Experiment: Motion of a Particle under a Central Force

$$F = -\frac{K}{r^3}\left(1 - \frac{\alpha}{r}\right)\mathbf{r}$$

Aim

To study the motion of a particle under the central force

$$F = -\frac{K}{r^3}\left(1 - \frac{\alpha}{r}\right)\mathbf{r},$$

and to show that the resulting orbit is a **precessing ellipse**, depending on the constants K and α.

Apparatus Required

Computer or microprocessor system capable of numerical computation (Python with `NumPy` and `Matplotlib` recommended).

You don't need to write Theory in the Record

Theory-This is for your understanding. You can just skim it. No need to write

Consider a particle of mass m moving under a central potential $V(r)$ which depends only on the distance r from a fixed center.

Potential Energy and Force Components

The given central force

$$\mathbf{F}(r) = -\frac{K}{r^3}\left(1 - \frac{\alpha}{r}\right)\mathbf{r}$$

corresponds to the potential energy function

$$V(r) = -\frac{K}{r} + \frac{K\alpha}{2r^2}.$$

Definitions of Constants:

- m: Mass of the particle (kg)
- K: Strength of the inverse-square (attractive) central force

- α: Small positive constant introducing deviation from the pure inverse-square law
- r: Radial distance from the center (m)
- $\mathbf{r}$: Position vector of the particle

Since V depends only on r, the force is purely radial:

$$F_\theta = 0, \qquad F_r = -\frac{dV}{dr} = -\frac{K}{r^2} + \frac{K\alpha}{r^3}.$$

Equation of Motion in Polar Coordinates

The radial and angular equations of motion are

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F_r(r), \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0.$$

The second equation implies conservation of angular momentum:

$$\boxed{L = mr^2\dot{\theta} = \text{constant.}}$$

Here L is the angular momentum of the particle about the center of force.

Substituting for $\dot{\theta}$ in the radial equation gives

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{K}{mr^2} + \frac{K\alpha}{mr^3}.$$

Reduction using $u = 1/r$

Let $u = 1/r$. Then

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}\,F_r\left(\frac{1}{u}\right).$$

Substituting $F_r = -K(1 - \alpha u)u^2$ gives

$$\frac{d^2u}{d\theta^2} + \underbrace{\left(1 + \frac{mK\alpha}{L^2}\right)}_{\beta^2} u = \frac{mK}{L^2},$$

where

$$\beta = \sqrt{1 + \frac{mK\alpha}{L^2}}$$

is the *precession parameter*.

Solution of the Orbit Equation

The general solution of the above differential equation is

$$u(\theta) = \frac{mK/L^2}{1 + \frac{mK\alpha}{L^2}} \left[1 + e\cos\bigl(\beta(\theta - \theta_0)\bigr)\right],$$

where

- e: Eccentricity-like parameter that determines the shape of the orbit,
- θ_0: Angular position of the periapsis (initial phase).

Hence the trajectory is

$$\boxed{r(\theta) = \frac{p_{\text{eff}}}{1 + e\cos\bigl(\beta(\theta - \theta_0)\bigr)}}, \quad p_{\text{eff}} = \frac{L^2 + mK\alpha}{mK}.$$

When $\alpha = 0$, $\beta = 1$ and the orbit is a closed Keplerian ellipse. When $\alpha \neq 0$, $\beta \neq 1$ and the orbit precesses.

Precession of the Orbit

The radial distance repeats when the angle advances by $2\pi/\beta$, so the apsidal precession per revolution is

$$\boxed{\Delta\varpi = 2\pi\left(1 - \frac{1}{\sqrt{1 + \dfrac{mK\alpha}{L^2}}}\right)}.$$

Thus, the orbit precesses in the direction of motion (prograde precession) for positive α.

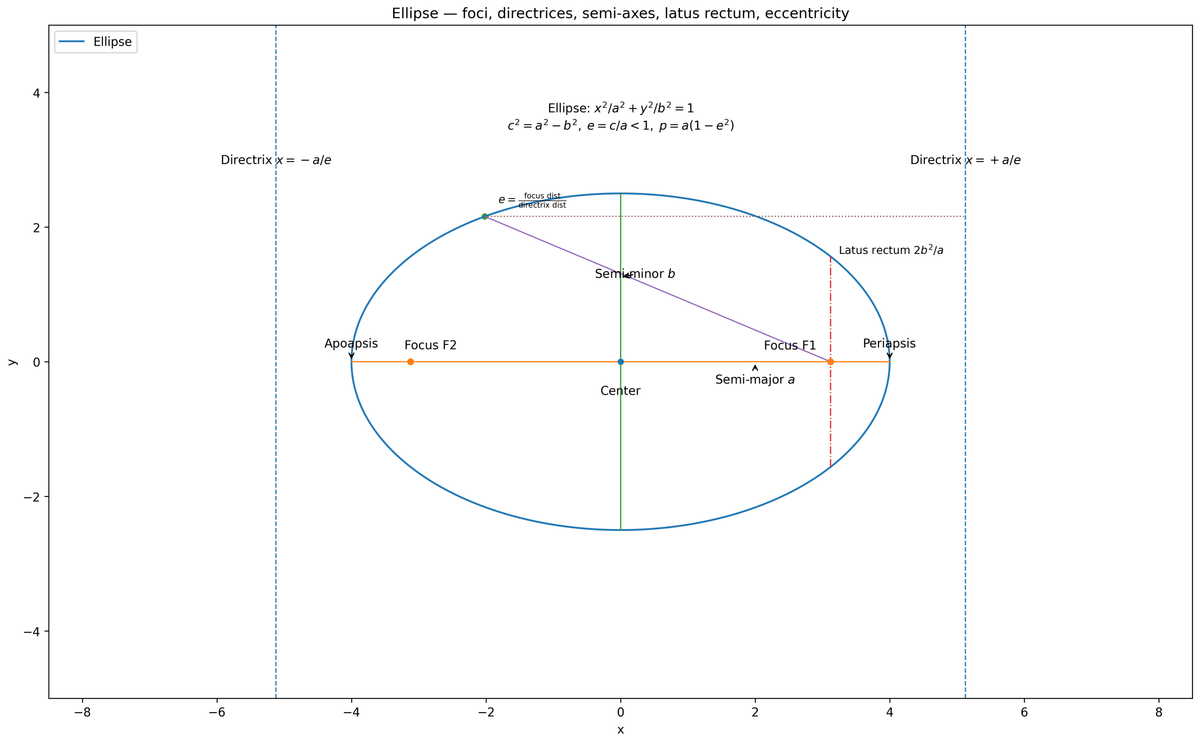

Figure 1: Caption

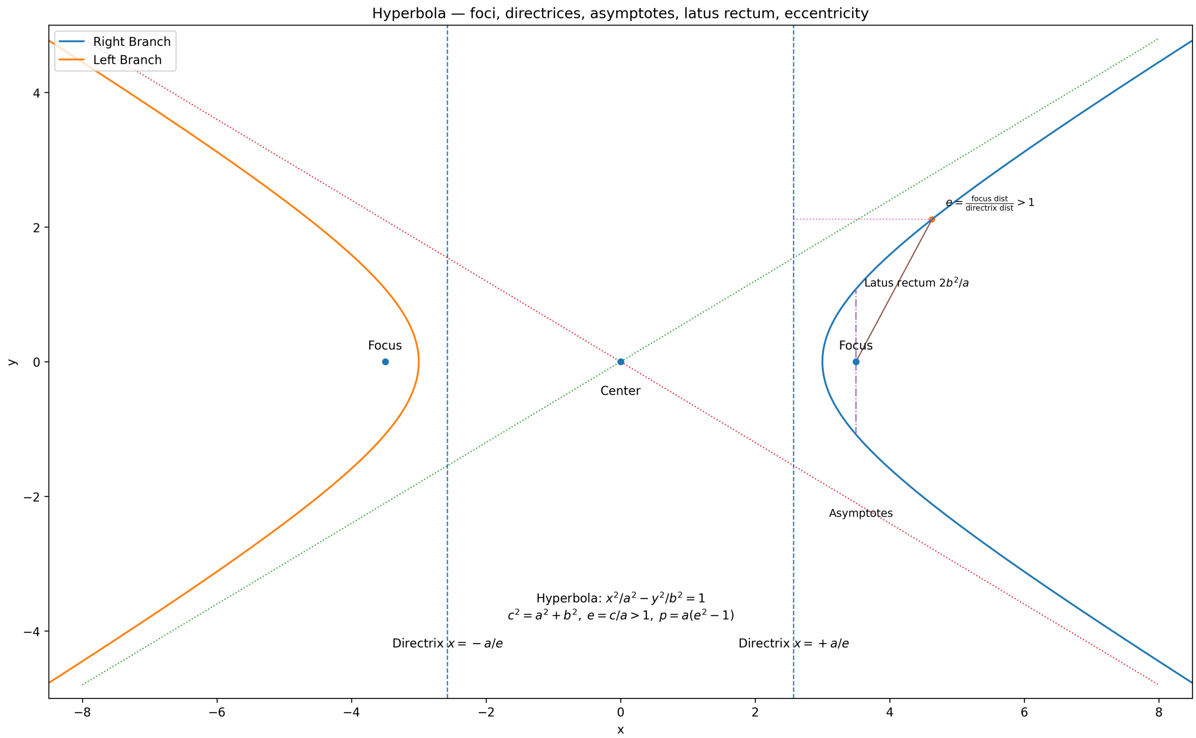

Figure 2: Caption

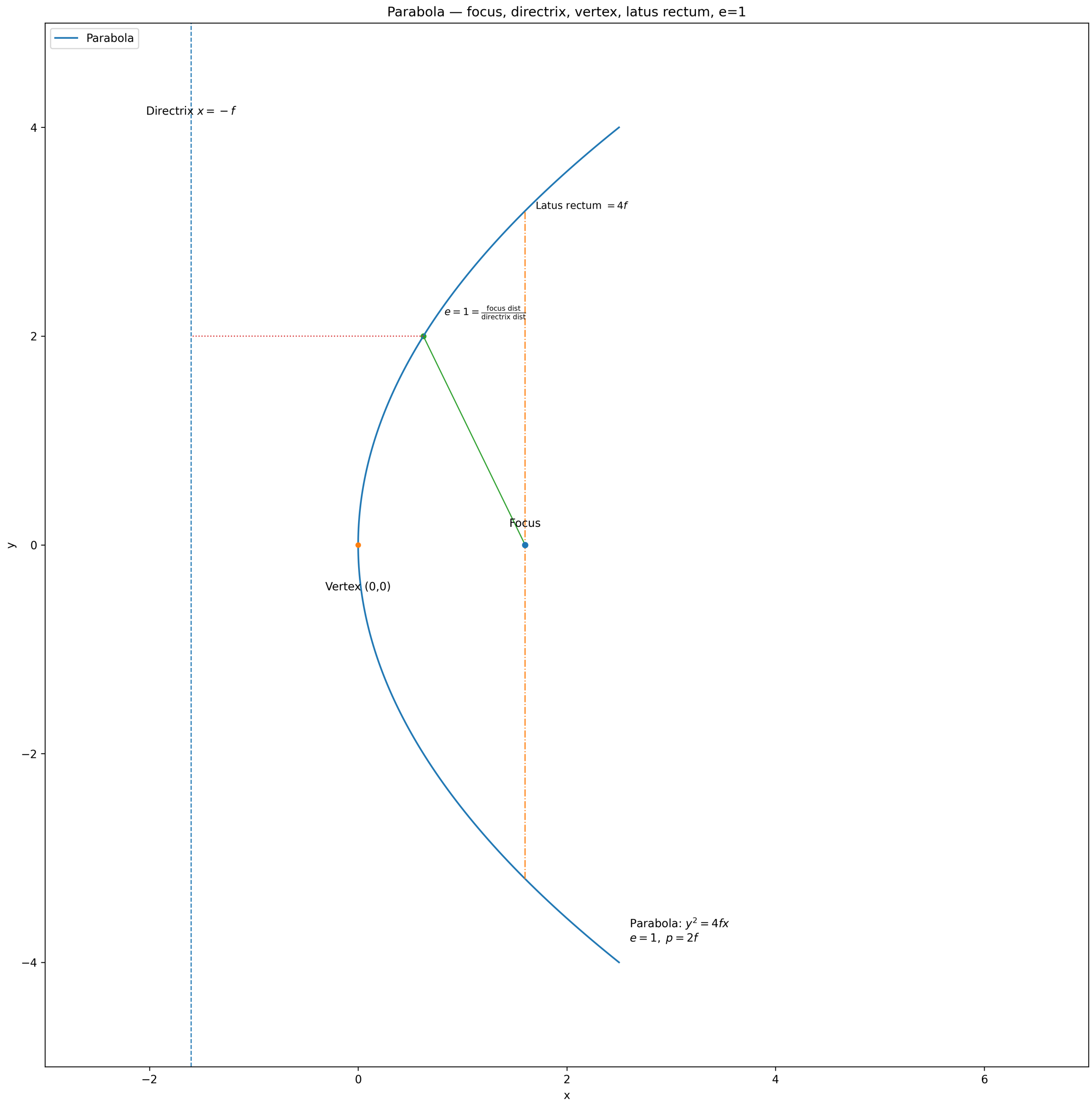

Figure 3: Caption

Procedure

1. Set the values of m, K, L, α and the eccentricity e.
2. Using the formula
$$r(\theta) = \frac{p_{\text{eff}}}{1 + e\cos\big(\beta(\theta - \theta_0)\big)},$$
compute r for a range of θ values (e.g. 0 to 12π).
3. Plot $r(\theta)$ as a parametric curve in Cartesian coordinates:
$$x = r\cos\theta, \quad y = r\sin\theta.$$
4. Observe the effect of varying α and K on the orbit.

Observations

Typical results obtained from simulation:

α	Nature of Orbit	Remarks
0.0	Closed ellipse	No precession (Keplerian)
0.02	Slightly precessing ellipse	Small prograde shift per orbit
0.05	Rosette-like orbit	Significant precession visible

Result

The motion of a particle under the central force
$$F = -\frac{K}{r^3}\left(1 - \frac{\alpha}{r}\right)\mathbf{r}$$
is a **precessing conic section**. The amount of precession per revolution is given by
$$\Delta\varpi = 2\pi\left(1 - \frac{1}{\sqrt{1 + \dfrac{mK\alpha}{L^2}}}\right).$$
The precession increases with K and α and decreases with larger angular momentum L.

Precautions

1. Use small step sizes in the computation for higher accuracy.
2. Ensure that the initial conditions correspond to a bound orbit ($e < 1$).
3. Avoid very large α values to prevent numerical divergence.
4. Check that angular momentum $L = mr^2\dot{\theta}$ remains constant throughout the computation.

Python Code for Conic Sections

Python Simulation Code: Central Force Motion

In the code avoid every that is in green. That is for your understanding

```
#
#Central Force Motion: F = -(K/r^3) * (1 - alpha/r) * r_vec
#------------------------------------------------------------
#This program visualizes the motion of a particle under the
#given central force. Instead of integrating with time, we
#use the analytic form of the orbit:

#    r(theta) = p_eff / [1 + e * cos(beta * (theta - theta0))]

#where:
#    p_eff = (L^2 + m*K*alpha) / (m*K)
#    beta = sqrt(1 + (m*K*alpha)/(L^2))
#    e = eccentricity-like parameter
#    theta0 = phase angle (set to 0 here)

#This gives a precessing conic (rosette-like orbit). When
#alpha = 0, the orbit reduces to a perfect Keplerian ellipse.
#
###############################################################

# ------------------------------------------------------------
#  Import required packages
# ------------------------------------------------------------
import numpy as np
import matplotlib.pyplot as plt

# ------------------------------------------------------------
#  Define constants and parameters
# ------------------------------------------------------------
m = 1.0          # mass of the particle (arbitrary units)
K = 1.0          # central force constant (attractive)
alpha = 0.03     # small positive number --> precession occurs
L = 1.0          # angular momentum (assumed constant)
e = 0.25         # eccentricity-like parameter (0 < e < 1)
theta0 = 0.0     # initial angular phase (r is max when theta = 0)

# ------------------------------------------------------------
#  Derived quantities from the theory
# ------------------------------------------------------------
beta = np.sqrt(1 + (m * K * alpha) / (L ** 2))
p_eff = (L ** 2 + m * K * alpha) / (m * K)

# ------------------------------------------------------------
```

```python
#  Compute r(theta) for a range of theta values
# ------------------------------------------------------------
theta = np.linspace(0, 12 * np.pi, 2000)
r = p_eff / (1 + e * np.cos(beta * (theta - theta0)))

x = r * np.cos(theta)
y = r * np.sin(theta)

# ------------------------------------------------------------
#  Plot in the xy-plane (spatial trajectory)
# ------------------------------------------------------------
plt.figure(figsize=(7, 6))
plt.plot(x, y, linewidth=1.2)
plt.scatter([0], [0], color='black', s=20, label='Center of Force')
plt.axis('equal')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Orbit under $F=-(K/r^3)(1-\\alpha/r)\\,\\mathbf{r}$')
plt.grid(True)
plt.legend()
plt.show()

# ------------------------------------------------------------
#  Plot r(theta) parametric curve (r vs theta)
# ------------------------------------------------------------
plt.figure(figsize=(7, 4.5))
plt.plot(theta, r, linewidth=1.2)
plt.xlabel(r'$\theta$ (radians)')
plt.ylabel(r'$r(\theta)$')
plt.title('Parametric plot $r(\\theta)$ showing precession')
plt.grid(True)
plt.show()

# ------------------------------------------------------------
#  Print theoretical precession per orbit
# ------------------------------------------------------------
Delta_varpi = 2 * np.pi * (1 - 1 / beta)

print(f''beta (precession parameter): {beta:.6f}'')
print(''Predicted apsidal advance per revolution (radians): {Delta_varpi:.6f}'')
print(f''In degrees: {np.degrees(Delta_varpi):.3f}'')
```

Listing 1: Simulation of a particle under the central force $F = -(K/r^3)(1-\alpha/r)\mathbf{r}$ showing precessing orbit.