Classical Mechanics
Notes for the dumb Jose joining MSc by Prof. Jose Mathew

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1 Introduction

In this lecture, we will explore the concepts of plane polar coordinates and how they compare to Cartesian coordinates. We will derive key equations for position, velocity, and acceleration in polar coordinates, and highlight the differences in unit vectors between these coordinate systems.

2 Plane Polar Coordinates vs Cartesian Coordinates

2.1 1. Constancy of Unit Vectors

Cartesian Coordinates: In the Cartesian system, the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ (or equivalently $\hat{x}$, $\hat{y}$, $\hat{z}$) are constant. They point in fixed directions:

- $\hat{i}$ along the x-axis,
- $\hat{j}$ along the y-axis,
- $\hat{k}$ along the z-axis.

These unit vectors do not change direction as you move from one point to another in space. This makes the Cartesian system particularly simple for problems where the geometry does not change with direction.

Curvilinear Coordinates: In curvilinear systems like the plane polar coordinate system, the unit vectors $\hat{r}$ (radial) and $\hat{\theta}$ (angular) are not constant. They change direction depending on the position of the point. Specifically:

- $\hat{r}$ points radially outward from the origin to the point of interest.
- $\hat{\theta}$ is perpendicular to $\hat{r}$, pointing in the direction of increasing θ.

This distinction introduces additional complexity, as the unit vectors themselves depend on the position of the point and change as the point moves.

2.2 2. Diagrams for Distinction

Cartesian Coordinates: A point $P(x, y)$ is represented with constant unit vectors $\hat{i}$ and $\hat{j}$.

Polar Coordinates: A point $P(r, \theta)$ with radial unit vector $\hat{r}$ and tangential unit vector $\hat{\theta}$. These vectors change direction as θ varies.

2.3 3. Unit Vectors in Plane Polar Coordinates in Terms of Cartesian Coordinates

The unit vectors in plane polar coordinates, $\hat{r}$ and $\hat{\theta}$, can be expressed in terms of Cartesian coordinates as:

$$\hat{r} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}$$
$$\hat{\theta} = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j}$$

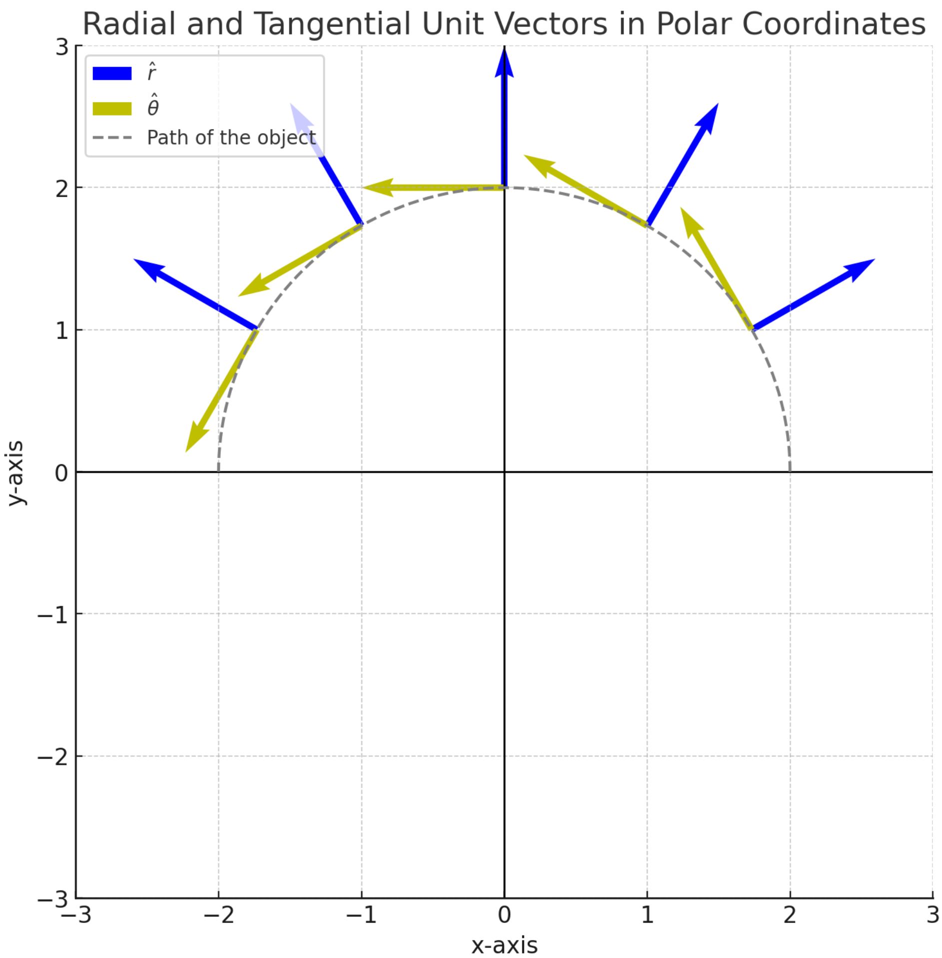

Figure 1: Radial and Tangential Unit Vectors in Polar Coordinates

Interpretation:

- $\hat{r}$ points radially outward.
- $\hat{\theta}$ is perpendicular to $\hat{r}$ in the counterclockwise direction.

2.4 4. Position Vector in Cartesian and Plane Polar Coordinates (Two-column Format)

Cartesian Coordinates	Plane Polar Coordinates
$\vec{r} = x\hat{i} + y\hat{j}$ $x = r\cos(\theta)$ $y = r\sin(\theta)$	$\vec{r} = r\hat{r}$ $r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

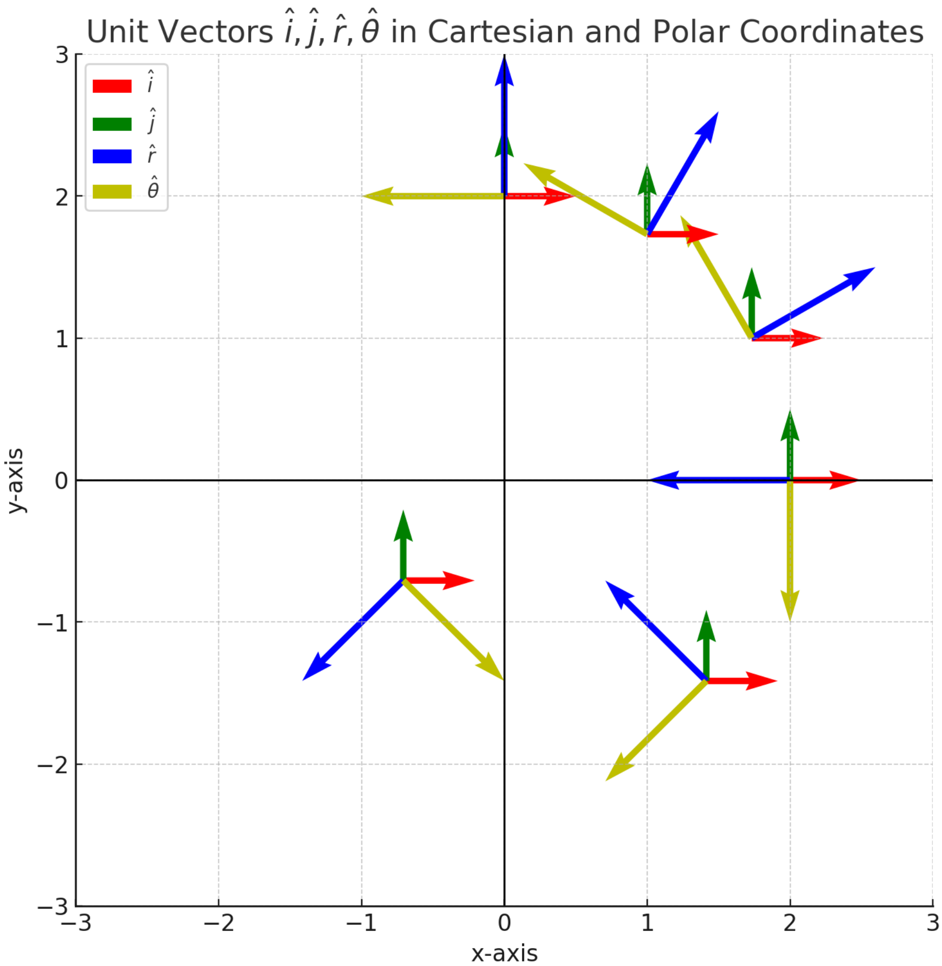

Figure 2: Constancy of Unit Vectors $hati, hatj$ vs. Varying $hatr, hattheta$

3 Velocity in Plane Polar Coordinates

To obtain the velocity in plane polar coordinates, we take the time derivative of the position vector $\vec{r} = r\hat{r}$.

Step-by-Step Derivation:

$$\vec{v} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

Now, since $\hat{r}$ is not constant, its derivative with respect to time can be found as follows:

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

Thus, the velocity becomes:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Where:

- $\dot{r}$ is the radial velocity,
- $r\dot{\theta}$ is the tangential velocity.

4 Acceleration in Plane Polar Coordinates

The acceleration is obtained by taking the time derivative of the velocity vector:

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)$$

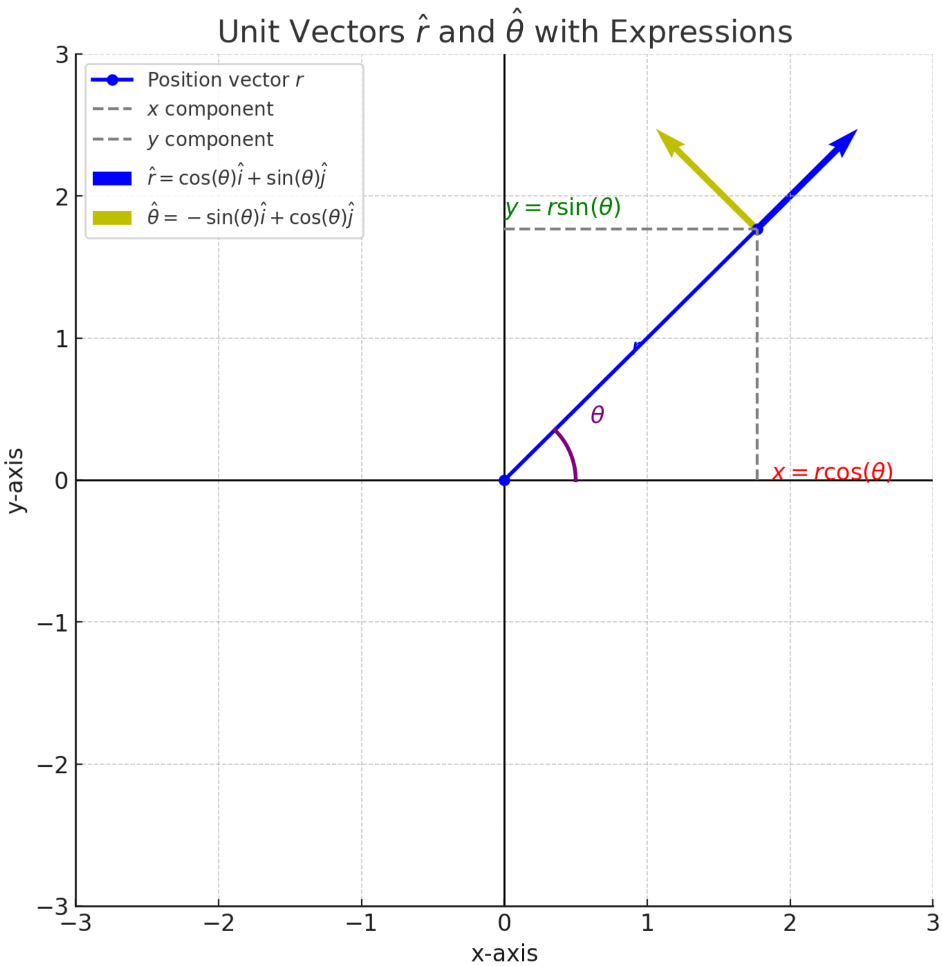

Figure 3: Dependence of r and $theta$ on x and y

Applying the product rule to each term:

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{d}{dt}\left(r\dot{\theta}\hat{\theta}\right)$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

Simplify:

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

5 Conclusion

We have discussed the differences between Cartesian and polar coordinates, the derivation of velocity and acceleration in polar coordinates, and the effect of the varying nature of polar unit vectors $\hat{r}$ and $\hat{\theta}$. Unlike the constant unit vectors in Cartesian coordinates, polar unit vectors change as the point of interest moves.

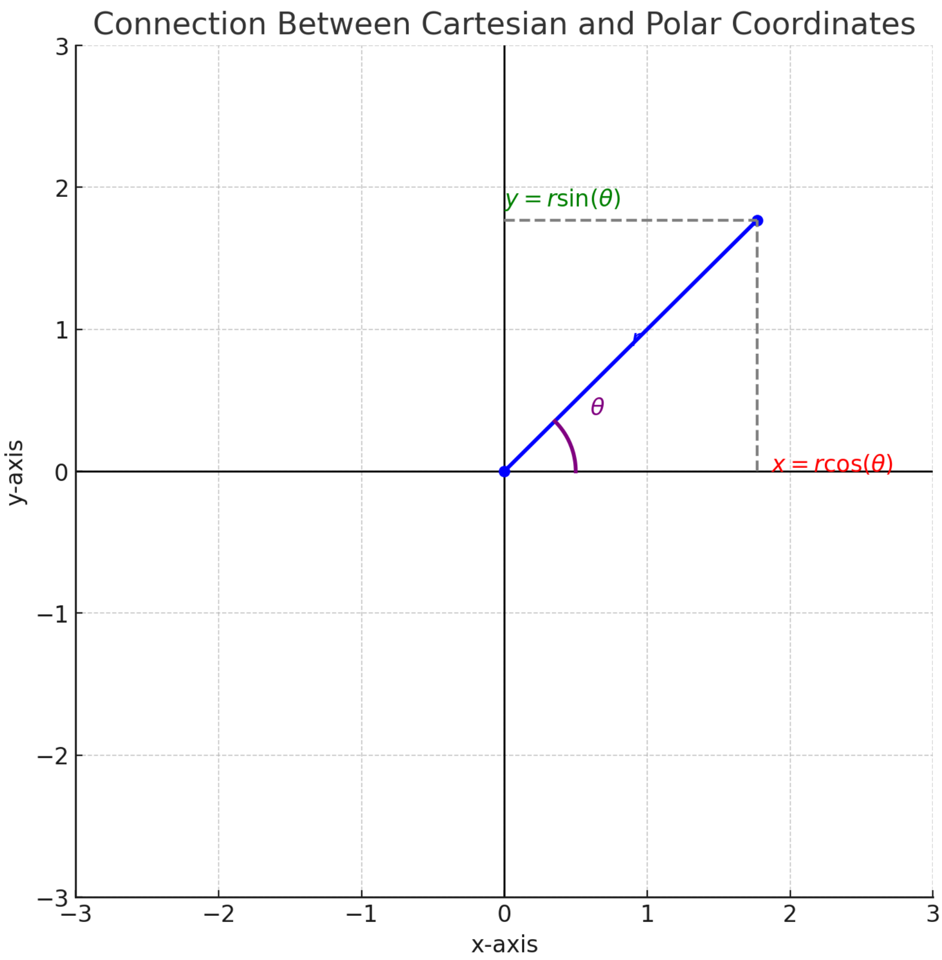

Figure 4: How x and y relate to *theta*

Exercises

5.1 Q1)

A velocity vector at the point $(1, 1)$ has a magnitude of $2.5\,\mathrm{m/s}$ and makes an angle of $30°$ with the horizontal. Express this vector in polar coordinates.

Solution:

Step 1: Calculate the Position Vector r in Polar Coordinates

The position vector r is the distance from the origin to the point $(1, 1)$. The formula for the magnitude of the position vector is:

$$r = \sqrt{x^2 + y^2}$$

Substituting $x = 1$ and $y = 1$:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.414\,\mathrm{m}$$

Step 2: Find the Angle θ for the Position Vector

The angle θ for the position vector can be found using:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Substituting $x = 1$ and $y = 1$:

$$\theta = \tan^{-1}(1) = 45^\circ$$

Step 3: Break Down the Velocity Vector into Radial and Tangential Components

In polar coordinates, the velocity vector can be decomposed into:

- **Radial component** v_r (along the direction of the position vector).
- **Tangential component** v_θ (perpendicular to the position vector).

Radial Component v_r

The radial component is the projection of the velocity vector along the position vector:

$$v_r = v\cos(\theta_v - \theta)$$

Substituting $v = 2.5\,\mathrm{m/s}$, $\theta_v = 30^\circ$, and $\theta = 45^\circ$:

$$v_r = 2.5\cos(30^\circ - 45^\circ) = 2.5\cos(-15^\circ)$$

$$v_r \approx 2.5 \times 0.9659 \approx 2.41\,\mathrm{m/s}$$

Tangential Component v_θ

The tangential component is the projection of the velocity vector perpendicular to the position vector:

$$v_\theta = v\sin(\theta_v - \theta)$$

Substituting $v = 2.5\,\mathrm{m/s}$, $\theta_v = 30^\circ$, and $\theta = 45^\circ$:

$$v_\theta = 2.5\sin(30^\circ - 45^\circ) = 2.5\sin(-15^\circ)$$

$$v_\theta \approx 2.5 \times (-0.2588) \approx -0.65\,\mathrm{m/s}$$

Step 4: Final Expression in Polar Coordinates

The velocity vector in polar coordinates is:

$$v_r \approx 2.41\,\mathrm{m/s}, \quad v_\theta \approx -0.65\,\mathrm{m/s}$$

Thus, the velocity vector in polar coordinates at the point $(1, 1)$ is:

$$v = (2.41\,\hat{r},\ -0.65\,\hat{\theta})\,\mathrm{m/s}$$

Q2: Direction of Radial Component of Velocity at Maximum Height of a Projectile

At the maximum height of a projectile, the radial component of velocity points in the direction of the change in position relative to the origin. Since the vertical velocity is zero at maximum height, the radial component of velocity points horizontally outward from the origin.

Q3: Direction of Tangential Component of Velocity at Maximum Height of a Projectile

At the maximum height of a projectile, the tangential component of velocity is perpendicular to the radial component and is zero because the object is momentarily at rest in the vertical direction.

6 Answer to Q2 and Q3. Radial and Tangential Components of the Ball's Velocity at Maximum Height

Given Data:

- Initial velocity, $v_0 = 100\,\mathrm{m/s}$
- Launch angle, $\theta_0 = 30^\circ$
- Gravitational acceleration, $g = 9.8\,\mathrm{m/s}^2$

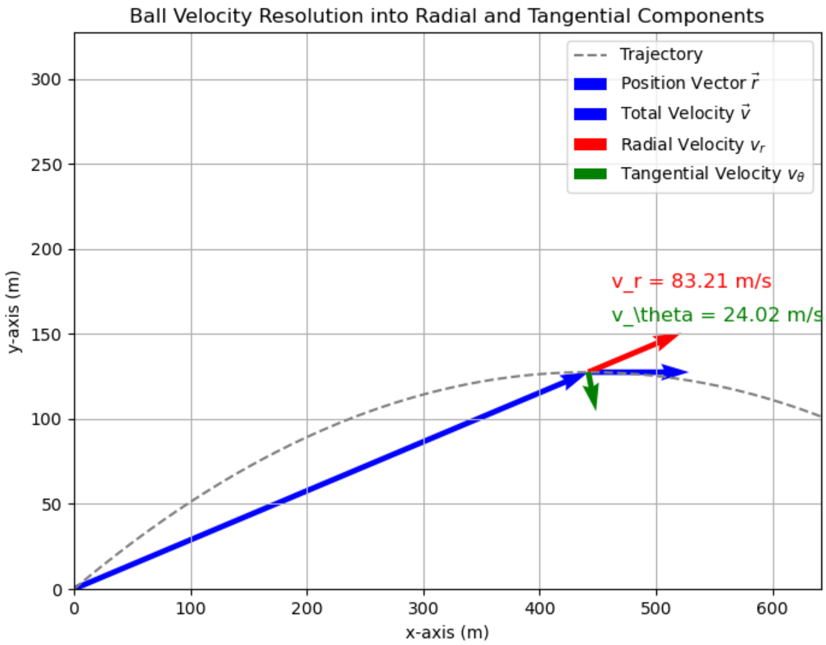

Figure 5: How x and y relate to *theta*

Step 1: Horizontal and Vertical Components of the Initial Velocity

The initial velocity has two components: horizontal (v_{0x}) and vertical (v_{0y}).
1. **Horizontal component of velocity**:

$$v_{0x} = v_0 \cos(\theta_0)$$

Substituting the values:

$$v_{0x} = 100 \times \cos(30^\circ) = 100 \times \frac{\sqrt{3}}{2} = 86.60\,\mathrm{m/s}$$

2. **Vertical component of velocity**:

$$v_{0y} = v_0 \sin(\theta_0)$$

Substituting the values:

$$v_{0y} = 100 \times \sin(30^\circ) = 100 \times \frac{1}{2} = 50\,\mathrm{m/s}$$

Step 2: Time to Reach Maximum Height

At maximum height, the vertical component of the velocity becomes zero ($v_y = 0$).

Using the first equation of motion to calculate the time to reach the maximum height:

$$v_y = v_{0y} - gt$$

Setting $v_y = 0$:

$$0 = 50 - 9.8t$$

$$t = \frac{50}{9.8} \approx 5.10\,\text{seconds}$$

Step 3: Position of the Ball at Maximum Height

1. **Horizontal position** at maximum height:

$$x_{\max} = v_{0x} \times t$$

Substituting the values:

$$x_{\max} = 86.60 \times 5.10 \approx 441.66\,\text{m}$$

2. **Vertical position** (maximum height):

$$y_{\max} = v_{0y} \times t - \frac{1}{2}gt^2$$

Substituting the values:

$$y_{\max} = 50 \times 5.10 - \frac{1}{2} \times 9.8 \times (5.10)^2$$

$$y_{\max} \approx 255 - 127.5 = 127.5\,\text{m}$$

Thus, the position of the ball at maximum height is:

$$(x_{\max}, y_{\max}) = (441.66\,\text{m}, 127.5\,\text{m})$$

Step 4: Radial and Tangential Components of Velocity at Maximum Height

At the maximum height, the velocity has only the horizontal component, which we already calculated to be $86.60\,\text{m/s}$.

1. **Radial Direction**: The radial component of the velocity is the component of the velocity in the direction of the position vector $\hat{r}$.

The angle of the position vector is:

$$\theta_r = \tan^{-1}\left(\frac{y_{\max}}{x_{\max}}\right) = \tan^{-1}\left(\frac{127.5}{441.66}\right) \approx 16^\circ$$

The radial velocity is the projection of the total velocity along the direction of the position vector:

$$v_r = v_x \cos(\theta_r)$$

Substituting the values:

$$v_r = 86.60 \times \cos(16^\circ) \approx 83.21\,\text{m/s}$$

2. **Tangential Direction**: The tangential component of the velocity is the component perpendicular to the position vector.

The tangential velocity is the projection of the total velocity perpendicular to the position vector:

$$v_\theta = v_x \sin(\theta_r)$$

Substituting the values:

$$v_\theta = 86.60 \times \sin(16^\circ) \approx 23.86\,\text{m/s}$$

However, since the tangential component is in the clockwise direction (below the horizontal), it will be negative:

$$v_\theta \approx -23.86\,\text{m/s}$$

Step 5: Conclusion

At maximum height:

- The **radial component** of the velocity is $v_r \approx 83.21\,\mathrm{m/s}$.
- The **tangential component** of the velocity is $v_\theta \approx -23.86\,\mathrm{m/s}$.

Q4: Express the Vector $3\hat{\imath} + 4\hat{\jmath}$ in Polar Coordinates

Given the vector $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$ at the point $(1, 1)$, we express it in polar coordinates.

Magnitude:

$$|\vec{v}| = \sqrt{3^2 + 4^2} = 5$$

Angle:

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

Thus, the vector in polar coordinates is:

$$\vec{v} = 5\,\hat{r} \quad \text{at}\,\theta = 53.13^\circ$$

Q5: Express the Vector $2\hat{r}$ in Cartesian Coordinates

The vector $2\hat{r}$ represents a radial vector with a magnitude of 2 units in the direction of $\hat{r}$. To express this in Cartesian coordinates, we use:

$$\hat{r} = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$$

For an arbitrary angle θ:

$$2\hat{r} = 2(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}) = 2\cos(\theta)\hat{\imath} + 2\sin(\theta)\hat{\jmath}$$

Q6: Express the Position Vector (PV) of a Ball at $(1, 2)$ in Cartesian and Polar Coordinates

In Cartesian coordinates, the position vector is:

$$\vec{r} = 1\hat{\imath} + 2\hat{\jmath}$$

In polar coordinates:

$$r = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.236$$

$$\theta = \tan^{-1}\left(\frac{2}{1}\right) = 63.43^\circ$$

Thus, the position vector in polar coordinates is:

$$\vec{r} = 2.236\,\hat{r} \quad \text{at}\,\theta = 63.43^\circ$$