LAGRANGE POINT

Project submitted to Mahatma Gandhi University, Kottayam in partial fulfilment of the requirement for the award of the degree

BACHELORS OF SCIENCE

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CERTIFICATE

This is to certify that the project work entitled "GRAVITATIONAL THREE BODY PROBLEM AND LAGRANGE POINT" bonafide piece of study done by **NANDANA SURESH, ATHULYA ANILKUMAR, NANDANA BHARATHAN,** under the guidance supervision **of DR. JOSE MATHEW,** Assistant Professor, Post Graduate and Research Department of Physics, and also no part has been presented for any other degree. The study was undertaken for a period of six months.

Name of the guide :

Signature of the guide :

Name of the Head of Department :

Signature of the Head of Department :

Name and signature of external expert:

DECLARATION

We, **Nandana Suresh, Athulya Anilkumar, Nandana Bharathan**, do hereby declare that the project titled **"Gravitational three body problem and Lagrange point"** is a bonafide record work done by us under the supervision of **Dr. JOSE MATHEW**, Assistant Professor of Post Graduate and Research Department of Physics, The Cochin College and is submitted to the Mahatma Gandhi University in partial fulfilment of the requirement for the award of the Degree of Bachelor of Physics. We also declare that this project has not been previously formed the basis forward of any other academic qualification, fellowship or other similar title of any other university or board.

PLACE: KOCHI

DATE:

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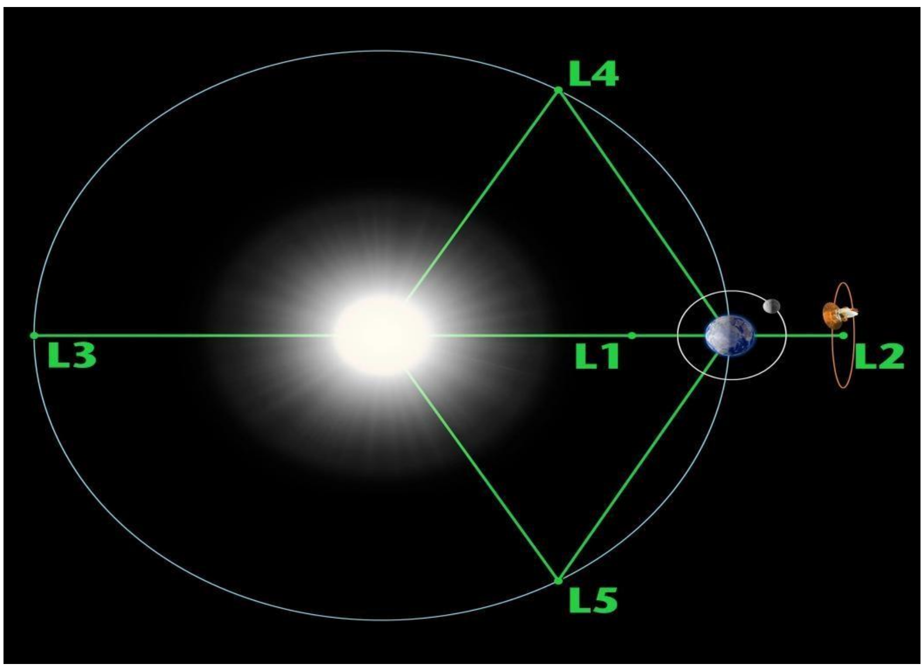
INTRODUCTION

Lagrange points or liberation points are points of equilibrium for a small-mass objects under the influence of gravitational force of two massive orbiting bodies. Normally two massive bodies exert an unbalanced gravitational force at a point, altering the orbit of whatever is at that point, the gravitational forces of the two large bodies are the centrifugal force balance each other, This can make Lagrange points an excellent location for satellites.

There are five special points where a small mass can orbit in a constant pattern with the two larger masses. For Earth-Sun system there are 5 Lagrangian points and they are L1, L2, L3, L4, L5. Of the five Lagrangian points, three are unstable and two are stable. The unstable Lagrangian points are L1, L2, & L3 along the line connecting the two large bodies. The stable Lagrangian points are L4 & L5 forms the apex of the two equilateral triangle that have the larger masses at the vertices. L4 leads the orbit of the Earth and L5 follows. This also similar in the case of Earth-Moon system.

As we already told earlier it is a 3body system and the 3rd body has smaller mass than the two other

bodies this mathematical problem is known as "General Three-Body Problem".
The centripetal forces needed to move the 3rd body at its orbit is equal to the gravitational pull of the two large bodies at the Lagrangian points.



Throughout the 1700s, a large number of mathematicians kept working on the three-body motion problem. As we can see, the analytical solutions available today are limited to restricted three-body problems. Despite perturbation theory as well as numerical methods using computers allows us to determine the forces involving planetary orbits from across all major solar systems with high precision planetary orbits involving the forces from across all major solar system bodies to be determined with high precision .In 1760, Leonhard Euler published a paper that identified the stationary points that are currently referred to as L1, L2, and L3.ItalianFrench mathematician Joseph Louis Lagrange discovered an intriguing anomaly in the solutions to the well-known three-body problem in 1772 while working on it. His initial goal was to devise a simple method for calculating the gravitational interaction between any number of bodies in a system. Since the conclusion drawn by Newtonian mechanics is that in a system like this, the bodies will orbit in a chaotic manner until they collide or one of the bodies is thrown out of the system, allowing equilibrium to be reached.

DESCRIPTION OF INDIVIDUAL

LAGRANGE POINT

L1

The closer an object is to the sun, the faster it will move. So, any spacecraft going around the Sun in an orbit smaller than Earth's will soon overtake our planet. However, there is a loophole: if the spacecraft is placed directly between the Sun and the Earth, Earth's gravity pulls it in the opposite direction and cancel some of the Sun's pill. With a weaker pull towards the Sun, the spacecraft need less speed to maintain its orbit, so it can slow down. If the distance is just right just hundredth of the distance to the Sun- the spacecraft will travel slowly enough to keep its position between the Sun and the Earth. And this is L1 which is good position to from which to monitor the Sun since the constant stream of particles from the Sun, the solar wind, reaches L1 about an hour before reaching the Earth. SOHO, the NASA solar watchdog is positioned here.

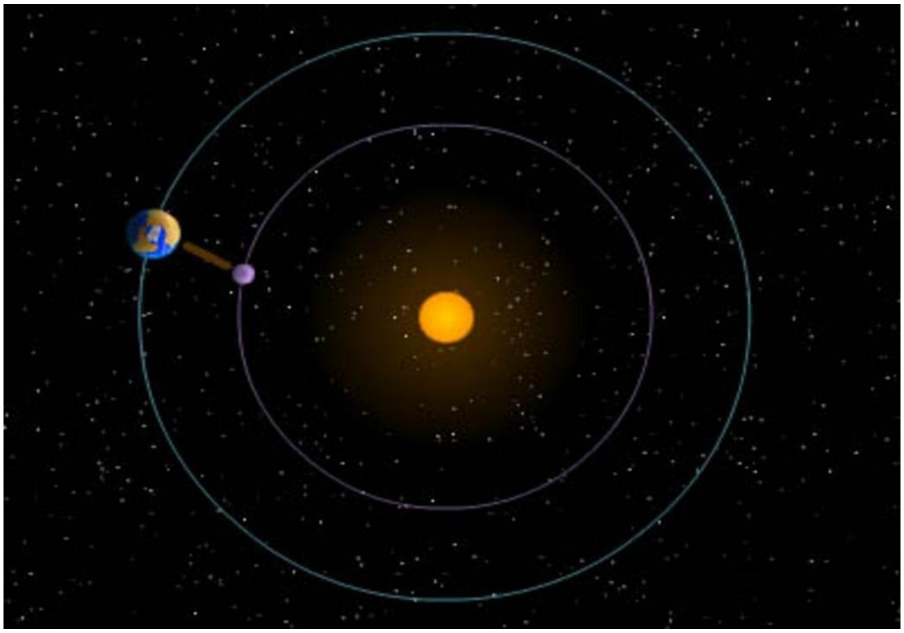


FIG. L1

L2

An effect similar to that which causes L1, also occurs on the 'night' side of Earth beyond Earth's orbit. A spacecraft placed there is more distant from the Sun and therefore should orbit it more slowly than Earth;

but the extra pull of our planet adds to that of the Sun's, and allows the spacecraft to move faster, keeping pace with the Earth. L2 is located 1.5 million kilometres directly behind the Earth as viewed from the Sun. L2 is a great place from which to observe the larger Universe. NASA has a number of missions making use of this region: Herschel, Planck, Gaia and the James Webb Space Telescope.

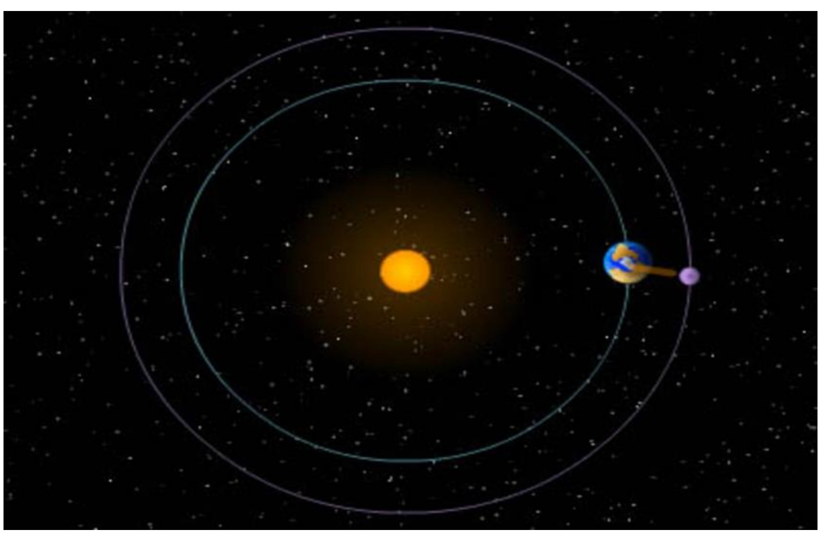


FIG. L2

L3

L3 lies behind the Sun, opposite Earth, just beyond our planets orbit. Objects in L3 cannot be seen from the Earth. It offers the potential to see the far side of the Sun.

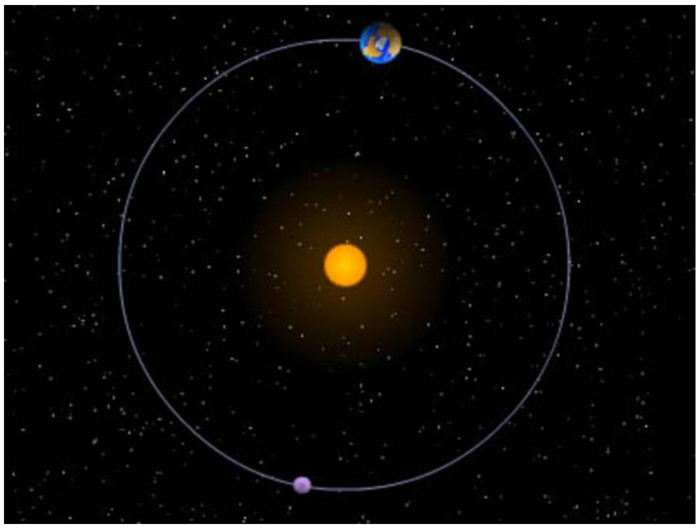


FIG. . L3

L4 & L5

As seen from the Sun, the L4 and L5 points lie at 60 degrees ahead of and behind Earth, close to its orbit. Unlike the other Lagrangian points, L4 & L5 are resistant to gravitational perturbations. Because of this stability, objects such as dust and asteroids tend to accumulate on these regions.

The procedure for finding Lagrange points need to consider two masses M1& M2 and r1 and r2 are their respective positions, then the total force exerted on a third mass m, at a position r, will be

THREE BODY PROBLEM

Today we know that the full three-body problem is chaotic, and so cannot be solved in closed form. Therefore, Lagrange had good reason to make some approximations. Moreover, there are many examples in our solar system that can be accurately described by the restricted three-body problem.

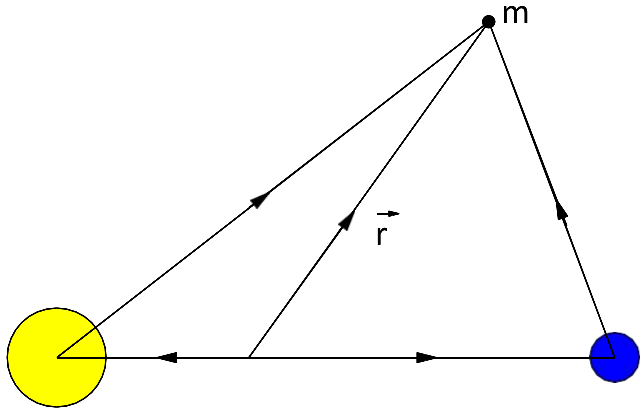


Figure 1: The restricted three-body problem

The procedure for finding the Lagrange points is fairly straightforward: We seek solutions to the equations of motion which maintain a constant separation between the three bodies. If M1 and M2 are the two masses, and $\vec{r}1$ and $\vec{r}2$ are their respective positions, then the total force exerted on a third mass m, at a position $\vec{r}$, will be

$$\vec{F} = -\frac{GM_1 m}{|\vec{r}-\vec{r_1}|^3}(\vec{r}-\vec{r_1}) - \frac{GM_2 m}{|\vec{r}-\vec{r_2}|^3}(\vec{r}-\vec{r_2})\,.$$

The catch is that both $\vec{r}1$ and $\vec{r}2$ are functions of time since M1 and M2 are orbiting each other. Undaunted, one may proceed and insert the orbital solution for $\vec{r}1(t)$ and $\vec{r}2(t)$ (obtained by solving the two-body problem for M1 and M2) and look solutions to the equation of motion

$$\vec{F}(t) = m\frac{d^2\vec{r}(t)}{dt^2}$$

that keep the relative positions of the three bodies fixed. It is these stationary solutions that are known as Lagrange points.

The easiest way to find the stationary solutions is to adopt a co-rotating frame of reference in which the two large masses hold fixed positions. The new frame of reference has its origin at the centre of mass, and an angular frequency given by Kepler's law:

$$\Omega^2 R^3 = G(M_1 + M_2).$$

Here R is the distance between the two masses. The only drawback of using a non-inertial frame of reference is that we have to append various pseudo-forces to the equation of motion. The effective force in a frame rotating with angular velocity $\vec{\Omega}$ is related to the inertial force $\vec{F}$ according to the transformation

$$\vec{F}_\Omega = \vec{F} - 2m\left(\vec{\Omega} \times \frac{d\vec{r}}{dt}\right) - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}).$$

The first correction is the Coriolis force and the second is the centrifugal force. The effective force can be derived from the generalized potential

$$U_\Omega = U - \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2}(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) \, ,$$

as the generalized gradient

$$\vec{F}_\Omega = -\nabla_{\vec{r}} U_\Omega + \frac{d}{dt}\left(\nabla_{\vec{v}} U_\Omega\right) \, .$$

The velocity dependent terms in the effective potential do not in uence the positions of the equilibrium points, but the are crucial in determining the dynamical stability of motion about the equilibrium points. A plot of U with ~v = 0, M1 = 10M2 = 1 and R = 10 is shown in Figure 2. The extreme of the generalized potential are labeled L1 through L5.

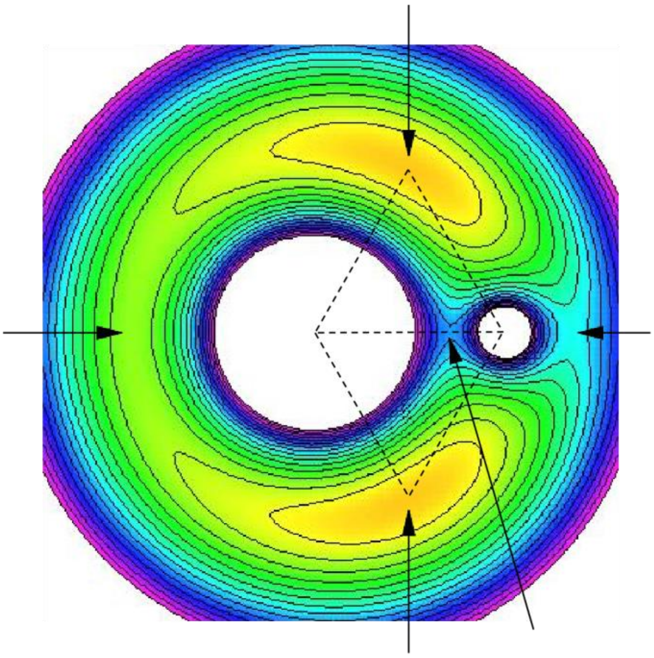


Figure 2: A contour plot of the generalized potential.

Choosing a set of Cartesian coordinates
from the center of mass with the z axis aliened with the angular velocity, we have

$$\vec{\Omega} = \Omega\,\hat{k}$$
$$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$$
$$\vec{r}_1 = -\alpha R\,\hat{\imath}$$
$$\vec{r}_2 = \beta R\,\hat{\imath}$$

Where

$$\alpha = \frac{M_2}{M_1 + M_2}, \quad \beta = \frac{M_1}{M_1 + M_2}.$$

To find the static equilibrium points we set the velocity $\vec{v}$ = d$\vec{r}$/dt to zero and seek solutions to the equation $\vec{F}_\Omega = \vec{0}$, where,

$$\begin{aligned}\vec{F}_\Omega &= \Omega^2\left(x - \frac{\beta(x+\alpha R)R^3}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha(x-\beta R)R^3}{((x-\beta R)^2+y^2)^{3/2}}\right)\hat{\imath}\\ &\quad \Omega^2\left(y - \frac{\beta y R^3}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha y R^3}{((x-\beta R)^2+y^2)^{3/2}}\right)\hat{\jmath}.\end{aligned}$$

Here the mass m has been set equal to unity without loss of generality. The brute-force approach for finding the equilibrium points would be to set the magnitude of each force component to zero, and solve the resulting set of coupled, fourteenth order equations for x and y. A more promising approach is to think about the problem physically, and use the symmetries of the system to guide us to the answer.

Since the system is reflection symmetric about the x axis, the y component of the force must vanish along this line. Setting y = 0 and writing x = R(u +) (so that u measures the distance from M2 in units of R), the condition for the force to vanish along the x-axis reduces to finding solutions to the three fifth-order equations

$u^2((1 \quad {}_1 s\ ) + 3u + 3u^2 + u^3) = \ {}_1 (s \ + 2s\ u + (1+ s$
$s\ )u^2 + 2u^3 + u^4)$; (10)

where s0 = sign(u) and s1 = sign(u + 1). The three cases we need to solve have (s0; s1) equal to (1; 1), (1; 1) and (1; 1). The case (1; cannot occur. In each case there is one real root to the quintic equation, giving us the positions of the rst three Lagrange points.

We are unable to nd closed-form solutions to equation (10) for general values of , so instead we seek approximate solutions valid in the limit 1. To lowest order in , we nd the rst three Lagrange points to be positioned at

$$L1: \quad \left(R\left[1-\left(\frac{\alpha}{3}\right)^{1/3}\right],0\right),$$

$$L2: \quad \left(R\left[1+\left(\frac{\alpha}{3}\right)^{1/3}\right],0\right),$$

$$L3: \quad \left(-R\left[1+\frac{5}{12}\alpha\right],0\right).$$

For the earth-sun system $\alpha \sim 3\times10^{-6}$, R = 1 AU $\sim1.5\times10^{8}$km, and the first and second Lagrange points are located approximately 1.5 million kilometres from the earth. The third Lagrange point home of the mythical planet X - orbits the sun just a fraction further out than the earth. Identifying the remaining two Lagrange points requires a little more thought. We need to balance the centrifugal force, which acts in a direction radially outward from the centre of mass, with the gravitational force exerted by the two masses. Clearly, force balance in the direction perpendicular to centrifugal force will only involve gravitational forces. This suggests that we should resolve the force into directions parallel and perpendicular to ~r. The appropriate projection vectors are $x\hat{\imath}+ y\hat{\imath}$ and $y\hat{\imath}- x\hat{\imath}$. The perpendicular projection yields

$$F_{\Omega}^{\perp} = \alpha\beta y\Omega^2 R^3 \left(\frac{1}{((x-R\beta)^2+y^2)^{3/2}} - \frac{1}{((x+R\alpha)^2+y^2)^{3/2}}\right).$$

Setting $F_{\Omega}= 0$ and $y \neq 0$ tells us that the equilibrium points must be equidistant from the two masses. Using this fact, the parallel projections implies to read other

$$F_{\Omega}^{\parallel} = \Omega^2 \frac{x^2 + y^2}{R} \left(\frac{1}{R^3} - \frac{1}{((x - R\beta)^2 + y^2)^{3/2}} \right) .$$

Demanding that the parallel component of the force vanish leads to the condition that the equilibrium points are at a distance R from each mass. In other words, L4 is situated at the vertex of an equilateral triangle, with the two masses forming the vertices. L5 is obtained by a mirror re ection of L4 about the x-axis. Explicitly, the fourth and Lagrange points have coordinates

$$L4: \quad \left(\frac{R}{2} \left(\frac{M_1 - M_2}{M_1 + M_2} \right), \frac{\sqrt{3}}{2} R \right) ,$$

$$L5: \quad \left(\frac{R}{2} \left(\frac{M_1 - M_2}{M_1 + M_2} \right), -\frac{\sqrt{3}}{2} R \right) .$$

CONCLUSION

In conclusion, the study of gravitational three body systems and Lagrange points provides valuable insights into the complex dynamics of celestial bodies. Through our investigation, we have discovered that Lagrange points represent stable equilibrium points where the gravitational forces of three celestial bodies balance perfectly. These points play a crucial role in understanding the stability of orbits and the placement of satellites in space. By analysing the behaviour of particles within these systems, we have gained a deeper understanding of celestial mechanics and the intricate dance of gravitational forces in the cosmos. Our findings underscore the importance of continued research in astrophysics to unravel the mysteries of the universe.