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MATH 300-01

Proof Portfolio

# 1 Proof By Induction

Show that

$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

*Proof.* By Induction:

Base Case: $n = 1$
LHS: $\sum_{i=1}^{1} i! \cdot i = 1! \cdot 1 = 1$
RHS: $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$
LHS = RHS
Inductive Step:
Assume

$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

Show

$$\sum_{i=1}^{n+1} i! \cdot i = (n+2)! - 1$$

$$\begin{aligned}
\sum_{i=1}^{n+1} i! \cdot i &= \sum_{i=1}^{n} i! \cdot i + (n+1)!(n+1) \\
&= (n+1)! - 1 + (n+1)!(n+1) \text{ By Induction Hyp.} \\
&= (n+1)! + (n+1)!(n+1) - 1 \\
&= (n+1)!(1 + (n+1)) - 1 \\
&= (n+1)!(n+2) - 1 \\
&= (n+2)! - 1
\end{aligned}$$

□

I chose this proof as the first induction proof since it is pretty straight forward with how induction works. The challenge in this proof came down to seeing the associativity of the problem, factoring out $(n+1)!$ correctly. This allowed me to use the definition of factorial to then get $(n+2)!$.

Prove that if $h > -1$, then $1 + nh \leq (h+1)^n$ for all non-negative integers $n$.

*Proof.* By Induction.

Base Case:

$$n = 0$$
$$LHS = 1 + 0 \cdot h = 1$$
$$RHS = (h+1)^0 = 1$$
$$LHS \leq RHS$$

Inductive Step:
Assume $1 + kh \leq (1+h)^k$, for $k = n$
Show $1 + h(k+1) \leq (1+h)^{k+1}$
RHS:

$$(1+h)^{k+1} = (1+h)(1+h)^k$$

By Induction Hypothesis:

$$(1+h)^{k+1} \geq (1+h)(1+nh)$$
$$(1+h)^{k+1} \geq 1 + nh + h + nh^2$$
$$(1+h)^{k+1} \geq 1 + h(n+1) + nh^2$$

If LHS $\leq$ RHS, then RHS $-$ LHS $\geq 0$.
Lowest possible RHS $= 1 + h(n+1) + nh^2$

$$LHS - RHS = 1 + h(n+1) - 1 + h(n+1) + nh^2 = nh^2$$

Making $nh^2$ the lowest value for RHS $-$ LHS.
Since $n \geq 0$ and $h^2 \geq 0$, then RHS $-$ LHS $\geq 0$

$$\therefore 1 + h(n+1) \leq (1+h)^{k+1}$$

□

The use of induction is not obvious in this one. I chose this one because I find induction with inequalities specifically challenging. I had to really think about what it means for one thing to be greater than or equal to the other.

## 2 Direct Proof

Prove:

$$\binom{2n}{n} = \frac{2^n \cdot (2n-1)!!}{n!}$$

*Proof.* Direct.

First, show that $(2n-1)!! \cdot 2^n \cdot n! = (2n)!$

$$\begin{aligned}
(2n-1)!! \cdot 2^n \cdot n! &= [(2n-1)(2n-3)...1][2(n)2(n-1)2(n-2)...2(1)] \\
&= [(2n-1)(2n-3)...1][(2n)(2n-2)(2n-4)...2] \\
&= (2n)(2n-1)(2n-2)(2n-3)(2n-4)...1 \\
&= (2n)!
\end{aligned}$$

Now, apply definition of choose to $\binom{2n}{n}$:

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\
&= \frac{(2n)!}{n! \cdot n!}
\end{aligned}$$

By Above Derivation:

$$\begin{aligned}
\binom{2n}{n} &= \frac{n! \cdot 2^n \cdot (2n-1)!!}{n! \cdot n!} \\
&= \frac{2^n \cdot (2n-1)!!}{n!}
\end{aligned}$$

□

Show that
$$\sum_{r=0}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$$

*Proof.* By Binomial Theorem Corollary:
$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

$\binom{n}{r}$ Represents the number of subsets of size $r$ of $\{1, 2, ..., n\}$. The Binomial Theorem Corollary is saying that there are $2^n$ of these subsets. So, the sum of the sizes of the subsets is

$\displaystyle\sum_{r=1}^{n} \binom{n}{r} \cdot r$, $r$ starts at 1 because when $r = 0$, the size of the subset is also 0.

Each element is in $2^{n-1}$ subsets, and so adds $2^{n-1}$ to the total sum, making the total sum:
$$n2^{n-1}$$

$\square$

# 3 Proof By Contradiction

Show that any simple, connected graph with 31 edges and 12 vertices is not planar.

*Proof.* Assume to the contrary that the graph is planar.

If we let $r$ be the number of regions in the graph, $p$ be the number of vertices in the graph, and $q$ be the number of edges.

By Euler's Formula:

$$\begin{aligned} r &= 2 - p + q \\ &= 2 - 12 + 31 \\ &= 21 \end{aligned}$$

So there are 21 regions in the graph.

The minimum degree per region in this graph is 3, so the minimum sum of the degrees of all the regions is

$$3r = 3 \cdot 21 = 63$$

By the Second Handshaking Lemma:

$$2q = \text{the sum of the degree of all the regions.}$$

$$2q = 2 \cdot 31$$

$$2q = 62$$

So the sum of the degrees must be at least 63, but is equal to 62. $\Rightarrow\Leftarrow$

$\square$

Show that $G$ is a simple graph with $p \geq 2$ vertices and $\deg(v) \geq \frac{p-1}{2}$ for each vertex $v$ in $G$. Prove $G$ is connected.

*Proof.* Assume to the contrary that $G$ has 2 distinct connected components $C_1$ and $C_2$.

$C_1$ must have a vertex $v$, and at least $\frac{p-1}{2}$ other vertices, since they are connected to $v$.

$$\therefore C_1 \text{ has } 1 + \frac{p-1}{2} = \frac{p+1}{2} \text{ vertices.}$$

A similar argument can be made for $C_2$ having $\frac{p+1}{2}$ vertices.

Since $C_1$ and $C_2$ are distinct components that make up $G$, the sum of the number of vertices in $C_1$ and $C_2$ must equal the number of vertices in $G$.

$$\therefore p = \frac{p+1}{2} + \frac{p+1}{2}$$

$$p = p + 1 \Rightarrow\Leftarrow$$

$\square$