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MATH 300-01

Proof Portfolio

1 Proof By Induction

Show that

$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

Proof. By Induction:

Base Case: $n = 1$

LHS: $\sum_{i=1}^{1} i! \cdot i = 1! \cdot 1 = 1$

RHS: $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

LHS = RHS

Inductive Step:

Assume

$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1$$

Show

$$\sum_{i=1}^{n+1} i! \cdot i = (n+2)! - 1$$

$$\begin{aligned}
\sum_{i=1}^{n+1} i! \cdot i &= \sum_{i=1}^{n} i! \cdot i + (n+1)!(n+1) \\
&= (n+1)! - 1 + (n+1)!(n+1) \text{ By Induction Hyp.} \\
&= (n+1)! + (n+1)!(n+1) - 1 \\
&= (n+1)!(1 + (n+1)) - 1 \\
&= (n+1)!(n+2) - 1 \\
&= (n+2)! - 1
\end{aligned}$$

□

Prove that if $h > -1$, then $1 + nh \leq (h+1)^n$ for all non-negative integers n.

Proof. By Induction.

Base Case:

$$n = 0$$

$$LHS = 1 + 0 \cdot h = 1$$

$$RHS = (h+1)^0 = 1$$

$$LHS \leq RHS$$

Inductive Step:
Assume $1 + kh \leq (1+h)^k$, for $k = n$
Show $1 + h(k+1) \leq (1+h)^{k+1}$
RHS:

$$(1+h)^{k+1} = (1+h)(1+h)^k$$

By Induction Hypothesis:

$$(1+h)^{k+1} \geq (1+h)(1+nh)$$

$$(1+h)^{k+1} \geq 1 + nh + h + nh^2$$

$$(1+h)^{k+1} \geq 1 + h(n+1) + nh^2$$

If LHS $\leq$ RHS, then RHS $-$ LHS ≥ 0.
Lowest possible RHS $= 1 + h(n+1) + nh^2$

$$LHS - RHS = 1 + h(n+1) - 1 + h(n+1) + nh^2 = nh^2$$

Making nh^2 the lowest value for RHS $-$ LHS.
Since $n \geq 0$ and $h^2 \geq 0$, then RHS $-$ LHS ≥ 0

$$\therefore 1 + h(n+1) \leq (1+h)^{k+1}$$

$\square$

2 Direct Proof

Prove:

$$\binom{2n}{n} = \frac{2^n \cdot (2n-1)!!}{n!}$$

Proof. Direct.

First, show that $(2n-1)!! \cdot 2^n \cdot n! = (2n)!$

$$\begin{aligned}
(2n-1)!! \cdot 2^n \cdot n! &= [(2n-1)(2n-3)...1][2(n)2(n-1)2(n-2)...2(1)] \\
&= [(2n-1)(2n-3)...1][(2n)(2n-2)(2n-4)...2] \\
&= (2n)(2n-1)(2n-2)(2n-3)(2n-4)...1 \\
&= (2n)!
\end{aligned}$$

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\
&= \frac{(2n)!}{n! \cdot n!}
\end{aligned}$$

By Above Derivation:

$$\begin{aligned}
\binom{2n}{n} &= \frac{n! \cdot 2^n \cdot (2n-1)!!}{n! \cdot n!} \\
&= \frac{2^n \cdot (2n-1)!!}{n!}
\end{aligned}$$

□

Show that

$$\sum_{r=0}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$$

Proof. By Binomial Theorem Corollary:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

$\binom{n}{r}$ Represents the number of subsets of size r of $\{1, 2, ..., n\}$. The Binomial Theorem Corollary is saying that there are 2^n of these subsets. So, the sum of the sizes of the subsets is

$\displaystyle\sum_{r=1}^{n} \binom{n}{r} \cdot r$, r starts at 1 because when $r = 0$, the size of the subset is also 0.

Each element is in 2^{n-1} subsets, and so adds 2^{n-1} to the total sum, making the total sum:

$$n2^{n-1}$$

□

3 Proof By Contradiction

Show that any simple, connected graph with 31 edges and 12 vertices is not planar.

Proof. Assume to the contrary that the graph is planar.

If we let r be the number of regions in the graph, p be the number of vertices in the graph, and q be the number of edges.

By Euler's Formula:

$$\begin{aligned} r &= 2 - p + q \\ &= 2 - 12 + 31 \\ &= 21 \end{aligned}$$

So there are 21 regions in the graph.

The minimum degree per region in this graph is 3, so the minimum sum of the degrees of all the regions is

$$3r = 3 \cdot 21 = 63$$

By the Second Handshaking Lemma:

$$2q = \text{the sum of the degree of all the regions.}$$

$$2q = 2 \cdot 31$$

$$2q = 62$$

So the sum of the degrees must be at least 63, but is equal to 62. $\Rightarrow\Leftarrow$

$\square$

Show that G is a simple graph with $p \geq 2$ vertices and $\deg(v) \geq \frac{p-1}{2}$ for each vertex v in G. Prove G is connected.

Proof. Assume to the contrary that G has 2 distinct connected components C_1 and C_2.

C_1 must have a vertex v, and at least $\frac{p-1}{2}$ other vertices, since they are connected to v.

$$\therefore C_1 \text{ has } 1 + \frac{p-1}{2} = \frac{p+1}{2} \text{ vertices.}$$

A similar argument can be made for C_2 having $\frac{p+1}{2}$ vertices.

Since C_1 and C_2 are distinct components that make up G, the sum of the number of vertices in C_1 and C_2 must equal the number of vertices in G.

$$\therefore p = \frac{p+1}{2} + \frac{p+1}{2}$$

$$p = p + 1 \Rightarrow\Leftarrow$$

$\square$