Homework 6 - Math 300 - Discrete Mathematics

In order to receive full credit problems and solutions must be clearly written and presented neatly. In writing up your solution, you should state the problem as explaining the solution. Merely giving answers to exercises will is not enough to receive credit. Your write up must show work and convince me that you understand the material. Late homework will not be graded. Starred (★) proof problems are eligible for inclusion in the proof portfolio assignment.

- Choose, complete, and write-up **3 exercises** and **3 proofs**.
- Exercises are worth up to 3 points, and proofs are worth up to 5 points.
- Put your name on each page you submit, and somewhere indicate who you worked with on this assignment.

Exercise 6.1: From the text, Exercise 2.1 (a) and (d).

Exercise 6.2: Use generating functions to solve:
$a_n = 6a_{n-1} - 8a_{n_2} + 3$; $a_0 = 1$, $a_1 = 0$ $a_n = 3a_{n-1} + 4^{n-1}$; $a_0 = 1$

Exercise 6.3: From the text, Exercise 2.14.

Exercise 6.4: Let $\{a_n\}$ be a sequence of real numbers. The **backwards differences** of this sequence are defined recursively:
The **first difference** $\bigtriangledown a_n$ is an new sequence defined by:$\bigtriangledown a_n = a_n - a_{n-1}$.
The $(k+1)$**st difference** $\bigtriangledown^{k+1} a_n$ is $\bigtriangledown^{k+1} a_n = \bigtriangledown^k a_n - \bigtriangledown^k a_{n-1}$. Find $\bigtriangledown a_n$ and $\bigtriangledown^2 a_n$ for $a_n = 2n$ and $a_n = n^2$.

Exercise 6.5: Solve the recurrence relation $T(n) = nT^2(n/2)$ with initial condition $T(1) = 6$. (Hint: Let $n = 2^k$ and then make the substitution $a_k = \log T(2^k)$.)

Exercise 6.6: Let a_n be the number of ways in which a rectangular box that contains $6n$ square tiles in three rows of length $2n$ can be split into two connected pieces of size $3n$ without cutting any tiles. One of the ways for $n = 3$ is shown. Find a_1, a_2, and a_3.

Taking $a_0 = 1$, find a closed form for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$.

Proof 6.1: From the text, Exercises 2.15.

Proof 6.2: From the text, Exercise 2.17.

Proof 6.3: From the description in the exercises, show $a_{n-1} = a_n - \triangledown a_n$ and $a_{n-2} = a_n - 2 \triangledown a_n + \triangledown^2 a_n$

Proof 6.4: Prove $\triangledown(\triangledown^i a_n) = \triangledown^{i+1} a_n$.

Proof 6.5: ★ Find and prove a formula for a_{n-k} in terms of a_n, $\triangledown a_n$, $\triangledown^2 a_n$, ... , $\triangledown^k a_n$.

Proof 6.6: ★ Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Proof 6.7: Derive Euler's formula for the Catalan numbers, $C_n = \frac{2 \cdot 6 \cdot 10 \cdots (4n-2)}{(n+1)!}$, and note that $(n+1)C_n = (4n-2)C_{n-1}$.

Proof 6.8: ★ Prove the following two statements about the Catalan numbers C_n.

$$C_n \geq 2^{n-1}$$

and

$$C_n \geq \frac{4^{n-1}}{n^2}$$

for all all positive integers $n \geq 1$. Which result is more precise? Prove it.

Proof 6.9: ★ Let T_n be the triangular numbers defined by $T_n = T_{n-1} + n$ with $T_0 = 0$ and n, p, and q be positive integers. Prove $T_n = pq$ if and only if $T_{n+p+q} = T_{n+p} + T_{n+q}$.

Proof 6.10: ★ Let T_n be the triangular numbers defined by $T_n = T_{n-1} + n$ with $T_0 = 0$ and n and k be positive integers. Prove $T_n = 3T_k$ if and only if $T_{2n+3k+2} = 3T_{n+2k+1}$.

Proof 6.11: ★ Let T_n be the triangular numbers defined by $T_n = T_{n-1} + n$ with $T_0 = 0$ and n and k be positive integers. Prove $T_n = 3k^2$ if and only if $T_{5n+12k+2} = 3(2n+5k+1)^2$.

Proof 6.12: ★ Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. Prove

$$\sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3}{2}\pi.$$

Proof 6.13: ★ Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. Prove

$$\sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi.$$