Homework 1 - Math 300 - Discrete Mathematics

In order to receive full credit problems and solutions must be clearly written and presented neatly. In writing up your solution, you should state the problem as explaining the solution. Merely giving answers to exercises will is not enough to receive credit. Your write up must show work and convince me that you understand the material. Late homework will not be graded. Starred (★) proof problems are eligible for inclusion in the proof portfolio assignment.

- Choose, complete, and write-up **3 exercises** and **3 proofs**.
- Exercises are worth up to 3 points, and proofs are worth up to 5 points.
- Put your name on each page you submit, and somewhere indicate who you worked with on this assignment.

Exercise 1.1: Find and explain the flaw or flaws in the following proof:
"Claim:" For every positive integer n, $\sum_{i=1}^{n} i = \frac{(n+\frac{1}{2})^2}{2}$.
Base Case: The formula is true for n=1.
Inductive Step: Suppose $\sum_{i=1}^{n} i = \frac{(n+\frac{1}{2})^2}{2}$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By induction hypothesis,

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \frac{(n+\frac{1}{2})^2}{2} + n + 1 \\ &= \frac{(n^2+n+\frac{1}{4})}{2} + n + 1 \\ &= \frac{(n^2+3n+\frac{9}{4})}{2} \\ &= \frac{(n+\frac{3}{2})^2}{2} \\ &= \frac{((n+1)+\frac{1}{2})^2}{2}\end{aligned}$$

completing the induction proof.

Exercise 1.2: Find and explain the flaw or flaws in the following proof:
"Claim:" For every positive integer n, if x and y are positive integers such that $\max(x, y) = n$ then $x = y$.
Base Case: Suppose $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, then $x = 1$ and $y = 1$. Hence $x = y$.
Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ with x and y positive integers, then $x = y$. Now let $\max(x, y) = k + 1$ when x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis $x - 1 = y - 1$. Hence $x = y$ completing the inductive step.

Exercise 1.3: A **bit string** of length k is a string of k binary digits (0 or 1). How many bit strings are there of length 6 or less? How many bit strings of length 6 begin and end with a 1? How many bit strings of length k begin and end with a 1?

Exercise 1.4: How many strings of 2 lowercase letters have the letter x in them? How many strings of 4 lowercase letters have the letter x in them? How many strings of k lowercase letters have the letter x in them?

Exercise 1.5: A small auditorium labels its seats with one of the 26 a-z letters and one of the ten digits 0-9. How many different seat labels are possible? What is the answer if each seat is labeled with one of the letters a-z and one of the number 1-100? What is the answer if each seat is labeled with two letters and two digits?

Exercise 1.6: How many different decimal strings of length 2 (strings of length two where each element of the string is one of the decimal digits 0-9) are there where no digit can be repeated? How many length 3 decimal strings are there like this. How many length 7 decimal strings are there like this? How many length 11 decimal strings are there like this? Give an example of one of the length 11 strings.

Exercise 1.7: Do research on the Pigeonhole Principle. State it, and give two examples of how it can be used to solve an counting problem. Are there other historical names for this counting principle?

Proof 1.1: Use induction to prove that $8^n - 3^n$ is divisible by 5 for all integers $n \geq 1$.

Proof 1.2: Use induction to prove that $n! \geq 2^{n-1}$ for for all integers $n \geq 1$.

Proof 1.3: Prove by induction: $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$

Proof 1.4: Use induction to prove:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Proof 1.5: Use induction to prove:

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

Proof 1.6: Use induction to prove:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Proof 1.7: Use induction to prove:

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

Proof 1.8: Use induction to prove:

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

Proof 1.9: Use induction to prove that the n^{th} derivative of $\ln x$ is

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n+1)!}{x^n}$$

Proof 1.10: Use induction to prove that the n^{th} derivative of xe^{-x} is

$$f^{(n)}(x) = (-1)^{n+1}e^{-x}(x-n)$$

for n a positive integer.

Proof 1.11: ★ Define $x_1 = 1$ and $x_2 = 2$ and $x_{n+2} = x_{n+1} + x_n$ for $n \geq 1$. Prove that $4^n x_n < 7^n$ for all positive integers n.

Proof 1.12: ★ If $n \geq 2$, prove the number of prime factors of n is less than $2\ln n$.