# Derivation of $E = mc^2$ from the Shift in Center of Mass

## Setup

Consider a box of mass $M$:

- Length of the box: $L$.
- The box is initially at rest in an inertial frame of reference.
- The box is floating in free space without external forces acting on it.

A photon is emitted from one end of the box:

- Let the photon be emitted from the left end of the box.
- The energy of the photon is $E$.
- The speed of light is $c$.
- The mass of the box doesn't change as the photon is massless.

## Photon Emission

The box recoils:

- By conservation of momentum, as the photon is emitted to the right, the box recoils to the left.
- The momentum of the photon is $p = \frac{E}{c}$.
- Thus, the recoil momentum of the box is $-\frac{E}{c}$.

## Center of Mass Considerations

Calculate the shift in the center of mass:

- Before the photon is emitted, the center of mass of the system is at the center of the box.
- After the photon is emitted, the box has shifted slightly to the left, and the photon is traveling to the right.
- The new center of mass must remain unchanged (by the principle of conservation of the center of mass in a closed system).

## Calculation

The photon travels from left to right:

- The photon travels a distance $L$ to the other end of the box.

  Calculate the time taken for the photon to reach the other end:

$$t = \frac{L}{c}$$

  Recoil velocity of the box:

$$v = \frac{E}{Mc}$$

  Displacement of the box:

$$d = v \cdot t = \frac{E}{Mc} \cdot \frac{L}{c} = \frac{EL}{Mc^2}$$

## Final Position

Final position of the box:

- After the photon is absorbed at the other end, the box stops recoiling.
- The box has shifted by $\Delta x = \frac{EL}{Mc^2}$.

## Energy-Mass Equivalence

Relate energy to mass:

- For the center of mass of the system to remain unchanged, the mass associated with the energy $E$ of the photon must account for the displacement.
- The displacement of the center of mass due to the energy $E$ being transferred across the box implies that $E$ can be considered as equivalent to a mass $m$ such that $m = \frac{E}{c^2}$.

Thus, we derive that energy $E$ is equivalent to mass $m$ by the famous equation:

$$E = mc^2$$