Area-Preserving Poincaré Mappings of the Unit Disk

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ABSTRACT. The best way to investigate the long-time behaviour of dynamical systems is to introduce an appropriate Poincaré mapping P and study its iterates.
Two cases of physical interest arise : Conservative and dissipative systems. While the latter has been considered by a great many authors, much less is known for the first one (according to Liouville's theorem, here the mapping leaves a certain measure in phase space invariant).
In this paper, we concentrate our attention on compact phase spaces (or, rather, surfaces of section). This assumption is mathematically useful and physically reasonable.
We consider the simplest possible (2-dimensional) systems whehre the phase space is the compact unit disk $\bar{D}$ in $\mathbb{R}^2$. A family of simple area-preserving mappings from $\bar{D}$ onto itselves will be given and discussed in detail.
It is shown that general characteristics of the dynamics are quite similar to those of e.g. the Hénon-Heiles system, while other features, as the structure of invariant curves, are different.

1. Introduction

Perhaps the only useful method to investigate the long-time behaviour of dynamical systems consists of introducing suitable surfaces of sections, i.e. sub-manifolds of co-dimension 1 in the phase space and studying the corresponding return mapping and its iterations.
In the physical application, two different cases arise : Dissipative and conservative flows.
In the first case, phase-space volumina shrink with time, and eventually the region occupied by possible states of the system reduces to a set of measure zero. Here during the last decades the concept of attractors, e.g. in the sense of Milnor (1985), comes into play.
Numerical studies have been carried out by Hénon and Heiles (1964) and Hénon (1976) for a special two-dimensional Poincaré mapping which (as well as its inverse) is polynomial of degree 2. The last fact allows one to apply algebraic and complex analytic methods which lead to an almost complete understanding of this mapping (Hubbard, 1988).

Celestial Mechanics **43** (1988), 185–191.

On the other hand, it seems to be much more difficult to develop natural concepts for exploring conservative dynamical systems.
In fact, numerical studies have been carried out, and conservative Hénon-type functions on $\mathbb{R}^2$ can also be given. Unfortunately, they seem t be more complicated than in the general dissipative case. In reality, however, the energy shell to which the possible states are confined is not of infinite extent. For that reason it is desirable to have a more appropriate numerical model of a conservative Poincaré mapping, namely one whose domain is a compact manifold.
In this paper we construct a special class of mappings of this kind.

2. General properties of Poincaré mappings

Let K be a compact Riemannian C^1-manifold with volume measure ("area") μ. Df denotes the Jacobian of f, and we define the set

$$(2.1) \qquad \Gamma_K = \{ f \in C^1(K \longrightarrow K) ; \det Df = 1 \}$$

of area-preserving smooth mappings from K to K.
Note that since each $f \in \Gamma_K$ is bijective, Γ_K is a group under composition.
To describe the action of f and its iterates on K, we have to consider the orbit

$$(2.2) \qquad B(x) = \{ f^n(x) : n \in \mathbb{N} \}$$

of a point $x \in K$ and the set H(x) of the limit points of the sequence

$$(2.3) \qquad (f^n(x) : n \in \mathbb{N}) \quad .$$

We call

$$(2.4) \qquad \pi(x) = \# H(x)$$

the period of x.
In case $\pi(x) = n < \infty$, $H(x) = B(x)$ and x is a periodic point of f.
If dim K = 2 only three possibilities arise :

(a) $|\operatorname{tr} Df^n(x)| < 2 \longrightarrow$ *elliptic* periodic point,
(b) $|\operatorname{tr} Df^n(x)| > 2 \longrightarrow$ *hyperbolic* periodic point,
(c) $|\operatorname{tr} Df^n(x)| = 2 \longrightarrow$ *parabolic* periodic point.

By the KAM-theorem (Arnold, 1963 , Arnold and Avez, 1968) in the vicinity of an elliptic point there exist f-invariant curves filling an open region, while close to hyperbolic points two invariant curves (stable and unstable manifolds) are found (Moser, 1956).
If $\pi(x)$ is infinite, several possibilities can be imagined : H(x) might be a Jordan curve (this corresponds to the existence of separating integrals in dynamical systems). Such points x will be called *laminar*.
If H(x) contains a non-empty open set, we describe x as *chaotic*. In contrast to Poincaré mappings with unbounded domains, in the compact case we have a natural measure of the degree of "chaoticity", namely

$$(2.5) \qquad \chi(x) = \frac{\mu(H(x))}{\mu(K)} \quad .$$

Furthermore, we define χ to be the area of all chaotic regions measured in units of $\mu(\overline{D})$.

In section 5 we shall apply these concepts to a special family of mappings in $\Gamma_{\bar{D}}$, where $\bar{D}$ is the compact unit disk in $\mathbb{R}^2$.

3. An area-preserving mapping of the compact unit disk

The simplest non-trivial examples illustrating the ideas of the last section can be found for 2-dimensional, simply connected phase-spaces with smooth boundaries. For the sake of clarity, we restrict our attention to analytic functions whose domain is the closed unit disk $\bar{D}$ in $\mathbb{R}^2$.
The well-known twist-mappings T_φ belong to this family and are constructed as follows :

$$(3.1) \qquad \varphi : [0,1] \longrightarrow \mathbb{R}^2$$

is an analytic function, and

$$(3.2) \qquad T_\varphi : \bar{D} \longrightarrow \bar{D}$$

rotates each point of distance r from the centre by the angle $\varphi(r^2)$.
For later applications we introduce the special symbol $f_{o,c}$ for T_φ with

$$(3.3) \qquad \varphi(r^2) = c \cdot r^2 .$$

In particular, $f_{o,o}$ is the identity on $\bar{D}$. All twist mappings leave the concentric circles invariant, so they are not interesting in their own right.
Therefore we have to construct more complicated area-preserving mappings.
It is useful to introduce a complex number

$$(3.4) \qquad z = x + i \cdot y$$

to describe the point with Cartesian coordinates (x,y).
Now take the point $a \in [0,1)$ on the non-negative real axis in D and its mirror image $b = 1/a$ with respect to the unit circle.
$f_{a,c}$ will be defined as an area-preserving function which maps the bundle of circles with foci (a,b) to the concentric bundle. This definition is made unique by the requirement that its restriction to the real interval [-1,+1] is a monotonously increasing real function.
Let the image of z under $f_{a,o}$ be z^*. The circles in the (a,b)-bundle are the lines of constant λ where

$$(3.5) \qquad \lambda = \left| \frac{z-a}{z-b} \right| ,$$

$$(3.6) \qquad \lambda^2 = \frac{(x-a)^2+y^2}{(x-b)^2+y^2} .$$

The circle with parameter λ has radius

$$(3.7) \qquad r = r(\lambda) = \frac{1-a^2}{a} \cdot \frac{\lambda}{1-\lambda^2}$$

and centre

$$(3.8) \qquad x_m = \frac{1-x^2-y^2}{a+b-2\cdot x} = \frac{a^2-\lambda^2}{a\cdot(1-\lambda^2)} \quad .$$

The image of the interior of this circle is the interior of the image circle, so both areas must be equal. Thus $| z^* | = r$ and

$$(3.9) \qquad z^* = (z - x_m) \cdot e^{-i\delta} \quad ,$$

where the rotating angle δ is determined by the demand that $f_{a,o}$ is area-preserving.
The area element at $z = (x,y)$ is

$$(3.10) \qquad dF = dx \wedge dy = \frac{i}{2} \cdot dz \wedge d\bar{z} \quad .$$

Putting

$$(3.11) \qquad x = x_m + r\cdot\cos a \quad , \qquad y = r\cdot\sin a \quad ,$$

$$(3.12) \qquad x^* = r\cdot\cos a^* \quad , \qquad y^* = r\cdot\sin a^* \quad ,$$

we find

$$(3.13) \qquad dF = r \cdot \cos a\cdot dx_m \wedge da + r\cdot dr \wedge da \quad ,$$

and the area element at the image point z^* is

$$(3.14) \qquad dF^* = r^*\cdot dr^* \wedge da^* = r\cdot dr \wedge da^* \ .$$

Since $f_{a,o}$ is area-preserving,

$$(3.15) \quad 0 = dF - dF^* = r\cdot\cos a\cdot dx_m \wedge da + r\cdot dr \wedge (da - da^*),$$

and we get with

$$(3.16) \qquad \delta = a - a^*$$

the differential equation

$$(3.17) \qquad 0 = \cos a \frac{dx_m}{d\lambda} + \frac{dr}{d\lambda} \cdot \frac{\partial\delta}{\partial a}$$

whose solution is

$$(3.18) \qquad \delta = - \sin a \cdot \frac{dx_m}{dr} = \frac{2\lambda}{1+\lambda^2} \cdot \sin a \quad .$$

In the next section we shall study the behaviour of the composite mapping

$$(3.19) \qquad f_{a,c}(x,y) = f_{o,c} [f_{a,o}(x,y)] \quad .$$

4. Numerical results

Fig. 1 gives an impression of the multitude of patterns arising from iterating $f = f_{a,c}$ for different values of the parameters.

Of course, by Brouwer's theorem, there must be at least one fixed point of f in $\bar{D}$. For the mappings considered here, even in the open disk D such a point z_1 (of elliptic type) was found.
Hence z_1 is surrounded by a system of analytic invariant curves inside a region $E(z_1)$ of apparently non-chaotic points. Each of these curves seems to possess a small neighbourhood which is completely contained in $E(z_1)$. Therefore, $E(z_1)$ is open and its boundary $\partial E(z_1)$ cannot be an analytic curve.
On the contrary, numerical calculations show that $\partial E(z_1)$ is a piecewise analytic curve consisting of finitely many arcs, each connecting two hyperbolic periodic points in the same orbit.
Consequently, these arcs are cyclically permuted by f.
They seem to be at the same time the stable manifold of one endpoint and the unstable manifold of the other. In this case no homoclinic points are found here. This stands in contrast to the structure of the outer branches of the invariant curves passing through the same periodic points.
The behaviour of the boundary curves seems to differ in this respect from what is found when studying mappings with unbounded domains as the Hénon type ones.
Although it is difficult to give a proof for the existence of chaotic points of f, the numerical results indicate that they are quite frequent. In Fig. 2 the variation of χ with the parameters a and c is shown.
χ depends sensitively on 'a' and increases monotonically with 'a' reflecting a growing degree of irregularity.

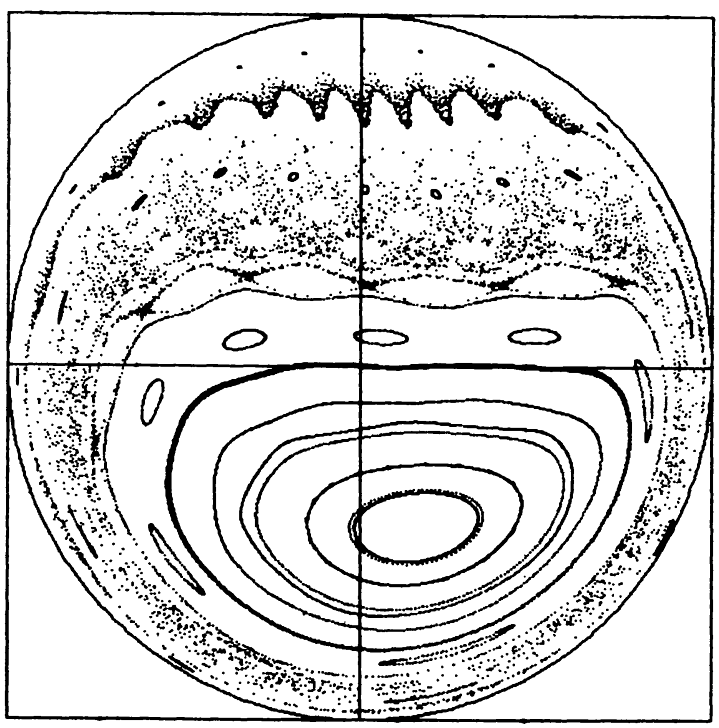

Fig. 1 : Mapping $f_{a,c}$ applied to a variety of initial points.

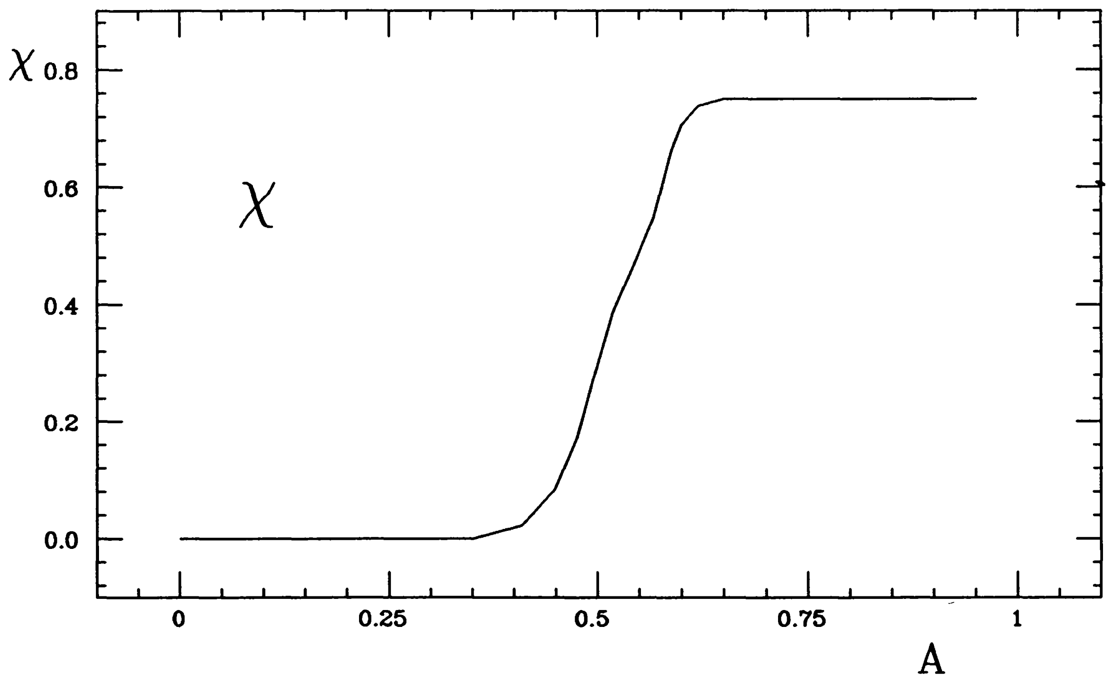

Fig. 2 : Chaoticity χ as a function of the parameter A, C=1.

5. Summary

We construct and study area-preserving Poincaré mappings with compact domain. The intention is to explore the dynamics of conservative physical systems.

The mappings given here are probably the first non-trivial examples of analytic functions of this kind on the unit disk. They are considered to be more realistic models than for instance Hénon type mappings which have unbounded phase spaces.

Acknowledgements

The numerical calculations have been carried out on the CONVEX of the Max-Planck-Institut für Radioastronomie (Bonn) and the VAX 780 of the Astronomische Institute der Universität Bonn.

W. Neutsch wants to thank the Deutsche Forschungsgemeinschaft for partial financial support. J. Kallrath acknowledges support through a stipend of the Cusanuswerk (Bonn).

Furthermore, W.N. is greatful to Dr. M. Robnik (Heidelberg) and Prof. Dr. J. H. Hubbard (New York, Bonn) for a number of interesting and helpful discussions.

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