6. Study the shape of the chaotic attractor for different initial conditions. Keep the drive force fixed at $F_D = 1.2$ and calculate the attractors found for several different initial values of θ. Show that you obtain the same attractor even for different initial conditions, provided that these conditions are not changed by too much. Repeat your calculations for different values of the time step to be sure that it is sufficiently small that it does not cause any structure in the attractor.

*7. Investigate how a strange attractor is altered by small changes in one of the pendulum parameters. Begin by calculating the strange attractor in Figure 3.7. Then change either the drive amplitude or drive frequency by a small amount and observe the changes in the attractor.

*8. Construct a very high-resolution plot of the chaotic attractor in Figure 3.7, concentrating on the region $\theta > 2$ rad. You should find that there is more structure in the attractor than is obvious on the scale plotted in Figure 3.7. In fact, an important feature of chaotic attractors is that the closer you look, the more structure you find. We will see later that this property is related to fractals. It turns out that a strange attractor is a fractal object. Hint: In order to get accurate results for a high resolution plot of the attractor, it is advisable, in terms of the necessary computer time, to use the Runge-Kutta method.

3.3 Routes to Chaos: Period Doubling

We have seen that at low driving forces the damped, nonlinear pendulum exhibits simple oscillatory motion, while at high drive it can be chaotic. This raises an obvious question: Exactly how does the transition from simple to chaotic behavior take place? It turns out that the pendulum exhibits transitions to chaotic behavior at several different values of the driving force. We have already observed in Figure 3.4 that one of these transitions must take place between $F_D = 0.5$ and 1.2. However, this transition is not the clearest one to study numerically, so we will instead consider the behavior at somewhat higher driving forces.[10]

Figure 3.8 shows results for θ as a function of time for several values of the driving force calculated using the Euler-Cromer program described earlier. At these high values of the drive the pendulum often swings all the way around its support; this can be seen from the vertical steps in θ as our program resets[11] this angle to keep it in the range $-\pi$ to π. These steps notwithstanding, the behavior in Figure 3.8 is a periodic, repeating function of t in all three cases (after the initial transients have damped away). The drive frequency used here was $\Omega_D = 2/3$ so the period of the driving force was $2\pi/\Omega_D = 3\pi$, and this is precisely the period of the motion found at $F_D = 1.35$. Hence, in this case, the pendulum moves at the same frequency as the driving force.

The behavior at $F_D = 1.44$ is a bit more subtle. While we again have periodic motion, the period is now *twice* the drive period. This can be seen most clearly by comparing the θ-t waveforms at $F_D = 1.35$ and 1.44, and noticing that in the latter case the bumps alternate in amplitude. Our pendulum has already surprised us on several occasions, and we might be tempted to add this behavior to our list of pendulum puzzles. However, this surprise is a very special and important one. When a nonlinear

[10] The pendulum exhibits an extraordinarily rich behavior, some of which we discuss later. Unfortunately, we only have time to explore a small portion of the chaotic regime here. We will, therefore, limit ourselves to one value of both the drive frequency and damping, and only certain ranges of the drive amplitude. The interested reader is encouraged to investigate other parameter values. Additional results are also described by Baker and Gollub (1990).

[11] This distraction can be avoided by plotting ω instead of θ. We will leave this to the exercises.

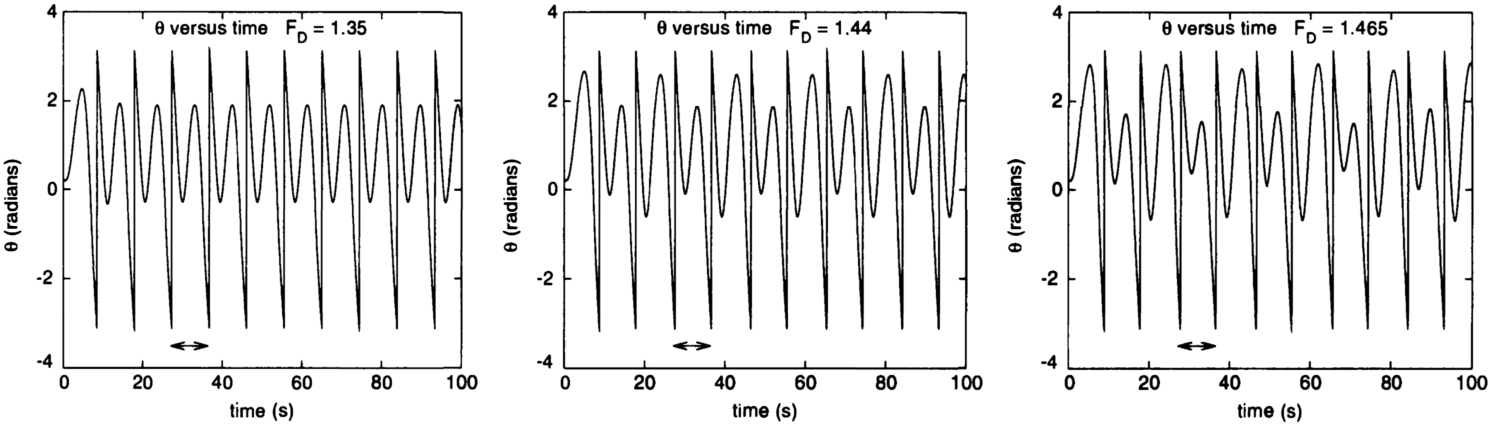

Figure 3.8: Results for θ as a function of time for our pendulum for several different values of the drive amplitude. The other parameters were the same as in Figure 3.4 except that here we used a time step of 0.01. The horizontal arrows show the period of the driving force. In the middle plot the period is twice the drive period, since the values at the maxima alternate between $\theta \approx 1.9$ and 2.6. On the right the period is four times the drive period, as the maxima alternate between the values $\theta \approx 1.69, 2.82, 1.52$, and 2.72.

system is excited or driven by a single frequency stimulus, the response is, in general, not limited to the driving frequency. If Ω_D is the drive frequency, the nonlinear response will usually contain components at $2\Omega_D$, $3\Omega_D$, etc., at all harmonics. This process is known as mixing and is manifest by the generation of responses at integer multiples of the driving frequency. Such a nonlinear response is standard and well understood, and its key property is that it contains frequencies that are equal to or greater than the drive frequency. Hence, the periods of these harmonics will be smaller than the drive period. In contrast, our pendulum is now exhibiting a response at $\Omega_D/2$ (a lower frequency!), a *subharmonic*, which is unlike any standard mixing effect.

Returning to Figure 3.8, a careful look at the results for $F_D = 1.465$ shows that they exhibit a period that is four times the driving period. The pattern should now be evident. If we were to increase the drive amplitude further, the period would double again as the pendulum would switch to a motion that has a period eight times that of the drive. This period-doubling cascade would continue if the drive were increased further.

But if the period keeps on doubling, what about the transition to chaos? A nice way to appreciate how this transition comes about is with what is known as a *bifurcation diagram*. In Figure 3.9 we show a bifurcation diagram for θ as a function of drive amplitude, which was constructed in the following manner. For each value of F_D we have calculated θ as a function of time. After waiting for 300 driving periods so that the initial transients have decayed away, we plotted θ at times that were in phase with the driving force[12] as a function of F_D. Here we have plotted points up to the 400th drive period. This process was then repeated for the range of values of F_D shown in the figure.[13]

To understand the bifurcation diagram we start at $F_D = 1.35$. We have already seen that in this case the motion has the same period as the drive, so if we observe θ at a particular time in the drive cycle we will always find the same value. Our bifurcation diagram thus consists of a single point; there is just

[12]Just as we did in constructing the Poincaré section in Figure 3.7.

[13]We have chosen to study this range of F_D because it exhibits period-doubling in an especially clear manner. Results for other values of F_D are shown by Baker and Gollub (1990), or you can calculate them for yourself.

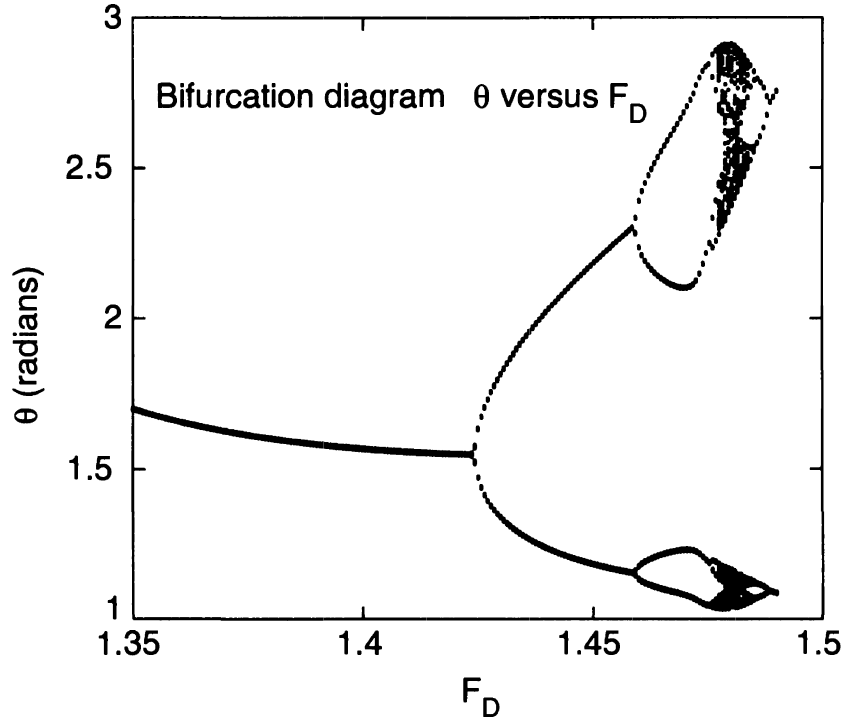

Figure 3.9: Bifurcation diagram for our pendulum. The parameters used for the calculation were the same as in Figure 3.8.

one value of θ for this value of F_D, although that point will be plotted many times. We will refer to this as period-1 behavior, since the θ-t waveform has the same period as the driving force. If the motion is period-2, then the values of θ that are plotted will alternate between two values. This is just the alternation we saw at $F_D = 1.44$ in Figure 3.8, and it leads to two points on the bifurcation diagram. The pattern should now be clear: Motion that is period n will yield n points on the bifurcation diagram for that value of F_D. From Figure 3.9 we see that the behavior is period 1 up to approximately $F_D = 1.424$, where there is a transition to period-2 motion. This persists up to the transition to period-4 behavior at $F_D \approx 1.459$. This process continues, although the resolution of our diagram makes it difficult to follow the behavior past period 8. This period-doubling cascade ultimately ends in a transition to chaotic behavior.

We have now obtained at least a qualitative understanding of how the transition from regular to chaotic behavior occurs for one particular system, the pendulum, in one particular range of parameters (i.e., drive force, drive frequency, damping strength, etc.). But how general is this behavior? Is it only found in the pendulum, or does it occur in other systems? These are the kinds of questions that physicists often ask. We tend to look for (and like to discover) patterns and principles that occur widely and apply to many different systems. For example, the motion of macroscopic objects can, with only a few exceptions, be described by Newton's laws of motion. Likewise, classical (as opposed to quantum) electromagnetic phenomena can be described by Maxwell's equations. Because of their near-universal applicability we feel that these "laws" of physics provide us with a better understanding of the world. We don't want to get too philosophical here, and we also don't want to be pressed into a discussion of what the term understand really means, but we hope that this gives the reader some appreciation for why it is important to look hard for universal aspects in any problem or result. With that in mind, we now restate the question posed at the beginning of this paragraph. Is the period-doubling route to chaos universal in any way, or do all systems have their own particular way of making this transition?

While there is not yet a complete theory of chaos in the sense of Newton's laws or Maxwell's equations, the answer to this question seems to be the following. Many systems have been found to exhibit chaotic behavior, but there appears to be only a few ways in which the transition from simple to chaotic behavior can occur. The periodic-doubling scenario that we have observed with the pendulum is one of these few known routes to chaos. You can then ask if this periodic-doubling procedure itself has any properties that are universal. The answer to this is yes (why else would we raise the question?!), as we will now explain.

Returning to our bifurcation diagram, Figure 3.9, we note that the spacing between period-doubling transitions becomes rapidly smaller as the order of the transition increases. For example, the period-2 regime extends from $F_D \approx 1.424$ to 1.459, while the period-4 regime extends only from about 1.459 to 1.476, and the same trend is found for higher periods.[14] Let us define F_n to be the value of the driving force at which the transition to period-2^n behavior takes place. The shrinkage of the size of the periodic windows can be described by a parameter δ_n, where

$$\delta_n \equiv \frac{F_n - F_{n-1}}{F_{n+1} - F_n} . \tag{3.10}$$

The observation that the windows become smaller as n increases means that $\delta_n > 1$. It has been found that as n becomes large, δ_n approaches a constant that is known as the Feigenbaum δ. That is, the rate of shrinkage approaches a constant in the limit $n \to \infty$. Moreover, it turns out that essentially all systems that exhibit the period-doubling route to chaos appear to possess the same value of $\delta \approx 4.669\ldots$. Hence, there are indeed some universal aspects associated with the transition to chaos. There are, as we have hinted above, several other known routes to chaos, and a few of them can also be found in the pendulum. However, rather than making this chapter the story of the pendulum, we will next consider several other chaotic systems.

Exercises

1. Calculate Poincaré sections for the pendulum as it undergoes the period-doubling route to chaos. Plot ω versus θ, with one point plotted for each drive cycle, as in Figure 3.7. Do this for $F_D = 1.4$, 1.44, and 1.465, using the other parameters as given in connection with Figure 3.8. You should find that after removing the points corresponding to the initial transient the attractor in the period-1 regime will contain only a single point. Likewise, if the behavior is period n, the attractor will contain n discrete points.

2. Calculate the bifurcation diagram for the pendulum in the vicinity of $F_D = 1.35$ to 1.5. Make a magnified plot of the diagram (as compared to Figure 3.9) and obtain an estimate of the Feigenbaum δ parameter.

*3. Investigate the bifurcation diagrams found for the pendulum with other values of the drive frequency and damping parameter. Warning: This can easily become an ambitious project!

[14] In order to see this clearly we would need to examine the bifurcation on a much finer scale than in Figure 3.9.

*4. A very popular mathematical model that exhibits chaotic behavior is the logistic map. This map is defined by the relation

$$x_{i+1} = \mu\, x_i\, (1 - x_i)\,. \tag{3.11}$$

You can interpret x_i as the size of a population of animals in generation i and μ as a parameter that determines their reproductive rate. This model has been adopted by many different fields, with different interpretations. Its connection with physics is a bit remote, but it is nevertheless studied extensively by physicists as a (mathematically) simple chaotic system. In the logistic map the parameter μ plays the role that the driving force plays in the pendulum, while x is roughly analogous to θ.

Begin by calculating x as a function of i for several values of μ in the range 0 to 4. Note that x is restricted to values in the range 0 to 1. You should find that for $\mu < 3$, x is a constant (independent of i) after the initial transient has decayed away. For larger values of μ you should observe a sequence of period-doubling transitions, with a chaotic regime beginning at $\mu \approx 3.57$. It is also interesting to compute the bifurcation diagram (the analog of Figure 3.9 for the pendulum). Estimate the value of the Feigenbaum δ parameter and compare it with that quoted above for the pendulum. The article by May (1976) and the book by Baker and Gollub (1990) contain good introductions to the logistic map.

3.4 The Lorenz Model

The pendulum model we have considered is a very simple system, yet it exhibits extremely rich behavior. It is thus not surprising that other slightly more complicated systems are also capable of chaotic behavior. When we think of chaotic or unpredictable behavior, an example that naturally comes to mind is the weather. Because of the economic importance of having accurate weather predictions, a good deal of effort has been devoted to this problem. While much of this effort has gone into computer modeling of Earth's atmosphere, much has also been devoted to understanding the weather problem from a more fundamental point of view. It was work of this kind by the atmospheric scientist E. N. Lorenz (1963) that provided a major contribution to the modern field of chaos.

Lorenz was studying the basic equations of fluid mechanics, which are known as the Navier-Stokes equations; they can be thought of as Newton's laws written in a form appropriate for a fluid. These are a complicated set of differential equations that describe the velocity, temperature, density, etc., as functions of position and time, and they are very difficult to solve analytically in cases of practical interest. Of course, this is just the type of problem where a computational approach can be useful, and that is precisely what Lorenz did. The specific situation he considered was the Rayleigh-Bénard problem, which concerns a fluid in a container whose top and bottom surfaces are held at different temperatures. It had long been known that as the difference between these two temperatures is increased, the fluid can undergo transitions from a stationary state (no fluid motion) to steady flow (nonzero flow velocities that are constant in time, also referred to as convection) to chaotic flow. Lorenz did his work more than 30 years ago, so the computational power available to him was not very impressive by today's standards. This