

# “How much is in a square?” An appraisal of relational thinking

IFIP WG2.1 meeting #81

Kloster Neustadt, Germany, 4th April 2024

J.N. OLIVEIRA



UNIVERSITY OF MINHO & INESC TEC

# Ageing

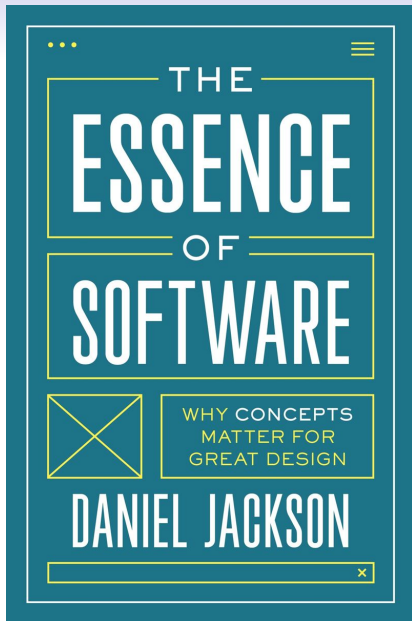
Quote from *Du und die Musik. Eine Einführung für alle Musikfreunde* by Frederich Herzfeld, Berlin, 1951:

---

*"The young are to the **content** what the old are to the **continent**".*

---

More and more interested in simple **patterns** (**shapes**, "continents") easy to **use** and **communicate** to a **wider audience**.



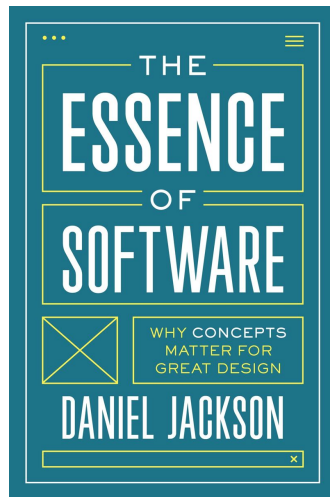
# The Essence of Software

"(...) The best services revolve around **a small number of concepts** that are **well designed and easy** (...) **to understand and use**, and their innovations often involve simple but compelling new concepts."



# The Essence of Software

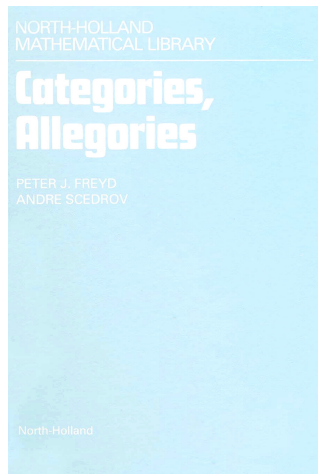
- ... **small number**
- ... **well defined**
- ... **easy to understand**
- ... **easy to use**



# Freyd & Ščedrov, 1990

"(...) A special feature of our approach is a general **calculus of relations** presented in part two.

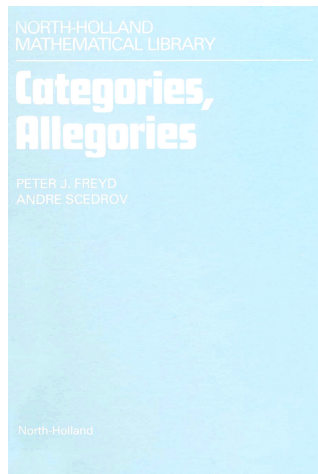
This calculus offers another, **often more amenable** framework for concepts and methods discussed in part one."



# Freyd & Ščedrov, 1990

"(...) A special feature of our approach is a general **calculus of relations** presented in part two.

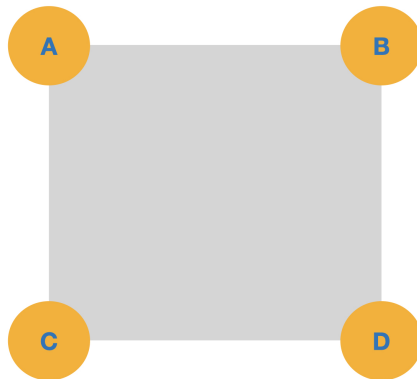
This calculus offers another, **often more amenable** framework for concepts and methods discussed in part one."

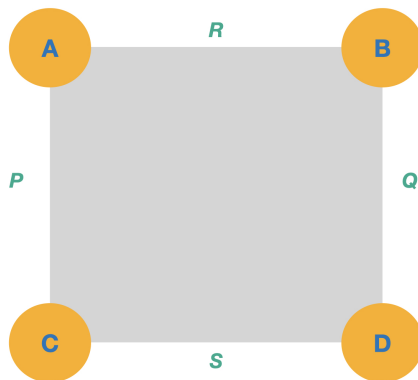


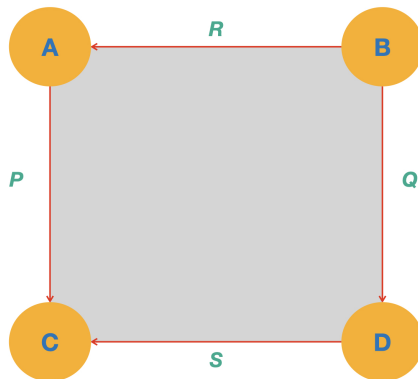
# Squares

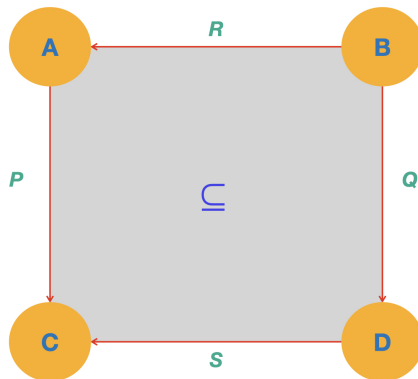




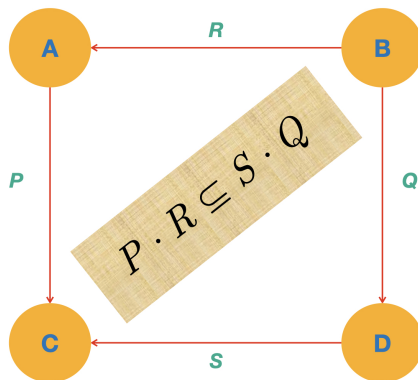






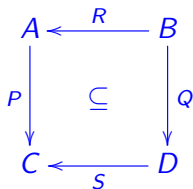


# “Magic” square



# “Magic” square

Four binary relations:



$$P \cdot R \subseteq S \cdot Q \quad (1)$$

**Terminology:**

$R$  — *pre-producer*

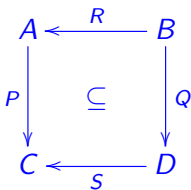
$P$  — *pre-consumer*

$Q$  — *post-producer*

$S$  — *post-consumer*

# “Magic” square

Pointfree:



$$P \cdot R \subseteq S \cdot Q$$

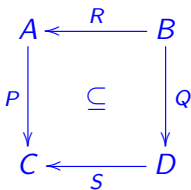
Pointwise:

$$\begin{array}{c}
 \exists \\
 \vdots \\
 P \cdot R \Rightarrow S \cdot Q \\
 \vdots \\
 \forall \quad c \quad b \quad c \quad b
 \end{array}$$

Diagram illustrating the pointwise interpretation of the subset relation. It shows the expression  $P \cdot R \Rightarrow S \cdot Q$  with variables  $a$  and  $d$  (green) and  $c$  and  $b$  (brown) connected by dotted lines to the corresponding nodes in the pointfree diagram.

# “Magic” square

Pointfree:



$$P \cdot R \subseteq S \cdot Q$$

Pointwise:

$$\begin{array}{c}
 \exists \\
 \begin{array}{ccccccc}
 & & a & & & d & \\
 & & \vdots & & & \vdots & \\
 P & \cdot & R & \Rightarrow & S & \cdot & Q \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \forall & c & b & & c & & b
 \end{array}
 \end{array}$$

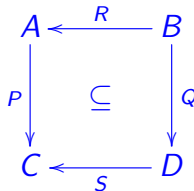


# “Magic” square

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

Perhaps not what you were expecting...

... but we can do **a lot** with



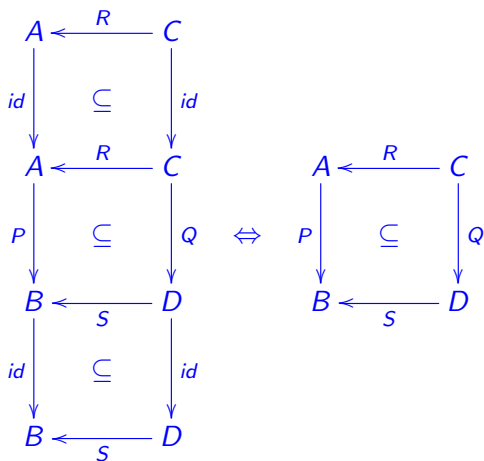
# Vertical composition

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xleftarrow{R} & C \\
 P \downarrow & \subseteq & \downarrow Q \\
 B & \xleftarrow{S} & D \\
 P' \downarrow & \subseteq & \downarrow Q' \\
 B' & \xleftarrow{S'} & D'
 \end{array}
 & \Rightarrow &
 \begin{array}{ccc}
 A & \xleftarrow{R} & C \\
 P' \cdot P \downarrow & \subseteq & \downarrow Q' \cdot Q \\
 B' & \xleftarrow{S'} & D'
 \end{array}
 \end{array}
 \quad (2)$$

# Horizontal composition

$$\begin{array}{ccccc}
 A & \xleftarrow{R} & C & \xleftarrow{R'} & C' \\
 \downarrow P & \sqsubseteq & \downarrow Q & \sqsubseteq & \downarrow Q' \\
 B & \xleftarrow{S} & D & \xleftarrow{S'} & D'
 \end{array}
 \Rightarrow
 \begin{array}{ccccc}
 A & \xleftarrow{R \cdot R'} & C & & \\
 \downarrow P & \sqsubseteq & & & \downarrow Q' \\
 B' & \xleftarrow{S \cdot S'} & D' & & 
 \end{array}
 \quad (3)$$

# Identity



(Similarly for horizontal.)

# Converse



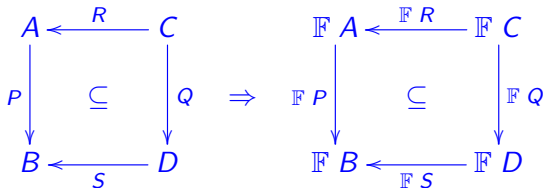
The converse of a square is its “*passive voice*” 😊

# Square (direct) sums

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xleftarrow{R} & C \\
 \downarrow P & \subseteq & \downarrow Q \\
 B & \xleftarrow{S} & D
 \end{array} & & \\
 + & & \\
 \begin{array}{ccc}
 A' & \xleftarrow{R'} & C' \\
 \downarrow P' & \subseteq & \downarrow Q' \\
 B' & \xleftarrow{S} & D'
 \end{array} & & \\
 = & & \begin{array}{ccc}
 A + A' & \xleftarrow{R+R'} & C + C' \\
 \downarrow P+P' & \subseteq & \downarrow Q+Q' \\
 B + B' & \xleftarrow{S+S'} & D + D'
 \end{array}
 \end{array}$$

# Functorial squares

Functor  $\mathbb{F}$ :



$\mathbb{F}$  should be monotonic and preserve converses — a **relator** (Backhouse and Backhouse, 2004).

# Squares with functions

Some relations  $f$  fit into the following squares:

$$\begin{array}{ccccc}
 A & \xleftarrow{id} & A & \xleftarrow{f^\circ} & B \\
 id \downarrow & \subseteq & \downarrow f & \subseteq & \downarrow id \\
 A & \xleftarrow{f^\circ} & B & \xleftarrow{id} & B
 \end{array} \tag{4}$$

Left square:  $\langle \forall a :: \langle \exists b :: b f a \rangle \rangle$   $f$  is **total**.

Right square:  $\langle \forall b, b' :: \langle \exists a :: b f a \wedge b' f a \rangle \Rightarrow (b = b') \rangle$   
 $f$  is **univocal**.

Such relations  $f$  are called **functions**.



# Squares with functions

Some relations  $f$  fit into the following squares:

$$\begin{array}{ccccc}
 A & \xleftarrow{id} & A & \xleftarrow{f^\circ} & B \\
 id \downarrow & \subseteq & \downarrow f & \subseteq & \downarrow id \\
 A & \xleftarrow{f^\circ} & B & \xleftarrow{id} & B
 \end{array} \tag{4}$$

Left square:  $\langle \forall a :: \langle \exists b :: b f a \rangle \rangle$

$f$  is **total**.

Right square:  $\langle \forall b, b' :: \langle \exists a :: b f a \wedge b' f a \rangle \Rightarrow (b = b') \rangle$

$f$  is **univocal**.

Such relations  $f$  are called **functions**.

# Squares with functions

Some relations  $f$  fit into the following squares:

$$\begin{array}{ccccc}
 A & \xleftarrow{id} & A & \xleftarrow{f^\circ} & B \\
 id \downarrow & \subseteq & \downarrow f & \subseteq & \downarrow id \\
 A & \xleftarrow{f^\circ} & B & \xleftarrow{id} & B
 \end{array} \tag{4}$$

Left square:  $\langle \forall a :: \langle \exists b :: b f a \rangle \rangle$

$f$  is **total**.

Right square:  $\langle \forall b, b' :: \langle \exists a :: b f a \wedge b' f a \rangle \Rightarrow (b = b') \rangle$

$f$  is **univocal**.

Such relations  $f$  are called **functions**.

# Squares with functions

Some relations  $f$  fit into the following squares:

$$\begin{array}{ccccc}
 A & \xleftarrow{id} & A & \xleftarrow{f^\circ} & B \\
 id \downarrow & \subseteq & \downarrow f & \subseteq & \downarrow id \\
 A & \xleftarrow{f^\circ} & B & \xleftarrow{id} & B
 \end{array} \tag{4}$$

Left square:  $\langle \forall a :: \langle \exists b :: b f a \rangle \rangle$   $f$  is **total**.

Right square:  $\langle \forall b, b' :: \langle \exists a :: b f a \wedge b' f a \rangle \Rightarrow (b = b') \rangle$   
 $f$  is **univocal**.

Such relations  $f$  are called **functions**.

# Squares with functions

Let  $f$  be a **function**. Then:



This the **shunting** rule

$$f \cdot R \subseteq Q \Leftrightarrow R \subseteq f^\circ \cdot Q \quad (5)$$

which, by taking converses, becomes:

$$R \cdot f^\circ \subseteq Q \Leftrightarrow R \subseteq Q \cdot f \quad (6)$$

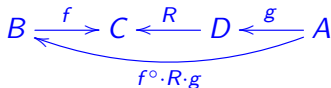
# “Nice” rules about functions

(Functional) **equality**:

$$f \subseteq g \Leftrightarrow f = g \Leftrightarrow g \subseteq f \quad (7)$$

**Existential quantifiers go away:**

$$b (f^\circ \cdot R \cdot g) a \Leftrightarrow (f b) R (g a) \quad (8)$$



Very useful in practice



## Squares with functions

A very common square with 2 **functions**:

$$\begin{array}{ccc}
 A & \xleftarrow{R} & B \\
 f \downarrow & \subseteq & \downarrow g \\
 C & \xleftarrow{S} & D
 \end{array}
 \qquad
 f \cdot R \subseteq S \cdot g
 \qquad
 (9)$$

This square captures a **higher-order relation** on functions:

$$f \ S^R \ g \iff f \cdot R \subseteq S \cdot g \qquad (10)$$

In words:

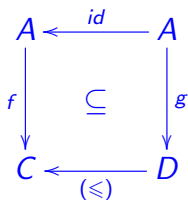
---

*“ $R$ -related inputs are mapped to  $S$ -related outputs”.*

---

# Squares with functions

Let  $R := id$ ,  $S := (\leq)$ :



$$f \subseteq (\leq) \cdot g$$

This square captures the  $(\leq)$ -pointwise-ordering of functions:

$$f (\leq)^{id} g \Leftrightarrow \langle \forall a :: f a \leq g a \rangle$$

In words:

---

*"The same input is mapped to  $(\leq)$ -related outputs".*

---

## “Higher-order” squares

Because of their role in *free theorems*, these squares will be referred to as **Reynolds squares**:

$$\begin{array}{ccc}
 A & \xleftarrow{R} & B \\
 f \downarrow & \subseteq & \downarrow g \\
 C & \xleftarrow{S} & D
 \end{array}
 \quad \text{that is to say,} \quad
 \frac{A \xleftarrow{R} B \quad C \xleftarrow{S} D}{C^A \xleftarrow{S^R} D^B}$$

Thus one is lead to **relational exponentials**  $S^R$  such that e.g.

$$(S^R)^\circ = (S^\circ)^{(R^\circ)} \quad (11)$$

$$id^{id} = id \quad (12)$$

etc. **NB**: We often write  $S \leftarrow R$  or  $R \rightarrow S$  instead of  $S^R$  when exponents get too nested.



# “Higher-order” squares

**Functions-only** Reynolds squares:

$$f (h \rightarrow k) g \Leftrightarrow f \cdot h = k \cdot g \quad (13)$$

In case of  $h^\circ$  instead of  $h$ ,

$$f (h^\circ \rightarrow k) g \Leftrightarrow f \cdot h^\circ \subseteq k \cdot g \quad (14)$$

we get a **higher-order function** (via shunting + equality):

$$(h^\circ \rightarrow k) g = k \cdot g \cdot h \quad (15)$$

Then:

$$(id \rightarrow k) g = k \cdot g \quad (16)$$

$$(h^\circ \rightarrow id) g = g \cdot h \quad (17)$$

cf. **covariant** and **contravariant** exponentials.

## “Higher-order” squares

In fully pointfree notation, the exponentials (16,17) become

$$k^{id} = (k \cdot)$$

$$id^{(h^\circ)} = (\cdot h)$$

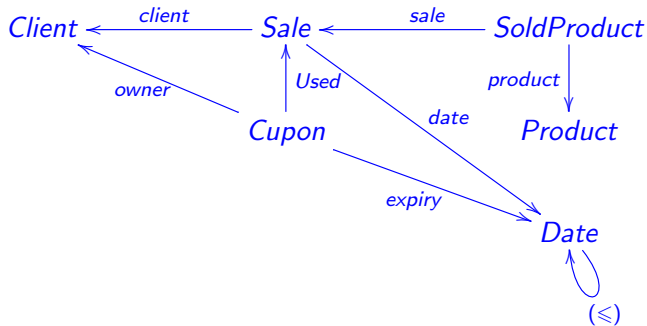
Then, by (11):

$$id^h = (\cdot h)^\circ \tag{18}$$

and so and so forth (very rich construction!).

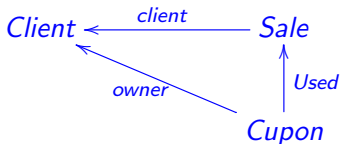
# Prosaic applications

“Chase” the squares:

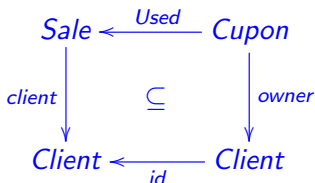


# “Chase” the squares

Pick



and try orienting it:



Indeed:

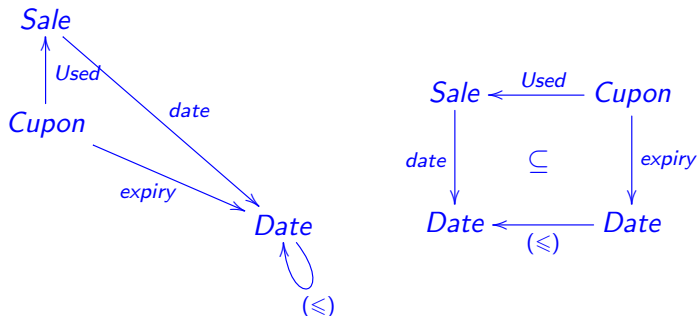
---

*Coupons can only be used by clients who own them.*

---

# “Chase” the squares

Now this one:



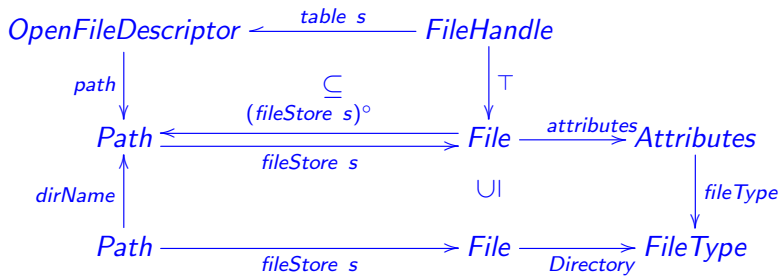

---

*Coupons cannot be used beyond their expiry date.*

---

# Formal modelling

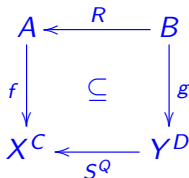
Model invariants as squares, cf. e.g.



(Verified File System (VFS) case study in (Oliveira and Ferreira, 2013).)

# Higher-order Reynolds squares

Exponential relations  $S^R$  can involve other exponentials, for instance  $(S^Q)^R$  i.e.  $R \rightarrow S^Q$ :



$$f (R \rightarrow S^Q) g$$

Let us unfold this, assuming all fresh variables universally quantified:

# Higher-order Reynolds squares

$$f (R \rightarrow S^Q) g \quad (19)$$

$$\Leftrightarrow \quad \{ \text{Reynolds square (9)} \}$$

$$f \cdot R \subseteq S^Q \cdot g$$

$$\Leftrightarrow \quad \{ \text{shunting (5) followed by "nice rule" (8)} \}$$

$$a R b \Rightarrow (f a) S^Q (g b)$$

$$\Leftrightarrow \quad \{ \text{(9) again} \}$$

$$a R b \Rightarrow ((f a) \cdot Q \subseteq S \cdot (g b))$$

$$\Leftrightarrow \quad \{ \text{(5) followed by (8) again} \}$$

$$a R b \Rightarrow c Q d \Rightarrow (f a c) S (g b d) \quad (20)$$



# Relational types

$S^R \cap id$  captures all Reynolds squares (9) in which  $f = g$ :

$$\begin{array}{ccc}
 A & \xleftarrow{R} & A \\
 f \downarrow & \subseteq & \downarrow f \\
 C & \xleftarrow{S} & C
 \end{array}
 \qquad
 f \cdot R \subseteq S \cdot f
 \qquad
 (21)$$

In this case we often abbreviate  $f (R \rightarrow S) f$  to  $f : R \rightarrow S$ , meaning that  $f$  has **relational type**  $R \rightarrow S$ .

---

*Note how type variables  $A$  and  $C$  in  $f : A \rightarrow C$  are straightforwardly replaced by relations  $R$  and  $S$  in  $f : R \rightarrow S$ .*

---

 **types “are” relations** (Voigtländer, 2019).

---

# Category

**Objects** — binary relations  $R, S, \dots$

**Morphisms** —  $R \xrightarrow{f} S$  as above (21)

 This category is named  $Rel_2$  in (Plotkin et al., 2000).

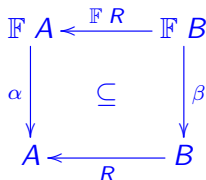
*Relational type*  $R \rightarrow S$  corresponds to the homset  $Rel_2(R, S)$ .

$Rel_2$  is Cartesian closed, meaning that homset  $R \rightarrow Q^S$  is, by uncurrying, isomorphic to  $R \times S \rightarrow Q$ .

**NB: “Tensor” product:**  $(y, x) (R \times S) (b, a) \Leftrightarrow y R b \wedge x S a$ .

# Algebraic squares

Let  $f, g := \alpha, \beta$  in a Reynolds square, where  $\alpha$  and  $\beta$  are  **$\mathbb{F}$ -algebras**:



In a succinct way, the square tells that  $R$  is a **logical relation** from  $\alpha$  to  $\beta$ .

Compare with:

**Definition 2.2.** Given a signature  $\Sigma$  and two models,  $M$  and  $N$ , of the language  $L$  generated by  $\Sigma$ , a (binary) logical relation from  $M$  to  $N$  consists of, for each type  $\sigma$  of  $L$ , a relation  $R_\sigma \subseteq M_\sigma \times N_\sigma$  such that

- for all  $f \in M_{\sigma \rightarrow \tau}$  and  $g \in N_{\sigma \rightarrow \tau}$ , we have  $f R_{\sigma \rightarrow \tau} g$  if and only if for all  $x \in M_\sigma$  and  $y \in N_\sigma$ , if  $x R_\sigma y$  then  $f(x) R_\tau g(y)$ ;
- for all  $(x_0, x_1) \in M_{\sigma \times \tau}$  and  $(y_0, y_1) \in N_{\sigma \times \tau}$ , we have  $(x_0, x_1) R_{\sigma \times \tau} (y_0, y_1)$  if and only if  $x_0 R_\sigma y_0$  and  $x_1 R_\tau y_1$ ;
- $* R_1 *$ ;
- $M(c) R_\sigma N(c)$  for every constant  $c$  in  $\Sigma$  of type  $\sigma$ .

(Plotkin et al. (2000) 'Lax Logical Relations', ICALP 2000: 85-102)

# Algebraic squares

In case  $R$  is a **function**  $h$  ( $R := h$ ),

$$\begin{array}{ccc}
 \mathbb{F} A & \xleftarrow{\mathbb{F} h} & \mathbb{F} B \\
 \alpha \downarrow & = & \downarrow \beta \\
 B & \xleftarrow{h} & A
 \end{array}$$

the square means

$$\alpha \cdot \mathbb{F} h = h \cdot \beta$$

by (7) and  $h$  is said to be a  $\mathbb{F}$ -**homomorphism**.

# Coalgebraic squares

Let  $f, g := \gamma, \phi$  in a Reynolds square, where  $\gamma$  and  $\phi$  are  $\mathbb{F}$ -coalgebras:

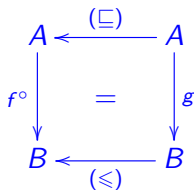
$$\begin{array}{ccc}
 A & \xleftarrow{R} & B \\
 \gamma \downarrow & \subseteq & \downarrow \phi \\
 \mathbb{F} A & \xleftarrow{\mathbb{F} R} & \mathbb{F} B
 \end{array}$$

$R$  is said to be a **bisimulation** between the two coalgebras, meaning:

$$\langle \forall a, b : a R b : (\gamma a) (\mathbb{F} R) (\phi b) \rangle$$

# Very special squares

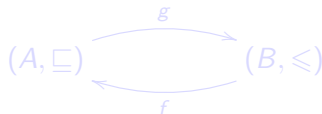
Assume two preorders  $(\sqsubseteq)$  and  $(\leq)$  in:



$$f^\circ \cdot (\sqsubseteq) = (\leq) \cdot g$$

$$f \, b \sqsubseteq a \Leftrightarrow b \leq g \, a \quad (22)$$

In this very special situation,  
 $f$  and  $g$  in

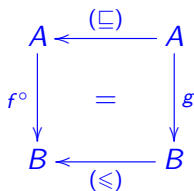


are said to be **Galois connected** (GC) and we write

$$f \vdash g \quad (23)$$

# Very special squares

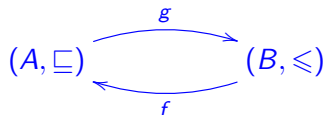
Assume two preorders  $(\sqsubseteq)$  and  $(\leq)$  in:



$$f \circ (\sqsubseteq) = (\leq) \cdot g$$

$$f \ b \sqsubseteq a \Leftrightarrow b \leq g \ a \quad (22)$$

In this very special situation,  
 $f$  and  $g$  in



are said to be **Galois connected** (GC) and we write

$$f \vdash g \quad (23)$$

# Free theorem squares

Let a parametric function  $f : \mathbb{F} X \rightarrow \mathbb{G} X$  be given.

Its **free theorem** states that  $f$  has **relational type**

$$f : \mathbb{F} R \rightarrow \mathbb{G} R \quad (24)$$

for any  $R$  relating its parameters, as shown in the corresponding square:

$$\begin{array}{ccc} \mathbb{F} A & \xleftarrow{\mathbb{F} R} & \mathbb{F} B \\ \downarrow f & \subseteq & \downarrow f \\ \mathbb{G} A & \xleftarrow{\mathbb{G} R} & \mathbb{G} B \end{array}$$

This extends to multi-parametric  $f$ , as shown next.



# Free theorem squares

**Example:** Haskell constant function  $\text{const} : a \rightarrow b \rightarrow a$ , that is  $\text{const} : A \rightarrow A^B$ .

By (24),  $\text{const}$  has relational type  $R \rightarrow R^S$ , that is:

$$\begin{array}{ccc}
 A & \xleftarrow{R} & C \\
 \text{const} \downarrow & \subseteq & \downarrow \text{const} \\
 A^B & \xleftarrow{R^S} & C^D
 \end{array}
 \quad \text{const} \cdot R \subseteq R^S \cdot \text{const} \quad (25)$$

Pointwise equivalent, recall (19,20):

$$a R c \Rightarrow b S d \Rightarrow (\text{const } a \ b) R (\text{const } c \ d)$$

for all  $a, b, c, d$ :

# Free theorem squares

**Example:** Haskell constant function  $\text{const} : a \rightarrow b \rightarrow a$ , that is  $\text{const} : A \rightarrow A^B$ .

By (24),  $\text{const}$  has relational type  $R \rightarrow R^S$ , that is:

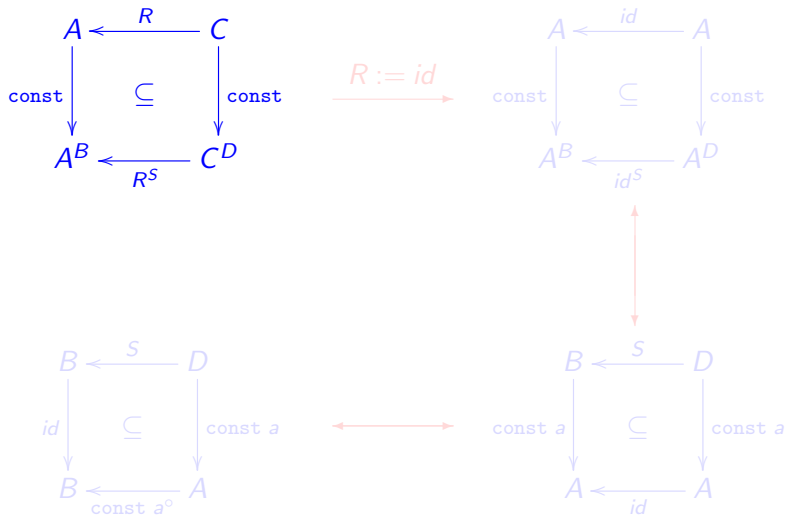
$$\begin{array}{ccc}
 A & \xleftarrow{R} & C \\
 \text{const} \downarrow & \subseteq & \downarrow \text{const} \\
 A^B & \xleftarrow{R^S} & C^D
 \end{array}
 \quad \text{const} \cdot R \subseteq R^S \cdot \text{const} \quad (25)$$

Pointwise equivalent, recall (19,20):

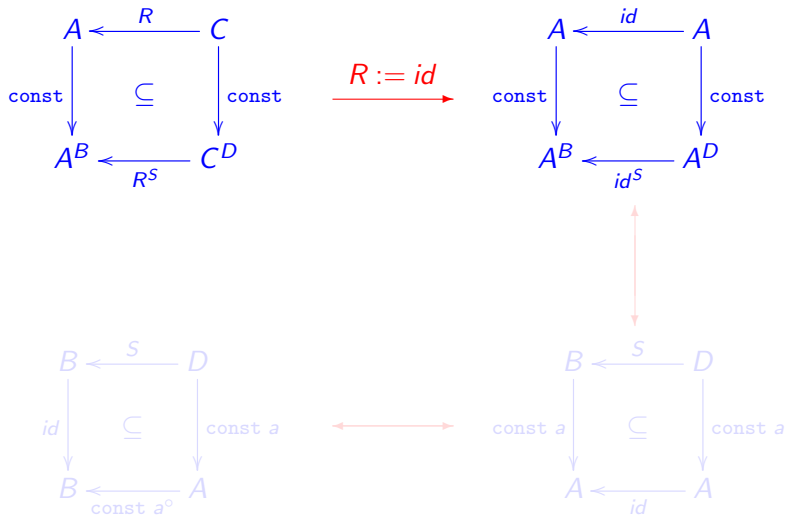
$$a R c \Rightarrow b S d \Rightarrow (\text{const } a \ b) R (\text{const } c \ d)$$

for all  $a, b, c, d$ :

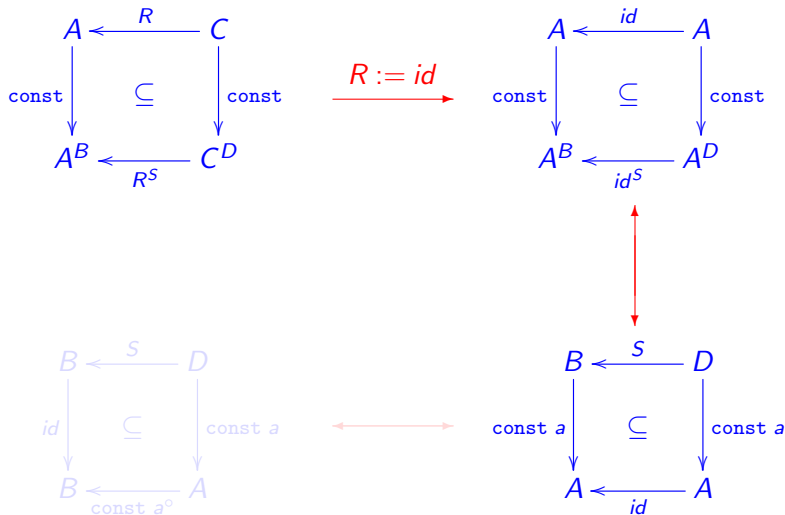
# Free theorem squares



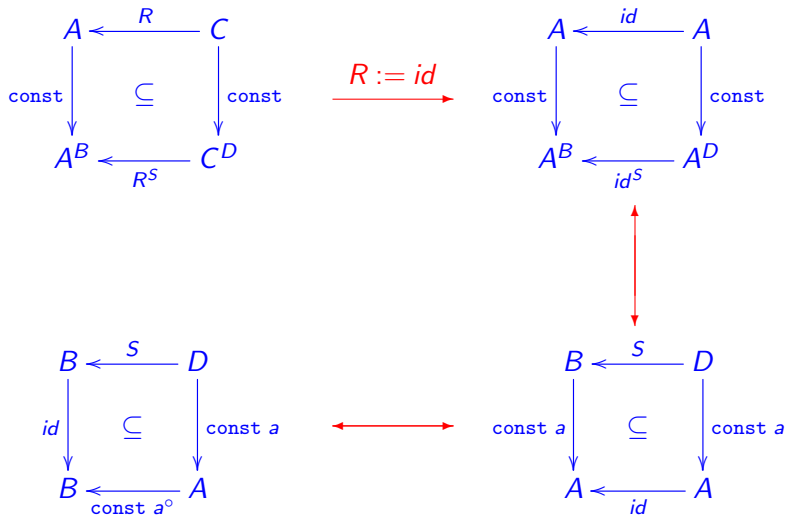
# Free theorem squares



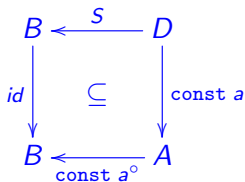
# Free theorem squares



# Free theorem squares



# Free theorem squares



$$S \subseteq (\text{const } a)^\circ \cdot (\text{const } a)$$

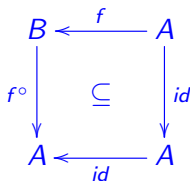
Thus  $(\text{const } a)^\circ \cdot \text{const } a$  is the largest possible  $S$ , i.e. the **top relation**  $\top$ :

$$(\text{const } a)^\circ \cdot (\text{const } a) = \top \quad (26)$$

Thus no other function can be **less injective** than  $\text{const } a$ .

# On Injectivity

Injective functions fit in the following square:



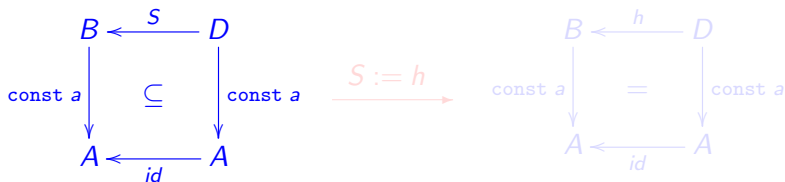
$f^\circ \cdot f$  is the **kernel** of  $f$ .

The kernel  $f^\circ \cdot f$  of a function tells how **injective** a function is.

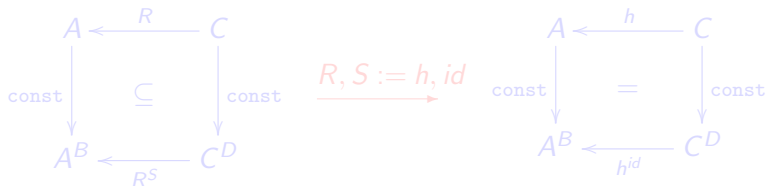
The larger the kernel the **least injective** a function is.



# Free theorem squares

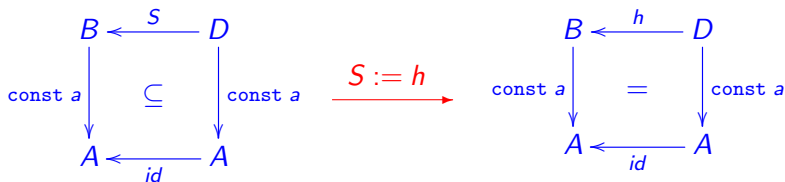


$$\text{const } a \cdot h = \text{const } a$$

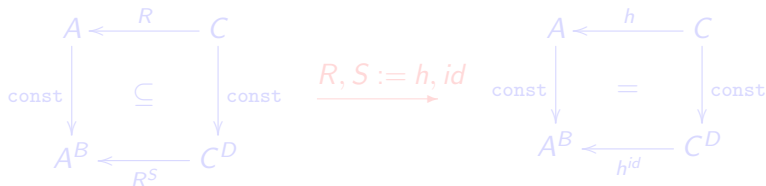


$$h \cdot (\text{const } a) = \text{const } (h a)$$

# Free theorem squares

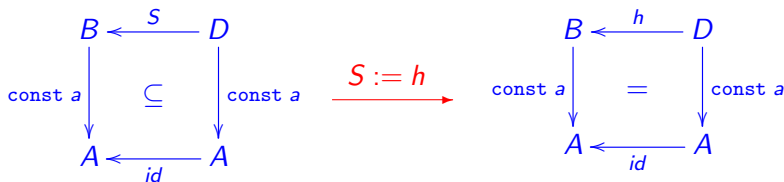


$$\text{const } a \cdot h = \text{const } a$$

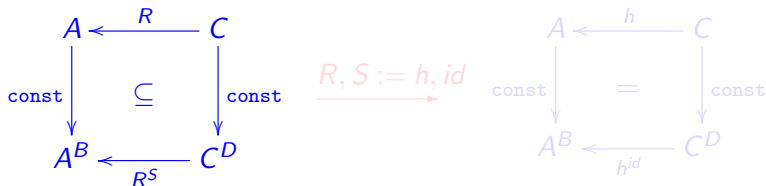


$$h \cdot (\text{const } a) = \text{const } (h a)$$

# Free theorem squares

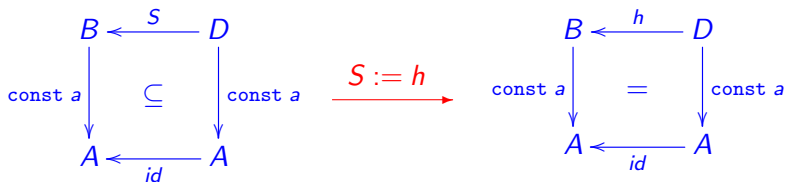


$$\text{const } a \cdot h = \text{const } a$$

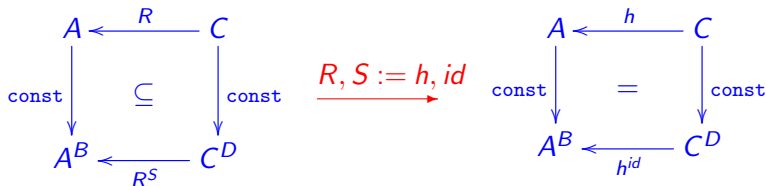


$$h \cdot (\text{const } a) = \text{const } (h a)$$

# Free theorem squares



$$\text{const } a \cdot h = \text{const } a$$



$$h \cdot (\text{const } a) = \text{const } (h a)$$

# Free theorem squares

**Example:**

$$\text{flip} :: (a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c \quad (27)$$

**Free theorem:**  $\text{flip} : Q^{SR} \rightarrow Q^{RS}$ , i.e.

$$g (R \rightarrow Q^S) f \Rightarrow (\text{flip } g) (S \rightarrow Q^R) (\text{flip } f)$$

that is:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xleftarrow{R} & X \\
 g \downarrow & \subseteq & \downarrow f \\
 C^B & \xleftarrow{Q^S} & Z^Y
 \end{array}
 & \Rightarrow &
 \begin{array}{ccc}
 B & \xleftarrow{S} & Y \\
 \tilde{g} \downarrow & \subseteq & \downarrow \tilde{f} \\
 C^A & \xleftarrow{Q^R} & Z^X
 \end{array}
 \end{array} \quad (28)$$

**Notation:**  $\tilde{f}$  abbreviates  $\text{flip } f$  where convenient.

# Free theorem squares

For  $Q := id$ ,  $S := id$  and  $R := r$  (a function):

$$\begin{aligned}
 & f (r \rightarrow id) g \Rightarrow \widetilde{f} (id \rightarrow id^r) \widetilde{g} \\
 \Leftrightarrow & \quad \{ (13) ; (16) \} \\
 & f \cdot r = g \Rightarrow \widetilde{f} \subseteq id^r \cdot \widetilde{g} \\
 \Leftrightarrow & \quad \{ id^r = (\cdot r)^\circ (18) ; \text{substitution of } g; \text{shunting (5)} \} \\
 & (\cdot r) \cdot \widetilde{f} = \widetilde{f \cdot r}
 \end{aligned}$$

This is the **fusion-law** of *flipping* — here obtained more directly than through an **adjunction**, as in e.g. (Oliveira, 2020).

# Free theorem squares

Since **types** are (higher-order) **squares**...

---

... *“how much is in a **type**”* ?

---

Quite a lot.

As we shall see by handling the types of the following functions:

$$\begin{aligned} \text{foldl} &:: \text{Foldable } t \Rightarrow (b \rightarrow a \rightarrow b) \rightarrow b \rightarrow t \, a \rightarrow b \\ \text{foldr} &:: \text{Foldable } t \Rightarrow (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow t \, a \rightarrow b \end{aligned} \quad (29)$$

(Hackage's Data.Foldable)

## foldl and foldr squares

Relational types (for  $\mathbb{T}$  in the **Foldable** class):

$$\mathbf{foldl} : (S \rightarrow S^R) \rightarrow (S \rightarrow S^{\mathbb{T}^R}) \quad (30)$$

$$\mathbf{foldr} : (R \rightarrow S^S) \rightarrow (S \rightarrow S^{\mathbb{T}^R}) \quad (31)$$

As seen above:

- Two squares in each type.
- The left one is a side-condition for the right one to hold.



# foldl squares

The squares of

$$\mathbf{foldl} : (S \rightarrow S^R) \rightarrow (S \rightarrow S^{\mathbb{T} R})$$

are:

$$\begin{array}{ccc}
 \begin{array}{ccc} B & \xleftarrow{S} & Y \\ g \downarrow & \subseteq & \downarrow f \\ B^A & \xleftarrow{S^R} & Y^X \end{array} & \Rightarrow & \begin{array}{ccc} B & \xleftarrow{S} & Y \\ \mathbf{foldl} \, g \downarrow & \subseteq & \downarrow \mathbf{foldl} \, f \\ B^{\mathbb{T} A} & \xleftarrow{S^{\mathbb{T} R}} & Y^{\mathbb{T} X} \end{array}
 \end{array} \quad (32)$$

## foldl squares

For  $R, S := id, h$  (hence  $X = A$ ), both  $S^{\mathbb{T} R}$  and  $S^R$  reduce to  $(h \cdot)$  by  $\mathbb{T} id = id$  and (16).

So the squares become equalities:

$$\begin{array}{ccc}
 \begin{array}{ccc} B & \xleftarrow{h} & Y \\ g \downarrow & & \downarrow f \\ B^A & \xleftarrow{(h \cdot)} & Y^A \end{array} & \Rightarrow & \begin{array}{ccc} B & \xleftarrow{h} & Y \\ \text{foldl } g \downarrow & & \downarrow \text{foldl } f \\ B^{\mathbb{T} A} & \xleftarrow{(h \cdot)} & Y^{\mathbb{T} A} \end{array}
 \end{array}$$

Pointwise:

$$h(f y x) = g(h y) x \quad \Rightarrow \quad h(\text{foldl } f y xs) = \text{foldl } g(h y) xs$$

---

*Fusion law of **foldl** proved in (Bird and Gibbons, 2020) for finite lists.*

---

# foldr

Repeating the above for **foldr** (31):

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xleftarrow{R} & X \\
 g \downarrow & \subseteq & \downarrow f \\
 B^B & \xleftarrow{S^S} & Y^Y
 \end{array}
 & \Rightarrow &
 \begin{array}{ccc}
 B & \xleftarrow{S} & Y \\
 \text{foldr } g \downarrow & \subseteq & \downarrow \text{foldr } f \\
 B^T A & \xleftarrow{S^T R} & Y^T X
 \end{array}
 \end{array}
 \quad (33)$$

Same right square as in (32), but the side-condition square is different:

$$g \cdot R \subseteq S^S \cdot f$$

# foldr squares

For  $R, S := id, h$  we get

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xleftarrow{id} & A \\
 g \downarrow & & \downarrow f \\
 B^B & \xleftarrow{h^h} & Y^Y
 \end{array}
 & \Rightarrow &
 \begin{array}{ccc}
 B & \xleftarrow{h} & Y \\
 \text{foldr } g \downarrow & & \downarrow \text{foldr } f \\
 B^{\mathbb{T}} A & \xleftarrow{(h \cdot)} & Y^{\mathbb{T}} A
 \end{array}
 \end{array}$$

where the side-condition square unfolds to:

$$\begin{aligned}
 & g (id \rightarrow h^h) f \\
 \Leftrightarrow & \quad \{ (11) \} \\
 & (g \ x) \ h^h \ (f \ x) \\
 \Leftrightarrow & \quad \{ (13) \} \\
 & (g \ x) \cdot h = h \cdot (f \ x)
 \end{aligned}$$

# foldr squares

Altogether, one gets:

$$\begin{array}{ccc}
 \begin{array}{ccc} B & \xleftarrow{h} & Y \\ g \times \downarrow & & \downarrow f \times \\ B & \xleftarrow{h} & Y \end{array} & \Rightarrow & \begin{array}{ccc} B & \xleftarrow{h} & Y \\ \text{foldr } g \downarrow & & \downarrow \text{foldr } f \\ B^{\mathbb{T} A} & \xleftarrow{(h \cdot)} & Y^{\mathbb{T} A} \end{array}
 \end{array}$$

that is:

$$(g \times) \cdot h = h \cdot (f \times) \Rightarrow \text{foldr } g \cdot h = (h \cdot) \cdot \text{foldr } f \quad (34)$$

Going fully pointwise,

$$g \times (h y) = h (f \times y) \Rightarrow h (\text{foldr } f \ e \ xs) = \text{foldr } g \ (h e) \ xs \quad (35)$$

---

**foldr**-fusion law proved in (Bird and Gibbons, 2020) for finite lists.

---

# Permutativity squares

Let  $f$  and  $g$  be the same function in (34), say  $s$ , and  $h := s a$

$$\begin{array}{ccc}
 B & \xleftarrow{s a} & B \\
 s x \downarrow & & \downarrow s x \\
 B & \xleftarrow{s a} & B
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 B & \xleftarrow{s a} & B \\
 \text{foldr } s \downarrow & & \downarrow \text{foldr } s \\
 B^{\mathbb{T}} A & \xleftarrow{(s a \cdot)} & B^{\mathbb{T}} A
 \end{array}$$

Then (34) becomes:

$$(s x) \cdot (s a) = (s a) \cdot (s x) \Rightarrow \text{foldr } s \cdot (s a) = (s a \cdot) \cdot \text{foldr } s \quad (36)$$

Property

$$(s x) \cdot (s a) = (s a) \cdot (s x) \quad (37)$$

is called (left) **permutativity** in (Danvy, 2023).



If  $s$  is *associative* and *commutative* then it is *permutative*.

# Is **foldl** equal to **foldr**?

Looking at

**foldl** :: *Foldable*  $t \Rightarrow (b \rightarrow a \rightarrow b) \rightarrow b \rightarrow t \rightarrow a \rightarrow b$

**foldr** :: *Foldable*  $t \Rightarrow (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow t \rightarrow a \rightarrow b$

the *type-wise distance* between **foldr** and **foldl** is the flip (27) of the first parameter.

So the “best fit” one can aim at is

$$\mathbf{foldl} \ f \stackrel{?}{=} \mathbf{foldr} \ \tilde{f} \tag{38}$$

**possibly valid** for a (as wide as possible) class of functions  $f$  and instances of class *Foldable*.

## But... no law relating both

Free theorems only relate pairs of folds, e.g. in (35):

$$g \times (h \ y) = h \ (f \times y) \Rightarrow h \ (\text{foldr } f \ e \ xs) = \text{foldr } g \ (h \ e) \ xs$$

Perhaps a **universal property** could be found?

For this we need to get rid of one of the **foldr**.

One way is to *assume* that, for some  $\alpha$  and  $\gamma$ ,

$$\text{foldr } \alpha \ \gamma = id \tag{39}$$

holds. Then  $(f, e := \alpha, \gamma)$ :

$$g \times (h \ y) = h \ (\alpha \times y) \Rightarrow h \ xs = \text{foldr } g \ (h \ \gamma) \ xs$$



## But... no law relating both

Free theorems only relate pairs of folds, e.g. in (35):

$$g \times (h \ y) = h \ (f \times y) \Rightarrow h \ (\text{foldr } f \ e \ xs) = \text{foldr } g \ (h \ e) \ xs$$

Perhaps a **universal property** could be found?

For this we need to get rid of one of the **foldr**.

One way is to *assume* that, for some  $\alpha$  and  $\gamma$ ,

$$\text{foldr } \alpha \ \gamma = id \tag{39}$$

holds. Then  $(f, e := \alpha, \gamma)$ :

$$g \times (h \ y) = h \ (\alpha \times y) \Rightarrow h \ xs = \text{foldr } g \ (h \ \gamma) \ xs$$

## Towards **foldr**-universal

Let us introduce  $z = h \gamma$  and drop  $xs$ :

$$\begin{cases} h \gamma = z \\ h (\alpha \times y) = g \times (h y) \end{cases} \Rightarrow h = \mathbf{foldr} \ g \ z \quad (40)$$

So,  $\mathbf{foldr} \ g \ z$  is the unique solution for  $h$  to the equations

$$\begin{cases} h \gamma = z \\ h (\alpha \times y) = g \times (h y) \end{cases}$$

Substituting this solution in the equations we get a definition for **foldr**:

$$\begin{cases} \mathbf{foldr} \ g \ z \ \gamma = z \\ \mathbf{foldr} \ g \ z \ (\alpha \times y) = g \times (\mathbf{foldr} \ g \ z \ y) \end{cases} \quad (41)$$

## Towards **foldr**-universal

Moreover, this definition is mathematically equivalent to (just replace  $h$  by **foldr**  $g$   $z$  and simplify):

$$h = \mathbf{foldr} \ g \ z \Rightarrow \begin{cases} h \ \gamma = z \\ h \ (\alpha \times y) = g \times (h \ y) \end{cases} \quad (42)$$

Altogether, (40) and (42) make up a universal property:

$$h = \mathbf{foldr} \ g \ z \Leftrightarrow \begin{cases} h \ \gamma = z \\ h \ (\alpha \times xs) = g \times (h \ xs) \end{cases} \quad (43)$$

(For lists, we can easily identify  $\gamma = []$  and  $\alpha \times xs = x : xs$ .)

What about **foldl**?

# Wikipedia

foldl would build. The extraneous intermediate list structure can be  
 eliminated with the [continuation-passing style](#) technique, `foldr f z`  
`xs == foldl (\k x-> k . f x) id xs z`; similarly, `foldl f`  
`z xs == foldr (\x k-> k . flip f x) id xs z` (`flip` is  
 only needed in languages like Haskell with its flipped order of arguments  
 to the combining function of `foldl` unlike e.g., in Scheme where the  
 same order of arguments is used for combining functions to both  
`foldl` and `foldr`).

[https://en.wikipedia.org/wiki/Fold\\_\(higher-order\\_function\)](https://en.wikipedia.org/wiki/Fold_(higher-order_function))

# Wikipedia

That is,

$$\widetilde{\text{foldl}} f = \text{foldr} (\lambda x k \rightarrow k \cdot \tilde{f} x) id \quad (44)$$

or

$$\begin{aligned} \widetilde{\text{foldl}} f &= \text{foldr} (\theta f) id \\ \text{where } (\theta f) x k &= k \cdot (\tilde{f} x) \end{aligned} \quad (45)$$

cf. the square

$$\begin{array}{ccc} B^B & \xleftarrow{\theta f} & A \\ (\cdot k) \downarrow & & \downarrow \tilde{f} \\ B^B & \xleftarrow{(k \cdot)} & B^B \end{array} \quad (46)$$

# Ok?

Let us unfold (45) via universal property (43):

$$\widetilde{\text{foldl } f} = \text{foldr } (\theta f) \text{ id}$$

$$\Leftrightarrow \{ \text{universal-foldr (43)} \}$$

$$\left\{ \begin{array}{l} \widetilde{\text{foldl } f} \gamma = \text{id} \\ \widetilde{\text{foldl } f} (\alpha \times xs) z = (\theta f) \times (\widetilde{\text{foldl } f} xs) \end{array} \right.$$

$$\Leftrightarrow \{ \text{definition of } \theta \text{ (46)} \}$$

$$\left\{ \begin{array}{l} \widetilde{\text{foldl } f} \gamma = \text{id} \\ \widetilde{\text{foldl } f} (\alpha \times xs) = \widetilde{\text{foldl } f} xs (\tilde{f} \times z) \end{array} \right.$$

$$\Leftrightarrow \{ \text{go pointwise on } z \text{ and unfold the flips} \}$$

$$\left\{ \begin{array}{l} \text{foldl } f z \gamma = z \\ \text{foldl } f z (\alpha \times xs) = \text{foldl } f (f z x) xs \end{array} \right.$$

# Universal-**foldl**

An advantage of defining **foldl** “as a **foldr**” (45) is that the universal property of the latter induces the universal property of the former:

$$\begin{aligned}
 k &= \mathbf{foldl} \ f \\
 \Leftrightarrow & \quad \{ \mathbf{foldl} \ f = \overbrace{\mathbf{foldr} \ (\theta \ f) \ id}^{(45)} ; \text{flipping} \} \\
 \tilde{k} &= \mathbf{foldr} \ \theta \ f \ id \\
 \Leftrightarrow & \quad \{ \text{universal-}\mathbf{foldr} \ (43) \text{ etc} \} \\
 & \quad \left\{ \begin{array}{l} \tilde{k} \ \gamma = id \\ \tilde{k} \ (\alpha \times xs) = (\theta \ f) \times (\tilde{k} \ xs) \end{array} \right.
 \end{aligned}$$

# Universal-**fold**

$$\Leftrightarrow \quad \{ \text{introduce } z \text{ and flip} \}$$

$$\begin{cases} k \ z \ \gamma = z \\ k \ z \ (\alpha \ x \ xs) = (\theta \ f) \ x \ (\tilde{k} \ xs) \ z \end{cases}$$

$$\Leftrightarrow \quad \{ (\theta \ f) \ x \ g = g \cdot (\tilde{f} \ x) \ (46) \}$$

$$\begin{cases} k \ z \ \gamma = z \\ k \ z \ (\alpha \ x \ xs) = \tilde{k} \ xs \ (f \ z \ x) \end{cases}$$

$$\Leftrightarrow \quad \{ \text{flipping} \}$$

$$\begin{cases} k \ z \ \gamma = z \\ k \ z \ (\alpha \ x \ xs) = k \ (f \ z \ x) \ xs \end{cases}$$



# Universal-**foldl**

Thus we get the universal-property of **foldl**:

$$k = \mathbf{foldl} \ f \Leftrightarrow \begin{cases} k \ z \ \gamma = z \\ k \ z \ (\alpha \ x \ xs) = k \ (f \ z \ x) \ xs \end{cases} \quad (47)$$

Ok — now we know something else about **foldl** and **foldr**.

Let us then address the question (38) that motivated this case-study:

---

*Under what conditions does **foldl**  $f = \mathbf{foldr} \ \tilde{f}$  hold?*

---

# Universal-**foldl**

Thus we get the universal-property of **foldl**:

$$k = \mathbf{foldl} \ f \Leftrightarrow \begin{cases} k \ z \ \gamma = z \\ k \ z \ (\alpha \ x \ xs) = k \ (f \ z \ x) \ xs \end{cases} \quad (47)$$

Ok — now we know something else about **foldl** and **foldr**.

Let us then address the question (38) that motivated this case-study:

---

*Under what conditions does  $\mathbf{foldl} \ f = \mathbf{foldr} \ \tilde{f}$  hold?*

---

# Equating **foldl** and **foldr**

The popular assumption is that **foldl**  $f$   $e$  and **foldr**  $\tilde{f}$   $e$  compute the same output for  $f$  **associative** and  $e$  its **unit**, see e.g. exercise 1.10 of (Bird and Gibbons, 2020).

However, we have that, for instance ( $\div$  is **div**),

$$\mathbf{foldl} \ (\div) \ 100000 \ [99, 2, 7] = 72 = \mathbf{foldr} \ (\hat{\div}) \ 100000 \ [99, 2, 7]$$

$$\mathbf{foldl} \ (\div) \ 10000 \ [99, 2, 7] = 7 = \mathbf{foldr} \ (\hat{\div}) \ 10000 \ [99, 2, 7]$$

and yet

- neither  $(\div)$  nor  $(\hat{\div})$  are associative
- the other parameter can be any number.

How do we explain this and similar examples?

# Equating **foldl** and **foldr**

We can use **foldl**-universal (47) to find an answer:

$$\mathbf{foldl} \ f = \mathbf{foldr} \ \tilde{f}$$

$$\Leftrightarrow \quad \{ \text{(47)} \}$$

$$\left\{ \begin{array}{l} \mathbf{foldr} \ \tilde{f} \ z \ \gamma = z \\ \mathbf{foldr} \ \tilde{f} \ z \ (\alpha \ x \ xs) = \mathbf{foldr} \ \tilde{f} \ (f \ z \ x) \ xs \end{array} \right.$$

$$\Leftrightarrow \quad \{ \text{flipping } f \ z \ x \}$$

$$\left\{ \begin{array}{l} \mathbf{foldr} \ \tilde{f} \ z \ \gamma = z \\ \mathbf{foldr} \ \tilde{f} \ z \ (\alpha \ x \ xs) = \mathbf{foldr} \ \tilde{f} \ (\tilde{f} \ x \ z) \ xs \end{array} \right.$$

# Back to the permutativity squares

Recall (36)

$$\begin{array}{ccc}
 B & \xleftarrow{s \times} & B \\
 s \ a \downarrow & & \downarrow s \ a \\
 B & \xleftarrow{s \times} & B
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 B & \xleftarrow{s \times} & B \\
 \text{foldr } s \downarrow & & \downarrow \text{foldr } s \\
 B^{\mathbb{T}} A & \xleftarrow{(s \times \cdot)} & B^{\mathbb{T}} A
 \end{array}$$

which, for  $s := \tilde{f}$ , becomes

$$\begin{array}{ccc}
 B & \xleftarrow{\tilde{f} \times} & B \\
 \tilde{f} \ a \downarrow & & \downarrow \tilde{f} \ a \\
 B & \xleftarrow{\tilde{f} \times} & B
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 B & \xleftarrow{\tilde{f} \times} & B \\
 \text{foldr } \tilde{f} \downarrow & & \downarrow \text{foldr } \tilde{f} \\
 B^{\mathbb{T}} A & \xleftarrow{(\tilde{f} \times \cdot)} & B^{\mathbb{T}} A
 \end{array}$$

This suits us because permuting **foldr**  $\tilde{f}$  with  $\tilde{f} \times$  will be useful.  
Let us see why:

# Equating **foldl** and **foldr**

$$\left\{ \begin{array}{l} \mathbf{foldr} \tilde{f} z \gamma = z \\ \mathbf{foldr} \tilde{f} z (\alpha \times xs) = \mathbf{foldr} \tilde{f} (\tilde{f} \times z) xs \end{array} \right.$$

$$\Leftrightarrow \quad \{ \text{(36) assuming permutativity: } (\tilde{f} \times) \cdot (\tilde{f} a) = (\tilde{f} a) \cdot (\tilde{f} \times) \}$$

$$\left\{ \begin{array}{l} \mathbf{foldr} \tilde{f} z \gamma = z \\ \mathbf{foldr} \tilde{f} z (\alpha \times xs) = \tilde{f} \times (\mathbf{foldr} \tilde{f} z xs) \end{array} \right.$$

$$\Leftrightarrow \quad \{ \text{definition of } \mathbf{foldr} \text{ (41)} \}$$

*True*

# Conclusion

We conclude that **foldl**  $f = \text{foldr } \tilde{f}$  holds for the instances of class *Foldable* such that **foldr**  $\alpha \gamma = id$  for some  $\alpha$  and  $\gamma$  (39), provided that  $\tilde{f}$  is **permutative**.

Back to

$$\text{foldl } (\div) 100000 [99, 2, 7] = 72 = \text{foldr } (\tilde{\div}) 100000 [99, 2, 7]$$

$$\text{foldl } (\div) 10000 [99, 2, 7] = 7 = \text{foldr } (\hat{\div}) 10000 [99, 2, 7]$$

How can we be sure  $(\hat{\div})$  is permutative?

# Equating **foldl** and **foldr**

Recall that the specification of  $x \div y$  is a **Galois connection**:

$$\begin{array}{ccc}
 A & \xleftarrow{(\leq)} & A \\
 (\times y)^\circ \downarrow & = & \downarrow (\div y) \\
 B & \xleftarrow{(\leq)} & B
 \end{array}
 \qquad
 a \times y \leq x \iff a \leq x \div y \qquad (48)$$

We can use (48) and **indirect equality** over  $(\leq)$  to prove

$$(\widetilde{\div} a) \cdot (\widetilde{\div} b) = (\widetilde{\div} b) \cdot (\widetilde{\div} a)$$

that is:

$$(x \div b) \div a = (x \div a) \div b$$



# Never underestimate indirect equality

$$y \leq (x \div b) \div a$$

$$\Leftrightarrow \quad \{ \text{Galois connection (48) twice} \}$$

$$(y \times a) \times b \leq x$$

$$\Leftrightarrow \quad \{ (\times) \text{ is } \mathbf{associative} \text{ and } \mathbf{commutative} \}$$

$$(y \times b) \times a \leq x$$

$$\Leftrightarrow \quad \{ \text{Galois connection (48) twice in the opposite direction} \}$$

$$y \leq (x \div a) \div b$$

$$\therefore \quad \{ \text{by indirect equality (Dijkstra, 2001)} \}$$

$$(x \div b) a = (x \div a) \div b$$

# Summary

Knowing that **permutativity** is enough for foldr/foldl “equality” is not new — see e.g. (Danvy, 2023).

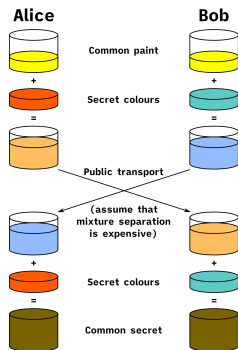
Danvy’s reasoning is, however, quite different: he **postulates** permutativity as side condition and then **proves** it in Coq by **list induction**.

Above, permutativity arose by **free-theorem** calculation.

Moreover, we’ve shown that a commutative + associative **lower adjoint**  $f$  in  $f \dashv g$  ensures permutative  $g$ , widening Olivier Danvy’s result.

# Permutativity

Diffie-Hellman key exchange (Merkle, 1978)<sup>1</sup>:



$$(+red) \cdot (+cyan) = (+cyan) \cdot (+red)$$

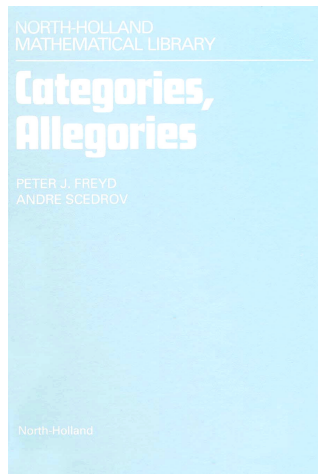
---

<sup>1</sup>Source: Wikipedia

# Summary

Following the advice of Freyd & Šcedrov:

*"This calculus [of relations] offers another, **often more amenable** framework for concepts and methods discussed in part one."*



# Annex

# On relational exponentials $S^R$

By MS vertical composition (2) one immediately infers that, as expected, it is monotone on the base and anti-monotonic on the exponent:

$$\begin{cases} R' \subseteq R \\ S \subseteq S' \end{cases} \Rightarrow S^R \subseteq S'^{R'}$$

We also know that  $id^{id} = id$  (12).

By horizontal composition (3) we get

$$S^R \cdot S'^{R'} \subseteq (S \cdot S')^{(R \cdot R')}$$

However, the converse inclusion does not hold and so relational exponentiation is not in general a (bi)relator — in a sense, relational exponentiation can be regarded as a “lax (bi)relator”.

Backhouse and Backhouse (2004) give conditions for strengthening (49) to an equality that include the cases involving functions and converses of functions used above.

# Data.Foldable

```
instance Foldable M where  
  foldMap = maybe mempty  
foldr _ z Nothing = z  
foldr f z (Just x) = f x z  
foldl _ z Nothing = z  
foldl f z (Just x) = f z x
```

Let  $\alpha x \_ = \text{Just } x$  and  $\gamma = \text{Nothing}$  and unfold **foldr**  $\alpha \gamma$ :

```
foldr  $\alpha$  Nothing Nothing = Nothing  
foldr  $\alpha$  Nothing (Just x) =  $\alpha x z = \text{Just } x$ 
```

So **foldr**  $\alpha \gamma = \text{id}$ .

- K. Backhouse and R.C. Backhouse. Safety of abstract interpretations for free, via logical relations and Galois connections. *SCP*, 15(1–2):153–196, 2004.
- R. Bird and J. Gibbons. *Algorithm Design with Haskell*. Cambridge University Press, 2020.
- O. Danvy. Folding left and right matters: Direct style, accumulators, and continuations. *Journal of Functional Programming*, 33:e2, 2023.
- E.W. Dijkstra. Indirect equality enriched, 2001. Technical note EWD 1315-0.
- R.C. Merkle. Secure communications over insecure channels. *Commun. ACM*, 21(4):294–299, 1978.
- J.N. Oliveira. A note on the under-appreciated for-loop. Technical Report TR-HASLab:01:2020 (PDF), HASLab/U.Minho and INESC TEC, 2020.
- J.N. Oliveira and M.A. Ferreira. Alloy meets the algebra of programming: A case study. *IEEE Trans. Soft. Eng.*, 39(3):305–326, 2013. .



- G. Plotkin, J. Power, D. Sannella, and R. Tennent. Lax logical relations. In Ugo Montanari, José D. P. Rolim, and Emo Welzl, editors, *Automata, Languages and Programming*, pages 85–102, Berlin, Heidelberg, 2000. Springer Berlin Heidelberg.
- J. Voigtländer. Free theorems simply, via dinaturality, 2019. arXiv cs.PL 1908.07776.