XY}. If we denote the variance of S_{XY} by L and write K for our estimate of this quantity, then using the notation of

the Appendix, we have

$$\mathrm{E}(K) - L = N^{-2}\mathrm{E}\left[\sum_k N_k \hat{C}_X(k)\hat{C}_Y(k) - \sum_k \sum_{A_k} \xi_\alpha \xi_\beta \eta_\alpha \eta_\beta\right]$$

$$= N^{-2}\left\{\sum_k \left[N_k^{-1} \sum_{A_k} a_{\alpha\beta} \sum_{A_k} b_{\alpha\beta} - \sum_{A_k} a_{\alpha\beta} b_{\alpha\beta}\right]\right\},$$

where $a_{\alpha\beta} = \mathrm{E}(\xi_\alpha \xi_\beta)$ and $b_{\alpha\beta} = \mathrm{E}(\eta_\alpha \eta_\beta)$. Hence

$$\mathrm{E}(K) - L = -N^{-2} \sum_k \sum_{A_k} (a_{\alpha\beta} - \bar{a}_k)(b_{\alpha\beta} - \bar{b}_k),$$

where

$$\bar{a}_k = \sum_{A_k} a_{\alpha\beta}/N_k \quad \text{and} \quad \bar{b}_k = \sum_{A_k} b_{\alpha\beta}/N_k.$$

Notice that if $\boldsymbol{\Sigma}_\mathbf{X} = \boldsymbol{\Sigma}_\mathbf{Y}$, then $a_{\alpha\beta} = b_{\alpha\beta}$, $\forall \alpha, \beta$, and the bias is negative. Also note that when $\mathrm{cov}(X_\alpha, \bar{X})$ is constant for all α, then $a_{\alpha\beta} = \bar{a}_k$ and the bias of K is zero. Such would be the case for a cyclic process.

Contrary to the interpretation given in the Reader Reaction, K, which indeed can be written as $\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_\mathbf{X}\hat{\boldsymbol{\Sigma}}_\mathbf{Y})$, was not constructed to be an unbiased estimate of $\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{X}\boldsymbol{\Sigma}_\mathbf{Y})$. Since $\hat{C}_X(k)$ is a downward biased estimate of $C_X(k)$, one would expect $\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_\mathbf{X}\hat{\boldsymbol{\Sigma}}_\mathbf{Y})$ to underestimate $\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{X}\boldsymbol{\Sigma}_\mathbf{Y})$ when both $\mathbf{X}$ and $\mathbf{Y}$ are positively autocorrelated. Our estimate K was constructed directly as an estimate of $\mathrm{var}(S_{XY})$, a quantity that is smaller than $\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{X}\boldsymbol{\Sigma}_\mathbf{Y})$, as extensively illustrated in Figure 1 of the Reader Reaction. Hence we do not believe that the differences shown in Figure 1 are particularly relevant in discussing the quality of our proposed estimate for $\mathrm{var}(S_{XY})$.

It is important to stress that using an estimate of M based on an estimate of σ_r^2 that has better bias properties provides no guarantee that the associated Type I error rate is closer to the nominal value. In our paper, we have presented substantial Monte Carlo evidence that the Type I error rates associated with our proposed estimate of M are accurate. In a further paper (Richardson and Clifford, 1991), we have also checked this accuracy for smaller lattice sizes (6×6, 8×8, 10×10). It would be very interesting to compare our Figure 2 with the equivalent figure obtained using the proposed estimate of M in the Reader Reaction.

2. Specific Points

We were aware of the paper by Bivand. This was one of the references that was removed when the historical review was shortened. Bivand showed, by a simulation study, that in the presence of spatial autocorrelation, the null distribution of the correlation coefficient differed from the distribution in the standard i.i.d. case. This confirmed the effect Student (1914) had identified in his early paper on "The elimination of spurious correlation due to position in time or space." Bivand did not offer any suggestions about how the Type I error rate might be corrected. Richardson (1985) contains a reference to Bivand and also documents additional historical material. A recent and different approach to testing association between spatial processes is outlined in Besag and Clifford (1989).

The "major remaining ambiguity" mentioned in the Reader Reaction seems to have arisen as a combination of bad wording on our part and notational changes introduced at a final stage of revision. On pages 125–126 we refer to the case of independent samples, and later "the general case." The general case is that of two independent *processes* $\mathbf{X}$ and $\mathbf{Y}$, each of which is spatially autocorrelated. By independent *samples* we mean the special case in which the elements of $\mathbf{X}$ are i.i.d., the elements of $\mathbf{Y}$ are i.i.d., and the $\mathbf{X}$ process is independent of the $\mathbf{Y}$ process. In other words, both $\mathbf{X}$ and $\mathbf{Y}$ are samples and the samples are independent. In both cases, we consider the variance of the random variable S_{XY} conditional on $\mathbf{X} = \mathbf{x}$ and here there is a notation error in the paper since uppercase Xs instead of lowercase are used in equation (2.3) and the line below, and also in equation (2.4). The important point that we are making in Section 2.3 is that to the first order of approximation, it does not matter whether you consider the distribution of the standardised covariance conditionally or marginally.

3. Conclusion

The Reader Reaction contains an alternative estimate of the effective sample size, which could be used in assessing the significance of the correlation between two spatial series. It would be quite feasible to carry out a Monte Carlo study to find out how well it performs. The fact that one has to

resort to Monte Carlo methods is a reflection of the weakness of the theoretical tools available in this area. We hope that others will be inspired to take up the challenges that this subject presents.

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Peter Clifford
Statistics Department
Oxford University
1 South Parks Road
Oxford OX1 3TG, United Kingdom

and

Sylvia Richardson and Denis Hémon
INSERM U170
16 Avenue Paul Vaillant-Coutourier
94807 Villejuif Cedex
France