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HASLab/INESC TEC & UM & INL

1 Linear relations — 20200920

Linear relations R are such that relational types

$$R \xleftarrow{(+)} R \times R \tag{1}$$

$$R \xleftarrow{s*} R \tag{2}$$

$$R \xleftarrow{0} 1 \tag{3}$$

hold. Note that the converse of a linear relation is also linear since, in general:

$$R \xleftarrow{f} \mathbb{F}\ R \ \equiv\ R^\circ \xleftarrow{f} \mathbb{F}\ R^\circ \tag{4}$$

Checking:

$$\begin{array}{ll}
& f \cdot \mathbb{F}\ R \ \subseteq\ R \cdot f \\
\equiv & \quad \{ \text{ converses } \} \\
& \mathbb{F}\ R^\circ \cdot f^\circ \ \subseteq\ f^\circ \cdot R^\circ \\
\equiv & \quad \{ \text{ shunting } \} \\
& f \cdot \mathbb{F}\ R^\circ \ \subseteq\ R^\circ \cdot f \\
\Box &
\end{array}$$

Axiom 3 Let us abbreviate “copy” by $\delta = \langle id, id \rangle$. Then Axiom 3 is the free theorem of δ:

$$\delta \cdot R \ \subseteq\ (R \times R) \cdot \delta$$

Its color-dual is

$$(+) \cdot (R \times R) \ \subseteq\ R \cdot (+) \tag{5}$$

which is a re-statement of (1). In the opposite direction, take (3), that is $\underline{0} \ \subseteq\ R \cdot \underline{0}$ saying that $(0, 0) \in R$, and color-dualize it:

$$! \cdot R \ \subseteq\ !$$

This is the free theorem of !.

Axiom 4 Now something more elaborate, Axiom 4 (’Wrong Way’) of the paper

$$\langle R, id \rangle \ \subseteq\ \langle id, R^\circ \rangle \cdot R \tag{6}$$

which holds via the Dedekind rule [1],

$$S \cdot R \cap T \ \subseteq\ (S \cap T \cdot R^\circ) \cdot R \tag{7}$$

instantiated to pairing,

$$\pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \ \subseteq\ (\pi_1^\circ \cap \pi_2^\circ \cdot S \cdot R^\circ) \cdot R$$

that is:

$$\langle R, S \rangle \ \subseteq\ \langle id, S \cdot R^\circ \rangle \cdot R \tag{8}$$

We get its dual

$$R \cdot (+) \cdot (id \times R^\circ) \ \subseteq\ (+) \cdot (R \times id) \tag{9}$$

via:

$$\begin{array}{ll}
& \langle R, id \rangle \ \subseteq\ \langle id, R^\circ \rangle \cdot R \\
\equiv & \quad \{ \ \} \\
& (R \times id) \cdot \delta \ \subseteq\ (id \times R^\circ) \cdot \delta \cdot R
\end{array}$$

Let us check this: first think of special case $R := f$, shunt immediately

$$
\begin{array}{ll}
& f \cdot (+) \;\subseteq\; (+) \cdot (f \times f) \\
= & \qquad \{ \; \} \\
& (+) \cdot (f \times f)^{\circ} \;\subseteq\; f^{\circ} \cdot (+) \\
\equiv & \qquad \{ \; \}
\end{array}
$$

Ok - f° linear...

from (5) we get $(+) \cdot (R \times R) \cdot (id \times R^{\circ}) \;\subseteq\; R \cdot (+) \cdot (id \times R^{\circ})$

$$R \cdot (+) \cdot (id \times R^{\circ}) \;\subseteq\; (+) \cdot (R \times id)$$

$\square$

Can be of help...

$$
\begin{array}{l}
\langle R, S \rangle \cdot T = \langle R \cdot T, S \cdot T \rangle \\
\qquad \Leftarrow\; R \cdot (\mathrm{img}\; T) \subseteq R \vee S \cdot (\mathrm{img}\; T) \subseteq S
\end{array}
\tag{10}
$$

References

1. R. Bird and O. de Moor. *Algebra of Programming*. Series in Computer Science. Prentice-Hall, 1997.