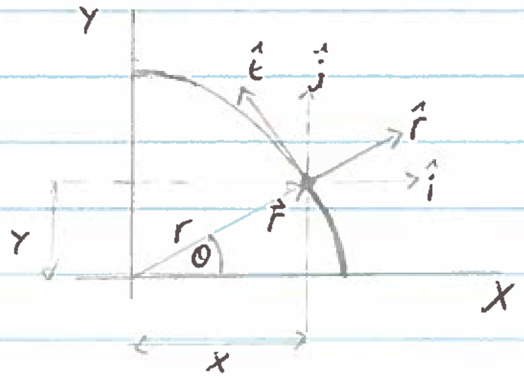
Ch 9 Circular Motion

Polar Coordinates

For an object in circular motion around a fixed point (set as the origin), x- & y- continually change
- but distance from origin (radius of circle), r, is con. while angle relative to x-axis, θ, changes.



~~At~~ Any point around the circle can be described in Cartesian (x, y) or polar (r, θ) coordinates.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Whereas Cartesian unit vectors (i, j) point along the x-axis & y-axis, so polar unit vectors point along the radius of the circle (from the origin) and tangential to the circle (CCW from x-axis).

l: $\hat{r} = (\cos\theta, \sin\theta)$ $\qquad |\hat{r}| = 1$

d: $\hat{t} = (-\sin\theta, \cos\theta)$ $\qquad |\hat{t}| = 1$

Note, $\hat{r} \cdot \hat{r} = \hat{t} \cdot \hat{t} = 1$ & $\hat{r} \cdot \hat{t} = 0$

Angular Coordinates & Angular Displacement

An object at constant r moving in circular motion produces a variation of θ in time ; $\theta(t)$

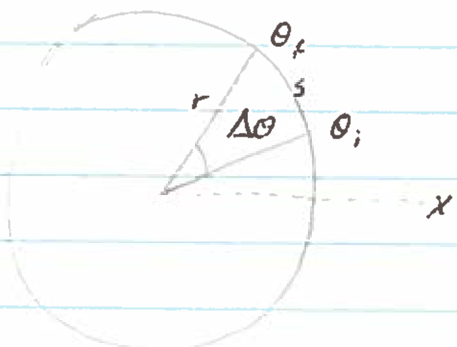
θ is measured CCW from x-axis.

measured in both degrees ($360°$ in a circle)
or radians (2π in a circle).

$\therefore$ 2π rads in $360°$

$$\therefore \quad 1 \text{ rad} = \frac{180°}{\pi} \approx 57.3°$$

– Use as a conversion factor.

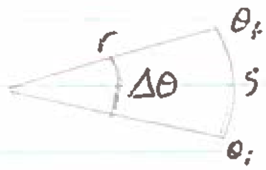


For a particle moving from θ_i to θ_f, it has moved by an angular displacement, $\Delta\theta$

$$\Delta\theta = \theta_f - \theta_i$$

(+ve : CCW motion
-ve : CW motion)

The linear distance travelled is called the arc length, s



$$s = r\Delta\theta$$

Or if $\theta_i = 0$: $s = r\theta$

Note: here, the angle must be converted into radians

Note: if $\Delta\theta = 2\pi$, $s = 2\pi r$ = circumference of circle.

Angular Velocity, ~~Angular~~ Frequency, Period

The ^av^ rate of change of angular position is the angular velocity, ω_{av}:

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous, $\Delta t \to 0$:

$$\omega = \frac{d\theta}{dt} \qquad \text{(measured in rad s}^{-1}\text{)}$$

Frequency, f, determines rate ~~[illegible]~~ at completion of a circular path, measured in Hz ($1\ Hz = 1\ s^{-1}$)
– how many times a circular path is completed:

$$f = \frac{\omega}{2\pi} \longrightarrow \omega = 2\pi f$$

The time taken to complete one rotation is the period, T (in s)

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \longrightarrow \omega = \frac{2\pi}{T}$$

Relating Angular & Linear Velocities

Linear velocity : $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(x\hat{i})}{dt} + \frac{d(y\hat{j})}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

As $x = r\cos\theta(t)$: $\vec{v} = \left(r\frac{d}{dt}\cos(t), \; r\frac{d}{dt}\sin\theta(t)\right)$
$\quad y = r\sin\theta(t)$

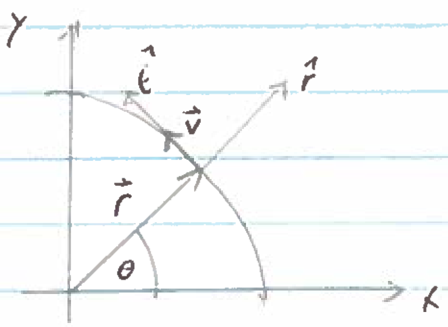
rule: $\frac{df}{d\theta}\frac{d\theta}{dt}$

$$\vec{v} = r\left(-\sin(\theta(t)), \; \cos\theta(t)\right)\frac{d\theta}{dt}$$

$$\vec{v} = r\hat{t}\,\omega$$

$$\vec{v} = r\omega\hat{t}$$

ie ~~Angular~~ Linear velocity has magnitude $|\vec{v}| = r\omega$ directed tangentially to the circular motion



Angular & Radial Acceleration

Average angular acceleration α_{av}:

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous: $\Delta t \to 0$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Linear acceleration: $\vec{a} = \frac{d\,\vec{v}(t)}{dt} = \frac{d\,(r\,\omega(t)\,\hat{t})}{dr}$

$$= r\left(\hat{t}\,\frac{d\omega}{dt} + \omega\,\frac{d\hat{t}}{dt}\right)$$

$\hat{t} = (-\sin\theta, \cos\theta)$

$$= r\alpha\hat{t} + r\omega\left(\frac{d\,(-\sin\theta(t))}{dt}, \frac{d\,(\cos\theta(t))}{dt}\right)$$

$$= r\alpha\hat{t} + r\omega\left(-\cos\theta\,\frac{d\theta}{dt}, -\sin\theta\,\frac{d\theta}{dt}\right)$$

$$= r\alpha\hat{t} - r\omega^2(\cos\theta, \sin\theta)$$

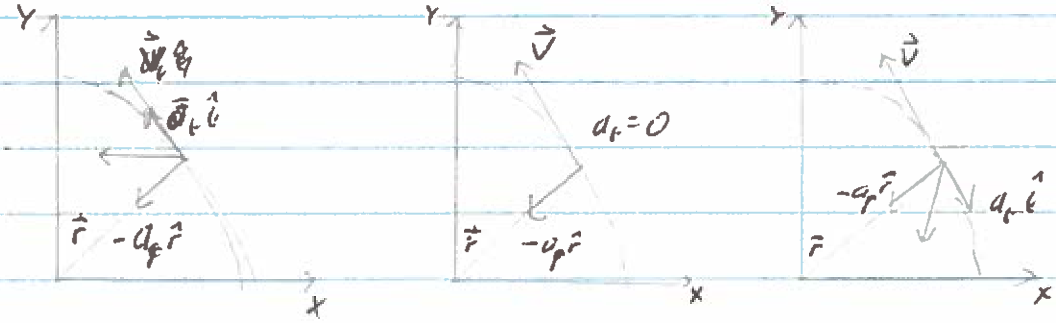
$$\vec{a} = r\alpha\hat{t} - r\omega^2\hat{r}$$

or $\vec{a} = a_t \hat{t} - a_r \hat{r}$

tangential acceleration, $\underline{a_t = r\alpha}$

radial acceleration, $\underline{a_r = \omega^2 r}$

$$a_r = \omega^2 r = \omega v = \frac{v^2}{r} \quad \text{as} \quad v = r\omega$$



$\alpha > 0$ speeding up

$\alpha = 0$ constant $|\vec{v}|$

$\alpha < 0$ slowing down

Magnitude of acceleration for circular motion with a_t, a_r components:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2}$$

$$a = r\sqrt{\alpha^2 + \omega^4}$$