Hamilton-Jacobi Equation for Hamilton's Principal Function

The **Hamilton-Jacobi equation** (HJE) for **Hamilton's Principal Function** is a cornerstone in classical mechanics and provides a bridge to quantum mechanics. Here's an explanation:

1. Context and Background

In classical mechanics, Hamilton's Principal Function, $S(q_1, q_2, \ldots, q_n, t)$, arises in the *principle of least action*. It represents the action along a path connecting initial and final configurations in phase space.

- **Action Principle:** The motion of a system can be derived by minimizing the action
$$\mathcal{S} = \int L \, dt,$$
where L is the **Lagrangian** of the system.
- Hamilton's formulation of mechanics involves the Hamiltonian H, which is related to the Lagrangian and describes the energy of the system.

The Hamilton-Jacobi equation is a reformulation of Hamilton's equations, where solving S gives a direct path to solving the equations of motion.

2. Hamilton's Principal Function

S depends on:

- The generalized coordinates q_i,
- The time t.

It satisfies the condition:

$$\frac{\partial S}{\partial t} = -H\left(q_1, q_2, \ldots, q_n; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}; t\right),$$

where H is the Hamiltonian expressed as a function of generalized coordinates q_i, their conjugate momenta p_i, and time t.

The conjugate momenta are related to S by:

$$p_i = \frac{\partial S}{\partial q_i}.$$

3. The Hamilton-Jacobi Equation

The Hamilton-Jacobi equation can be written as:

$$H\left(q_1, q_2, \dots, q_n; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, t\right) + \frac{\partial S}{\partial t} = 0.$$

This is a **partial differential equation** (PDE) for $S(q,t)$. Solving this equation gives the Hamilton's Principal Function, which encapsulates the dynamics of the system.

4. Interpretation

- The **solution** $S(q,t)$ is equivalent to solving the equations of motion for the system.
- Once S is known, the generalized momenta p_i and trajectories $q_i(t)$ can be determined.
- The HJE reduces the problem of solving a set of coupled second-order differential equations (Newton's laws) to solving a first-order PDE.

5. Applications

- **Classical Mechanics:** Direct determination of motion without explicit integration of Hamilton's equations.
- **Quantum Mechanics:** Forms the basis of Schrödinger's equation under certain conditions.
- **Geometrical Optics:** Related to Fermat's principle in optics.
- **General Relativity:** Plays a role in deriving geodesic equations in curved spacetime.

Hamilton's Principal Function and Its Derivation

1. Hamilton's Principal Function as the Action

Hamilton's Principal Function, S, is defined as the **action** along a path in the configuration space:

$$S(q_1, q_2, \dots, q_n, t) = \int_{t_1}^{t} L(q_i, \dot{q}_i, t)\, dt,$$

where:

- q_i are the generalized coordinates,
- $\dot{q}_i = \frac{dq_i}{dt}$ are their time derivatives,

- L is the Lagrangian of the system.

This integral depends on the path taken between initial and final points.

2. Total Differential of S

Taking the total differential of S:

$$dS = \frac{\partial S}{\partial t} dt + \sum_i \frac{\partial S}{\partial q_i} dq_i.$$

3. Relation Between S and the Lagrangian

From the definition of S, we have:

$$S = \int_{t_1}^{t} L \, dt.$$

Differentiating S with respect to time:

$$\frac{\partial S}{\partial t} = L.$$

4. Relation Between S and Conjugate Momentum

The conjugate momentum p_i is defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Using the principle of least action, the variation of the action S leads to Hamilton's equations. To find the connection between p_i and S, observe the dependence of S on the coordinates q_i. From the total differential of S:

$$dS = \sum_i \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial t} dt.$$

Comparing this with the canonical Hamiltonian formulation, we identify:

$$p_i = \frac{\partial S}{\partial q_i}.$$

5. Hamilton-Jacobi Equation

From Hamilton's equations, the Hamiltonian H is related to S as:

$$H\left(q_i, p_i, t\right) = \sum_i p_i \dot{q}_i - L.$$

Since $p_i = \frac{\partial S}{\partial q_i}$, substituting into the expression for H and using $\frac{\partial S}{\partial t} = -H$, we get:

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0.$$

This is the Hamilton-Jacobi equation, with S encapsulating the dynamics.

Summary of Key Relationships

1. **Hamilton's Principal Function:**
$$S = \int L\,dt.$$

2. **Conjugate momentum:**
$$p_i = \frac{\partial S}{\partial q_i}.$$

3. **Hamilton-Jacobi equation:**
$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0.$$

Why is Hamilton's Principal Function Independent of $\dot{q}_i$ When L Depends on $\dot{q}_i$?

Hamilton's Principal Function $S(q_i, t)$ is independent of $\dot{q}_i$ (the generalized velocities), even though the Lagrangian $L(q_i, \dot{q}_i, t)$ depends on $\dot{q}_i$. Here's the reasoning:

1. Definition of S

Hamilton's Principal Function is defined as:

$$S(q_i, t) = \int_{t_1}^{t} L(q_i, \dot{q}_i, t)\,dt.$$

This integral is evaluated along a specific trajectory in the configuration space that satisfies the equations of motion (derived from the principle of least action). Thus, S is not a general function of q_i, $\dot{q}_i$, and t, but instead depends only on the final generalized coordinates q_i and time t, after the trajectory is determined.

2. Trajectory Dependence on $\dot{q}_i$

The Lagrangian $L(q_i, \dot{q}_i, t)$ explicitly depends on $\dot{q}_i$. However:

- The trajectory of the system is uniquely determined by the equations of motion (Euler-Lagrange equations) and the boundary conditions.
- Along this trajectory, $\dot{q}_i$ is no longer an independent variable; it is a function of q_i, t, and possibly initial conditions.

Thus, when integrating L along the trajectory, the dependence on $\dot{q}_i$ is "absorbed" into the dependence of S on q_i and t.

3. Reduction to Canonical Coordinates

Hamilton's Principal Function S is constructed in such a way that it serves as a generator of the canonical transformation connecting the generalized coordinates q_i and their conjugate momenta p_i. The conjugate momenta are defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Since $\dot{q}_i$ appears only implicitly in S via $p_i = \frac{\partial S}{\partial q_i}$, S itself does not depend explicitly on $\dot{q}_i$.

4. Role of $\dot{q}_i$ in the Variational Principle

The dependence of L on $\dot{q}_i$ ensures that the equations of motion (Euler-Lagrange equations) can be derived from the action $\mathcal{S}$:

$$\mathcal{S} = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt.$$

Once the equations of motion are solved, the explicit dependence on $\dot{q}_i$ vanishes because the velocities are determined by the trajectory $q_i(t)$.

5. Summary

- **Lagrangian Dependence:** L depends on $\dot{q}_i$ because it describes the kinetic and potential energies of the system in terms of both positions q_i and velocities $\dot{q}_i$.
- **Hamilton's Principal Function Dependence:** $S(q_i, t)$ is the integrated action along a physical trajectory. As such, it depends only on the generalized coordinates q_i and time t, not on the generalized velocities $\dot{q}_i$, which are implicitly accounted for in the trajectory.

Derivation of the Hamilton-Jacobi Equation

We derive the Hamilton-Jacobi Equation (HJE) step by step, explicitly replacing L with $\frac{dS}{dt}$.

1. Hamilton's Principal Function

Hamilton's Principal Function $S(q_i, t)$ is defined as:

$$S(q_i, t) = \int L\, dt,$$

where $L(q_i, \dot{q}_i, t)$ is the Lagrangian. The total time derivative of S is:

$$\frac{dS}{dt} = L(q_i, \dot{q}_i, t).$$

2. Hamiltonian Definition

The Hamiltonian is defined as:

$$H = \sum_i p_i \dot{q}_i - L,$$

where:

- $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the conjugate momentum,
- $\dot{q}_i = \frac{dq_i}{dt}$ is the generalized velocity.

Now, replace L with $\frac{dS}{dt}$ in this definition. The total time derivative of S is:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i.$$

Substituting $\frac{dS}{dt}$ into the Hamiltonian, we get:

$$H = \sum_i p_i \dot{q}_i - \left(\frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i \right).$$

3. Simplification Using $p_i = \frac{\partial S}{\partial q_i}$

Since $p_i = \frac{\partial S}{\partial q_i}$, the term $\sum_i p_i \dot{q}_i$ cancels with $\sum_i \frac{\partial S}{\partial q_i} \dot{q}_i$. This leaves:

$$H = -\frac{\partial S}{\partial t}.$$

4. Substituting into the Hamiltonian

The Hamiltonian H is a function of q_i, p_i, and t. Substituting $p_i = \frac{\partial S}{\partial q_i}$, we express H as:

$$H \left(q_i, \frac{\partial S}{\partial q_i}, t \right).$$

Using the relation $H = -\frac{\partial S}{\partial t}$, we obtain:

$$\frac{\partial S}{\partial t} + H \left(q_i, \frac{\partial S}{\partial q_i}, t \right) = 0.$$

5. Final Form of the Hamilton-Jacobi Equation

The final form of the Hamilton-Jacobi Equation is:

$$\frac{\partial S}{\partial t} + H \left(q_i, \frac{\partial S}{\partial q_i}, t \right) = 0,$$

where:

- $S(q_i, t)$ is Hamilton's Principal Function,
- $p_i = \frac{\partial S}{\partial q_i}$.

6. Summary of Key Steps

1. Start with $S(q_i, t) = \int L\, dt$.
2. Replace $L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i$.
3. Substitute into the Hamiltonian definition:
$$H = \sum_i p_i \dot{q}_i - L.$$
4. Cancel terms using $p_i = \frac{\partial S}{\partial q_i}$.
5. Arrive at the Hamilton-Jacobi Equation:
$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0.$$

Harmonic Oscillator Problem Using the Hamilton-Jacobi Method

1. The Harmonic Oscillator Hamiltonian

The Hamiltonian of a one-dimensional harmonic oscillator is given by:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

where:

- q: Generalized coordinate,
- p: Conjugate momentum,
- m: Mass of the oscillator,
- ω: Angular frequency of the oscillator.

2. Hamilton-Jacobi Equation

The Hamilton-Jacobi equation (HJE) for Hamilton's Principal Function $S(q, t)$ is:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

Substitute H into the equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = 0.$$

3. Separation of Variables

Assume a solution of the form:

$$S(q,t) = W(q) - Et,$$

where $W(q)$ is the time-independent part of S, and E is the energy of the system.

Substituting into the HJE:

$$-\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = 0.$$

Simplify:

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2.$$

Rearranging gives:

$$\left(\frac{\partial W}{\partial q}\right)^2 = 2mE - m^2\omega^2 q^2.$$

4. Solve for $W(q)$

Take the square root of both sides:

$$\frac{\partial W}{\partial q} = \pm\sqrt{2mE - m^2\omega^2 q^2}.$$

Integrate:

$$W(q) = \int \pm\sqrt{2mE - m^2\omega^2 q^2}\, dq.$$

Perform the integration (using standard techniques for a quadratic under the square root):

$$W(q) = \pm\frac{m\omega}{2}\left(q\sqrt{\frac{2E}{m\omega^2} - q^2} + \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m\omega^2}{2E}}q\right)\right).$$

5. Hamilton's Characteristic Function and Motion

The total Hamilton's Principal Function is:

$$S(q,t) = W(q) - Et.$$

From $W(q)$, the momentum is:

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \pm\sqrt{2mE - m^2\omega^2 q^2}.$$

6. Action-Angle Variables

The Hamilton-Jacobi method naturally introduces **action-angle variables** for periodic systems like the harmonic oscillator. The action J is given by:

$$J = \oint p \, dq.$$

Substitute $p = \sqrt{2mE - m^2\omega^2 q^2}$:

$$J = \int_{-q_{\max}}^{q_{\max}} \sqrt{2mE - m^2\omega^2 q^2} \, dq.$$

Perform the integral (this is the area of an ellipse in phase space):

$$J = \frac{E}{\omega}.$$

The angular frequency ω is then directly related to the action.

Summary of Results

1. The energy of the harmonic oscillator:

$$E = \frac{1}{2} m\omega^2 q^2 + \frac{p^2}{2m}.$$

2. The Hamilton-Jacobi equation is solved by:

$$S(q, t) = W(q) - Et,$$

   where $W(q)$ is derived as above.

3. The system can be described in terms of action-angle variables for periodic motion.

Logic Behind the Variable Separable Form in the Hamilton-Jacobi Method

1. The Logic Behind the Separable Form

The chosen separable form for $S(q, t)$:

$$S(q, t) = W(q) - Et,$$

is guided by the following considerations:

a. The Structure of the Hamilton-Jacobi Equation

The Hamilton-Jacobi equation is:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

- H, the Hamiltonian, often depends only on the generalized coordinate q, the conjugate momentum p, and possibly t.
- A natural way to handle the explicit dependence on t is to separate it from the spatial variables.

b. Energy Conservation

For systems with a conserved energy E (time-independent Hamiltonians), $S(q,t)$ can often be expressed as:

$$S(q,t) = W(q) - Et,$$

where:

- $W(q)$ is the time-independent part of S,
- E represents the total energy, acting as a separation constant.

This separation reflects the symmetry of the system: the time evolution is independent of the spatial configuration q.

2. Can Other Variable Separable Forms Be Used?

Yes, other separable forms can be used, but the specific form depends on:

- The structure of the Hamiltonian,
- The nature of the system's constraints and symmetries.

a. Time-Independent Systems

If H is time-independent, the form $S(q,t) = W(q) - Et$ is natural because t separates cleanly as the Hamiltonian itself defines E. Any other form would likely complicate the PDE unnecessarily.

b. Time-Dependent Systems

For time-dependent Hamiltonians $H(q,p,t)$, separable forms such as:

$$S(q,t) = W(q) + F(t),$$

might be more appropriate, where $F(t)$ is determined based on the time-dependence of H. This is seen, for example, in systems with external time-dependent forces.

c. Multidimensional Systems

In multidimensional systems with separable coordinates $q_1, q_2, \ldots$, S can be separated into components:

$$S(q_1, q_2, \ldots, t) = \sum_i W_i(q_i) - Et,$$

where $W_i(q_i)$ corresponds to the motion in the i-th coordinate.

d. Arbitrary Forms

For systems with more complex constraints, other separable forms might emerge, but they would generally follow from:

- The symmetries of the system (e.g., spherical, cylindrical),
- The conserved quantities that allow separation (e.g., energy, angular momentum).

3. Why the Standard Form Works for the Harmonic Oscillator

In the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

- H is time-independent,
- Energy E is conserved.

The form $S(q, t) = W(q) - Et$ leverages these properties:

- t is treated independently due to energy conservation,
- q and p are connected through $W(q)$ and the HJE.

4. Why Not Use Arbitrary Separable Forms?

Other separable forms may be theoretically possible, but:

- They might introduce unnecessary complexity in solving the HJE.
- The standard form $S(q, t) = W(q) - Et$ is chosen because it aligns directly with conserved quantities (like E) and simplifies the PDE into an ODE for $W(q)$.

Summary

- The standard form $S(q,t) = W(q) - Et$ is motivated by the time-independence of the Hamiltonian and energy conservation.
- Other separable forms are possible but are typically tailored to the system's symmetries, constraints, or time dependence.
- The choice of form is guided by the goal of simplifying the Hamilton-Jacobi equation while reflecting the physical properties of the system.

Hamilton's Characteristic Function and Principal Function

Hamilton's Characteristic Function and Principal Function are closely related but have distinct purposes. Here's a detailed explanation:

1. Hamilton's Principal Function ($S(q_i, t)$)

Hamilton's Principal Function $S(q_i, t)$ is defined as:

$$S(q_i, t) = \int L \, dt,$$

where L is the **Lagrangian** of the system.

Key Characteristics:

- $S(q_i, t)$ depends explicitly on the generalized coordinates q_i and the time t.
- It solves the **time-dependent Hamilton-Jacobi Equation (HJE)**:

  $$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0,$$

  where H is the Hamiltonian.
- S describes the full dynamics of the system, including time evolution.

2. Hamilton's Characteristic Function ($W(q_i)$)

Hamilton's Characteristic Function $W(q_i)$ is the time-independent version of $S(q_i, t)$. It is used for systems where the Hamiltonian is **time-independent**.

Definition:

Hamilton's Characteristic Function $W(q_i)$ solves the **time-independent Hamilton-Jacobi Equation**:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E,$$

where:

- E is the total energy of the system,
- $H(q_i, p_i)$ is the Hamiltonian.

Key Characteristics:

- $W(q_i)$ depends only on the generalized coordinates q_i and constants of motion (like E).
- It does not involve time t explicitly.
- W describes the geometry of trajectories in phase space rather than their evolution in time.

3. Relation Between S and W

Hamilton's Principal Function $S(q_i, t)$ and Hamilton's Characteristic Function $W(q_i)$ are related for systems with time-independent Hamiltonians. In such cases:

$$S(q_i, t) = W(q_i) - Et,$$

where:

- $W(q_i)$ is Hamilton's Characteristic Function,
- E is the total energy,
- t is time.

This shows that S is a combination of $W(q_i)$ and the time-dependent term $-Et$.

4. When to Use Each Function

Function	**When to Use**
Hamilton's Principal Function (S)	For solving problems with time-dependent Hamiltonians ($H =$
Hamilton's Characteristic Function (W)	For time-independent Hamiltonians ($H = H(q_i, p_i$

5. Example: Harmonic Oscillator

Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2.$$

Time-Independent HJE (for W):

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 = E.$$

Solving this gives $W(q)$, the characteristic function:

$$W(q) = \pm \int \sqrt{2mE - m^2\omega^2 q^2}\, dq.$$

Time-Dependent HJE (for S):

Using $S(q,t) = W(q) - Et$, the principal function is:

$$S(q,t) = \pm \int \sqrt{2mE - m^2\omega^2 q^2}\, dq - Et.$$

6. Summary

- **Hamilton's Principal Function (S)**: Time-dependent, describes the full dynamics of the system.
- **Hamilton's Characteristic Function (W)**: Time-independent, focuses on the geometry of trajectories.

They are related via:

$$S(q_i, t) = W(q_i) - Et,$$

for time-independent systems.

Separation of Variables in the Hamilton-Jacobi Equation

The Hamilton-Jacobi Equation (HJE) is:

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0,$$

where:

- $S(q_i, t)$ is Hamilton's Principal Function,

- H is the Hamiltonian.

For **time-independent systems**, the HJE simplifies to:

$$H\left(q_i, \frac{\partial S}{\partial q_i}\right) = E,$$

where E is the total energy of the system.

1. Separation of Variables in the HJE

Separation of variables is a method to solve the HJE by assuming S can be expressed as a sum of functions, each depending on a single variable.

Assumption:

For a system with n generalized coordinates $q_1, q_2, \ldots, q_n$, assume:

$$S(q_1, q_2, \ldots, q_n, t) = W(q_1, q_2, \ldots, q_n) - Et,$$

where:

- $W(q_1, q_2, \ldots, q_n)$ is Hamilton's Characteristic Function,
- E is the total energy.

Further assume W can be written as a sum:

$$W(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} W_i(q_i),$$

where W_i depends only on q_i.

Substitution into the Time-Independent HJE:

Substitute $S = W - Et$ into the HJE:

$$H\left(q_i, \frac{\partial S}{\partial q_i}\right) = E.$$

If H is separable, this becomes:

$$\sum_{i=1}^{n} H_i\left(q_i, \frac{\partial W_i}{\partial q_i}\right) = E,$$

where H_i depends only on q_i and $\frac{\partial W_i}{\partial q_i}$.

Separation:

Each term H_i is equated to a constant α_i, such that:

$$H_i \left(q_i, \frac{\partial W_i}{\partial q_i} \right) = \alpha_i,$$

and:

$$\sum_{i=1}^{n} \alpha_i = E.$$

This transforms the HJE into n simpler, independent equations.

2. Cyclic (Ignorable) Coordinates

A **cyclic coordinate** (or ignorable coordinate) is a coordinate q_j that does not appear explicitly in the Hamiltonian H. This property simplifies the HJE.

Case of a Cyclic Coordinate:

If q_j is cyclic:

$$H = H(q_1, q_2, \ldots, p_j, \ldots),$$

and p_j (the conjugate momentum) is constant:

$$p_j = \frac{\partial S}{\partial q_j} = \text{constant}.$$

Impact on Separability:

For a cyclic coordinate q_j, the corresponding term in $W(q_i)$ is linear in q_j:

$$W_j(q_j) = p_j q_j.$$

This simplifies the separation of variables. The total function W becomes:

$$W(q_1, q_2, \ldots) = p_j q_j + \sum_{i \neq j} W_i(q_i).$$

3. Example: Particle in a Central Potential

Hamiltonian:

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r).$$

Cyclic Coordinate:

- ϕ is cyclic because it does not appear in $V(r)$.
- Thus:

$$p_\phi = \frac{\partial S}{\partial \phi} = \text{constant}.$$

Separation of Variables:

Assume:

$$S(r, \phi, t) = W_r(r) + W_\phi(\phi) - Et.$$

Substitute into the HJE and separate $W_r(r)$ and $W_\phi(\phi)$:

$$\frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r) = E.$$

This leads to:

1. An equation for $W_\phi(\phi) = p_\phi \phi$,
2. A radial equation for $W_r(r)$.

4. Summary of Key Steps

1. **Separability in the HJE:**
   - Assume $S(q_i, t) = W(q_i) - Et$.
   - Further assume $W(q_i) = \sum_{i=1}^{n} W_i(q_i)$.
   - Substitute into the HJE and separate into n independent equations.
2. **Cyclic Coordinates:**
   - A cyclic coordinate q_j does not appear in H, making p_j constant.
   - For a cyclic coordinate, $W_j(q_j) = p_j q_j$, simplifying the solution.
3. **Applications:**
   - This method is widely used for systems with symmetry, such as central force problems, harmonic oscillators, and planetary motion.

Action-Angle Variables

The **action-angle variables** are a set of canonical coordinates used to describe systems with periodic or quasi-periodic motion, such as harmonic oscillators, planetary orbits, or particles in central potentials. These variables simplify the dynamics of such systems.

1. Key Concepts

(a) Action Variable (J_i)

- The **action variable** is a conserved quantity associated with periodic motion.

- It is defined as the integral of the conjugate momentum p_i over one complete cycle of its periodic motion:

$$J_i = \oint p_i \, dq_i,$$

where q_i is the generalized coordinate and p_i is the conjugate momentum.

- J_i is constant for integrable systems.

(b) Angle Variable (θ_i)

- The **angle variable** represents the phase of the motion and changes linearly with time.

- It evolves as:

$$\theta_i(t) = \theta_{i0} + \omega_i t,$$

where $\omega_i = \frac{\partial H}{\partial J_i}$ is the angular frequency, and H is the Hamiltonian.

2. Why Use Action-Angle Variables?

- **Simplification**: For integrable systems, the equations of motion in action-angle variables are simple. The Hamiltonian H depends only on the action variables:

$$H = H(J_1, J_2, \dots).$$

The angle variables evolve linearly:

$$\dot{\theta}_i = \omega_i = \frac{\partial H}{\partial J_i}.$$

- **Periodic Systems**: These variables are ideal for systems where motion is periodic or quasi-periodic, as the angle variable θ_i captures the periodicity.

- **Canonical Transformation**: The transformation from (q_i, p_i) to (J_i, θ_i) is canonical, preserving the structure of Hamilton's equations.

3. How Are They Defined?

(a) Action Variable (J_i)

The action variable J_i is the area enclosed by the trajectory in phase space for the i-th degree of freedom:

$$J_i = \oint p_i \, dq_i.$$

(b) Angle Variable (θ_i)

The angle variable θ_i parameterizes the position within the periodic trajectory. It is defined such that:

$$\theta_i = \frac{\partial W}{\partial J_i},$$

where W is the generating function of the canonical transformation to action-angle coordinates.

4. Example: Harmonic Oscillator

Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

(a) Action Variable (J):

The trajectory in phase space is an ellipse. The action variable is the area enclosed by this ellipse:

$$J = \oint p\,dq.$$

Using energy conservation:

$$p = \sqrt{2mE - m^2\omega^2 q^2},$$

we compute J:

$$J = \int_{-q_{\max}}^{q_{\max}} \sqrt{2mE - m^2\omega^2 q^2}\,dq = \frac{E}{\omega}.$$

(b) Angle Variable (θ):

The angle variable evolves linearly with time:

$$\theta(t) = \omega t + \theta_0,$$

where ω is the angular frequency.

5. Applications

- **Perturbation Theory**: Used to study small deviations from integrable systems.
- **Celestial Mechanics**: Describes planetary orbits in a central force field.
- **Quantum Mechanics**: Quantization of the action variable J_i leads to quantum conditions:

$$J_i = n_i h, \quad n_i \in \mathbf{Z}.$$

6. Summary

- **Action Variables (J_i):** Conserved quantities that are integrals of motion for periodic systems.
- **Angle Variables (θ_i):** Periodic variables that describe the phase of the motion.
- Action-angle variables simplify the study of periodic and quasi-periodic systems by reducing the dynamics to simple linear evolution in the angle variables.