

TAREA II

ECUACIONES DIFERENCIALES SEPARABLES

1. Encuentre la solución general de los siguientes ecuaciones diferenciales

a) $\frac{dy}{dx} + y^{-2} = xy^{-2}$ ED es separable

$$\frac{dy}{dx} = xy^{-2} - y^{-2} \rightarrow \underbrace{y^{-2}(x-1)}_{n(y) g(x)} = \frac{dy}{y^{-2}} = x-1 dx \rightarrow \int y^2 dy = \int x-1 dx$$

$$\rightarrow \frac{y^3}{3} + C_1 = \frac{x^2}{2} - x + C_2 \rightarrow \frac{y^3}{3} = \frac{x^2}{2} - x + C_2 - C_1 \rightarrow y^3 = \left(\frac{x^2}{2} - x + C\right)^3$$

$$\rightarrow y^3 = 3\frac{x^2}{2} - 3x + C \rightarrow y = \sqrt[3]{\frac{3}{2}x^2 - 3x + C} \quad \begin{array}{l} \text{Sol. General} \\ \diagdown \end{array}$$

b) $x^2 \frac{dy}{dx} + y = 0$

$$x^2 \frac{dy}{dx} = -y \rightarrow \frac{dy}{dx} = \frac{-y}{x^2} \quad \begin{array}{l} \text{ED es} \\ \text{separable} \end{array} \rightarrow dy = \frac{-y}{x^2} dx \rightarrow \frac{dy}{y} = \frac{1}{x^2} dx$$

$$\rightarrow \int \frac{dy}{y} = \int \frac{1}{x^2} dx \rightarrow \ln|y| = -\int x^{-2} dx \rightarrow \ln|y| = -\frac{1}{x} + C$$

$$\rightarrow e^{\ln|y|} = e^{\frac{1}{x} + C} \rightarrow y = e^{\frac{1}{x}} \cdot e^C \rightarrow y = e^{\frac{1}{x}} \cdot C \quad \begin{array}{l} \text{Solución general} \\ \text{implícita} \\ \diagdown \end{array}$$

c) $\frac{dy}{dx} = 3x^2 + 3x^2y^2$

$$\frac{dy}{dx} = 3x^2 + 3x^2y^2 \rightarrow \frac{dy}{dx} = \frac{3x^2(1+y^2)}{g(x) n(y)} \rightarrow dy = 3x^2(1+y^2) dx$$

$$\frac{dy}{1+y^2} = 3x^2 dx \rightarrow \int \frac{dy}{1+y^2} = \int 3x^2 dx \rightarrow \tan^{-1}(y) = 3\frac{x^3}{3} + C$$

$$\rightarrow \tan^{-1}(y) = x^3 + C \rightarrow \tan(\tan^{-1}(y)) = \tan(x^3 + C) = y \quad \begin{array}{l} \text{Sol. implícita} \\ \diagdown \end{array}$$

$$\rightarrow y(x) = \tan(x^3 + C) \quad \begin{array}{l} \text{Sol. General} \\ \diagdown \end{array}$$

$$d) \frac{dy}{dx} = xy + x - 3y - 3$$

$$\rightarrow \frac{dy}{dx} = xy + x - 3y - 3 \rightarrow \frac{dy}{dx} = x(y-1) - 3(y+1) \rightarrow \frac{dy}{dx} = (y+1)(x-3)$$

$$\rightarrow \frac{dy}{y+1} = x-3dx \rightarrow \int \frac{dy}{y+1} = \int x-3dx = \underbrace{\ln|1+y|}_{\text{sol. implícita}} = \frac{x^2}{2} - 3x + C$$

$$\rightarrow e^{\ln|1+y|} = e^{\frac{x^2}{2} - 3x + C} \rightarrow 1+y = e^{\frac{x^2}{2}} \cdot e^{-3x} \cdot e^C \rightarrow y = e^{\frac{x^2}{2}} \cdot e^{-3x} \cdot \underbrace{e^C - 1}_C$$

$$\rightarrow y(x) = ce^{\frac{x^2}{2} - 3x - 1}$$

Sol. general

2. Resuelva los siguientes problemas de valor inicial:

$$a) \frac{dy}{dx} = y-1 \text{ sujeto a } y(0)=0$$

Sol. implícita

$$\frac{dy}{dx} = y-1 \rightarrow \frac{dy}{y-1} = dx \rightarrow \int \frac{dy}{y-1} = \int dx = \overbrace{\ln|y-1|} = x + C$$

$$\rightarrow e^{\ln|y-1|} = e^{x+C} \rightarrow y-1 = e^x \cdot e^C \rightarrow y(x) = ce^x + 1$$

Sol. general

Se busca solución que cumpla $y(0)=0$

$$ce^0 + 1 = 0 \rightarrow c = -1$$

comprobación

$$\rightarrow y(x) = -e^x + 1$$

Solución del PVI

$$b) \frac{dy}{dx} = \frac{y-1}{x+3} \text{ sujeto a } y(0)=0$$

Sol. implícita

$$\frac{dy}{dx} = \frac{y-1}{x+3} \rightarrow \frac{dy}{y-1} = \frac{dx}{x+3} \rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3} \rightarrow \ln|y-1| = \ln|x+3| + C$$

$$\rightarrow \ln|y-1| = \ln|x+3| + C \rightarrow \ln|y-1| = C \cdot (x+3) \Rightarrow y-1 = \pm e^C(x+3) \rightarrow y = C(x+3) + 1$$

$$\rightarrow y-1 = C(x+3) \rightarrow 0-1 = C(0+3)$$

Sol. general

$$\rightarrow -1 = C(3)$$

$$\rightarrow C = -\frac{1}{3}$$

PVI →

$$\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$$

• Se busca solución que cumpla $y(0)=0$

$$y(x) = -\frac{1}{3}(x+3)+1 \quad \text{Solución del PVI}$$

$$y(0) = -\frac{1}{3}(0+3)+1 = -1+1=0 \quad \text{comprobación}$$

c) $(1+x)dy - ydx = 0, y(1)=2$

$$\rightarrow \frac{(1+x)dy - ydx}{dx} = \frac{0}{dx} \rightarrow (1+x)\frac{dy}{dx} - y\frac{dx}{dx} = 0$$

$$\rightarrow (1+x)\frac{dy}{dx} - y = 0 \rightarrow (1+x)\frac{dy}{dx} = y \rightarrow \frac{dy}{dx} = \frac{y}{1+x}$$

$$\rightarrow \frac{dy}{y} = \frac{dx}{1+x} \rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x} \rightarrow \underbrace{\ln|y|}_{\text{Sol. Implícita}} = \ln|1+x| + C$$

$$\rightarrow e^{\ln|y|} = e^{\ln|1+x|+C} \rightarrow |y| = |1+x| \cdot e^C \rightarrow y = \pm (1+x)e^C$$

$$y(x) = e^C + e^Cx = y(x) = C + Cx \rightarrow \underbrace{y(x) = C(1+x)}_{\text{sol general}}$$

• Se busca solución que cumpla $y(1)=2$

$$C(1+1) = 2 \quad e=1$$

$$C(2) = 2$$

$$C = \frac{2}{2} = 1$$

$$y(x) = 1(1+x) \quad \text{Sal. del PVI}$$

$$y(1) = 1(1+1) = 1(2) = 2 \quad \text{Comprobación}$$

$$u) \frac{dy}{dx} = 4y - y^2, \text{ sujeto a } y(0) = 0 \quad M - y^2 + 4y - y$$

$$\frac{dy}{dx} = 4y - y^2 \rightarrow \frac{dy}{4y - y^2} = dx \rightarrow \int \frac{dy}{4y - y^2} = \int dx = \int \frac{dy}{4 - (y-2)^2} = x + C$$

$$\int \frac{du}{a^2 - u^2} = x + C = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| = x + C \rightarrow \frac{1}{2(2)} \ln \left| \frac{2+y-2}{2-y+2} \right| = x + C$$

$$u = y-2$$

$$du = dy$$

$$a = 2$$

$$\rightarrow \frac{1}{4} \ln \left| \frac{y}{4-y} \right| = x + C \rightarrow \ln \left| \frac{y}{4-y} \right| = 4(x+C) \rightarrow \ln \left| \frac{y}{4-y} \right| = 4x + 4C$$

$$\rightarrow e^{\ln \left| \frac{y}{4-y} \right|} = e^{4x+4C} \rightarrow \frac{y}{4-y} = e^{4x} \rightarrow 4-y \left(\frac{y}{4-y} \right) = e^{4x}(4-y)$$

$$\rightarrow y(x) = 4ce^{4x} - ce^{4x}y \rightarrow y + ce^{4x}y = 4ce^{4x} \rightarrow y(1+ce^{4x}) = 4ce^{4x}$$

$$\rightarrow y(x) = \frac{4ce^{4x}}{1+ce^{4x}}$$

Sol. general

• Se busca solución que cumpla $y(0) = 0$

$$\frac{4ce^{4(0)}}{1+ce^{4(0)}} = 0 \rightarrow \frac{4c}{1+c} = 0 \rightarrow 4c = 0(1+c) \rightarrow 4c = 0 \rightarrow c = \frac{0}{4} = 0$$

$$\therefore y(x) = \frac{4(0)e^{4x}}{1+0e^{4x}} = \frac{0}{1+0} = 0 \rightarrow y(x) = 0$$

Sol. del PVI

③ Encuentra la solución general de la ecuación diferencial $\frac{dy}{dx} + xy = 2x$

a) $\frac{dy}{dx} + xy - 2x \rightarrow \frac{dy}{dx} = 2x - xy = \frac{dy}{dx} = x(2-y) \rightarrow dy = x(2-y)dx$

$$\rightarrow \frac{dy}{2-y} = xdx \rightarrow \int \frac{dy}{2-y} = \int xdx = \ln|2-y| = \frac{x^2}{2} + C$$

$$\rightarrow e^{\ln|2-y|} = e^{\frac{x^2}{2} + C} \rightarrow |2-y| = e^{\frac{x^2}{2}} \cdot e^C \rightarrow 2-y = \pm e^{\frac{x^2}{2}}$$

$$\rightarrow y(x) = -((e^{\frac{x^2}{2}} - 2)) \rightarrow y(x) = +ce^{\frac{x^2}{2}} + 2 \quad \text{sol. general}$$

b) $y(0) = 4$

• Se busca solución que cumpla $y(0) = 4$

$$\begin{aligned} Ce^{\frac{1}{2}(0)} + 2 &= 4 \\ Ce^0 + 2 &= 4 \\ C(1) + 2 &= 4 \\ C + 2 &= 4 \end{aligned} \quad \left. \begin{aligned} C &= 4-2 \\ C &= 2 \end{aligned} \right\} \rightarrow y(x) = 2e^{\frac{1}{2}x^2} + 2$$

Sol del PVI

④ Encuentre la solución general de los ED's separables, con parámetros

a) $\frac{dy}{dx} + xy = Ax$, donde A es una constante positiva

$$\frac{dy}{dx} + xy = Ax \rightarrow \frac{dy}{dx} = Ax - xy \rightarrow \frac{dy}{dx} = x(A-y) \rightarrow dy = x(A-y)dx$$

$$\rightarrow \frac{dy}{A-y} = xdx \rightarrow \int \frac{dy}{A-y} = \int xdx \rightarrow \ln|A-y| = -\frac{x^2}{2} + C$$

$$\rightarrow e^{\ln|A-y|} = e^{-\frac{x^2}{2} + C} \rightarrow |A-y| = e^{-\frac{x^2}{2}} \rightarrow A-y = \pm e^{-\frac{x^2}{2}}$$

$$\rightarrow y(x) = A + ce^{-\frac{x^2}{2}}$$

b) $m \frac{dv}{dt} = -kv - mg$, donde m, k y g son constantes positivas

$$m \frac{dv}{dt} = -kv - mg \rightarrow \frac{dv}{dt} = \frac{-kv - mg}{m} \rightarrow \frac{dv}{dt} = \frac{-kv}{m} - \frac{mg}{m}$$

$$\rightarrow \frac{dv}{dt} + \frac{k}{m}v = -g \rightarrow \frac{dv}{dt} = \underbrace{-\left(g + \frac{k}{m}v\right)}_{h(v)} \rightarrow \frac{dv}{(g + \frac{k}{m}v)} = -1 dt$$

$$\int \frac{dv}{(g + \frac{k}{m}v)} = \int -1 dt \rightarrow \frac{m}{k} \int \frac{dv}{(g + \frac{k}{m}v)} = \int dt$$

$$v = g + \frac{k}{m}v \quad dv = \frac{1}{k}dt$$

$$dv = \frac{k}{m}dt$$

$$\rightarrow \frac{m}{k} \ln|g + \frac{k}{m}v| = -t + C \rightarrow e^{\ln|g + \frac{k}{m}v| - \frac{t+C}{m}} = e^{\ln|g + \frac{k}{m}v| - \frac{t}{m}}$$

Sol. genial implícita

$$\rightarrow g + \frac{k}{m}v = ce^{-\frac{kt}{m}} \rightarrow g + \frac{k}{m}v = ce^{-\frac{kt}{m}} - g \quad \frac{mc}{n} = c$$

$$\rightarrow v = \frac{ce^{-\frac{kt}{m}} - g}{\frac{m}{n}} \rightarrow v(t) = \frac{m(ce^{-\frac{kt}{m}} - g)}{k} = v(t) = \frac{m}{k}ce^{-\frac{kt}{m}} - \frac{mg}{k} \rightarrow$$

$$\rightarrow v(t) = ce^{-\frac{kt}{m}} - \frac{mg}{k} \quad \begin{array}{l} \text{Sol. general} \\ \text{ED separable} \end{array}$$

$$c) a \frac{dy}{dx} + by = m \rightarrow a \frac{dy}{dx} = m - bx \rightarrow \frac{dy}{dx} = \frac{m - bx}{a} \rightarrow \frac{dy}{dx} = \frac{1}{a} (m - bx) \rightarrow \frac{dy}{m - bx} = \frac{1}{a} dx$$

$$\rightarrow \frac{dy}{by - m} = -\frac{dx}{a} \rightarrow \int \frac{dy}{by - m} = -\int \frac{dx}{a} \quad \begin{cases} u = by - m \\ du = b dy \\ dy = \frac{du}{b} \end{cases} \rightarrow \frac{1}{b} \int \frac{dy}{by - m} = -\frac{1}{a} \int dx$$

$$\rightarrow \frac{1}{b} \ln|by - m| = -\frac{1}{a} x + C \rightarrow \ln|by - m| = \frac{1}{a} x + C \rightarrow \ln|by - m| = \frac{b(-\frac{1}{a}x + C)}{1}$$

Sol. implícita

$$\rightarrow \ln|by - m| = \frac{b}{a}x + bc \rightarrow e^{\ln|by - m|} = e^{\frac{b}{a}x + bc} \rightarrow |by - m| = e^{\frac{b}{a}x} \cdot e^bc$$

$$\rightarrow by - m = \pm e^{\frac{b}{a}x} \rightarrow by = ce^{\frac{-b}{a}x} + m \rightarrow y(x) = \frac{ce^{\frac{-b}{a}x} + m}{b}$$

$$\rightarrow y(x) = \underbrace{\frac{c}{b}e^{\frac{-b}{a}x} + \frac{m}{b}}_{c} \rightarrow y(x) = ce^{\frac{-b}{a}x} + \frac{m}{b}$$

Sol. general