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MATH 300-01

<div align="center">Proof Portfolio</div>

1 Proof By Induction

Show that
$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

Proof. By Induction:

Base Case: $n = 1$

LHS: $\sum_{i=1}^{1} i! \cdot i = 1! \cdot 1 = 1$

RHS: $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

LHS = RHS

Inductive Step:

Assume
$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

Show
$$\sum_{i=1}^{n+1} i! \cdot i = (n+2)! - 1$$

$$\sum_{i=1}^{n+1} i! \cdot i = \sum_{i=1}^{n} i! \cdot i + (n+1)!(n+1)$$
$$= (n+1)! - 1 + (n+1)!(n+1) \text{ By Induction Hyp.}$$
$$= (n+1)! + (n+1)!(n+1) - 1$$
$$= (n+1)!(1 + (n+1)) - 1$$
$$= (n+1)!(n+2) - 1$$
$$= (n+2)! - 1$$

$\square$

I chose this proof as the first induction proof since it is pretty straight forward with how induction works. The challenge in this proof came down to seeing the associativity of the problem, factoring out $(n + 1)!$ correctly. This allowed me to use the definition of factorial to then get $(n + 2)!$.

Prove that if $h > -1$, then $1 + nh \leq (h + 1)^n$ for all non-negative integers n.

Proof. By Induction.
  Base Case:
$$n = 0$$
$$LHS = 1 + 0 \cdot h = 1$$
$$RHS = (h + 1)^0 = 1$$
$$LHS \leq RHS$$

Inductive Step:
Assume $1 + kh \leq (1 + h)^k$, for $k = n$
Show $1 + h(k + 1) \leq (1 + h)^{k+1}$
RHS:
$$(1 + h)^{k+1} = (1 + h)(1 + h)^k$$

By Induction Hypothesis:

$$(1 + h)^{k+1} \geq (1 + h)(1 + nh)$$

$$(1 + h)^{k+1} \geq 1 + nh + h + nh^2$$
$$(1 + h)^{k+1} \geq 1 + h(n + 1) + nh^2$$

If LHS $\leq$ RHS, then RHS $-$ LHS ≥ 0.
Lowest possible RHS $= 1 + h(n + 1) + nh^2$

$$LHS - RHS = 1 + h(n + 1) - 1 + h(n + 1) + nh^2 = nh^2$$

Making nh^2 the lowest value for RHS $-$ LHS.
Since $n \geq 0$ and $h^2 \geq 0$, then RHS $-$ LHS ≥ 0

$$\therefore 1 + h(n + 1) \leq (1 + h)^{k+1}$$

$\square$

The use of induction is not obvious in this one. I chose this one because I find induction with inequalities specifically challenging. I had to really think about what it means for one thing to be greater than or equal to the other.

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2   Direct Proof

Prove:
$$\binom{2n}{n} = \frac{2^n \cdot (2n-1)!!}{n!}$$

Proof. Direct.

First, show that $(2n-1)!! \cdot 2^n \cdot n! = (2n)!$

$$
\begin{aligned}
(2n-1)!! \cdot 2^n \cdot n! &= [(2n-1)(2n-3)...1][2(n)2(n-1)2(n-2)...2(1)] \\
&= [(2n-1)(2n-3)...1][(2n)(2n-2)(2n-4)...2] \\
&= (2n)(2n-1)(2n-2)(2n-3)(2n-4)...1 \\
&= (2n)!
\end{aligned}
$$

Now, apply definition of choose to $\binom{2n}{n}$:

$$
\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\
&= \frac{(2n)!}{n! \cdot n!}
\end{aligned}
$$

By Above Derivation:

$$
\begin{aligned}
\binom{2n}{n} &= \frac{n! \cdot 2^n \cdot (2n-1)!!}{n! \cdot n!} \\
&= \frac{2^n \cdot (2n-1)!!}{n!}
\end{aligned}
$$

$\square$

This proof was particularly interesting because it required a strong understanding of the double factorial and how to use it in this problem. I did this in two parts because I wanted to make the proof more readable and easier to follow. Finding out what to prove in the first part was the particularly tricky part of this proof.

Show that

$$\sum_{r=0}^{n} r \binom{n}{r} = n \cdot 2^{n-1}$$

Proof. By Binomial Theorem Corollary:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

$\binom{n}{r}$ Represents the number of subsets of size r of $\{1, 2, ..., n\}$. The Binomial Theorem Corollary is saying that there are 2^n of these subsets. So, the sum of the sizes of the subsets is

$\sum_{r=1}^{n} \binom{n}{r} \cdot r$, r starts at 1 because when $r = 0$, the size of the subset is also 0.

Each element is in 2^{n-1} subsets, and so adds 2^{n-1} to the total sum, making the total sum:

$$n2^{n-1}$$

$\square$

I chose this proof because it required an understanding of what the equation was trying to count and how that relates to what the binomial coefficients are counting. Figuring these out was difficult, but once it clicked it all made sense.

3 Proof By Contradiction

Show that any simple, connected graph with 31 edges and 12 vertices is not planar.

Proof. Assume to the contrary that the graph is planar.

If we let r be the number of regions in the graph, p be the number of vertices in the graph, and q be the number of edges.

By Euler's Formula:

$$\begin{aligned} r &= 2 - p + q \\ &= 2 - 12 + 31 \\ &= 21 \end{aligned}$$

So there are 21 regions in the graph.

The minimum degree per region in this graph is 3, so the minimum sum of the degrees of all the regions is

$$3r = 3 \cdot 21 = 63$$

By the Second Handshaking Lemma:

$$2q = \text{the sum of the degree of all the regions.}$$

$$2q = 2 \cdot 31$$

$$2q = 62$$

So the sum of the degrees must be at least 63, but is equal to 62. $\Rightarrow\Leftarrow$

$\square$

This proof by contradiction is interesting because it requires knowledge on how planar graphs work, and knowing both Euler's formula and the Second Handshaking Lemma. It is a good example of finding two true facts about a mathematical object that must be true given certain parameters, and seeing how they contradict when assuming to the contrary.

Show that G is a simple graph with $p \geq 2$ vertices and $\deg(v) \geq \frac{p-1}{2}$ for each vertex v in G. Prove G is connected.

Proof. Assume to the contrary that G has 2 distinct connected components C_1 and C_2.

C_1 must have a vertex v, and at least $\frac{p-1}{2}$ other vertices, since they are connected to v.

$$\therefore C_1 \text{ has } 1 + \frac{p-1}{2} = \frac{p+1}{2} \text{ vertices.}$$

A similar argument can be made for C_2 having $\frac{p+1}{2}$ vertices.

Since C_1 and C_2 are distinct components that make up G, the sum of the number of vertices in C_1 and C_2 must equal the number of vertices in G.

$$\therefore p = \frac{p+1}{2} + \frac{p+1}{2}$$

$$p = p + 1 \Rightarrow\Leftarrow$$

$\square$

In this proof, an understanding of the degree of a graph and what it means to not be connected is crucial. When the understanding came that a graph that isn't connected must have distinct connected components, I was able to use that as my contradictory assumption. From there, showing how many vertices each component has required critical thinking, and when they finally added up the statement at the end is so obviously contradictory.