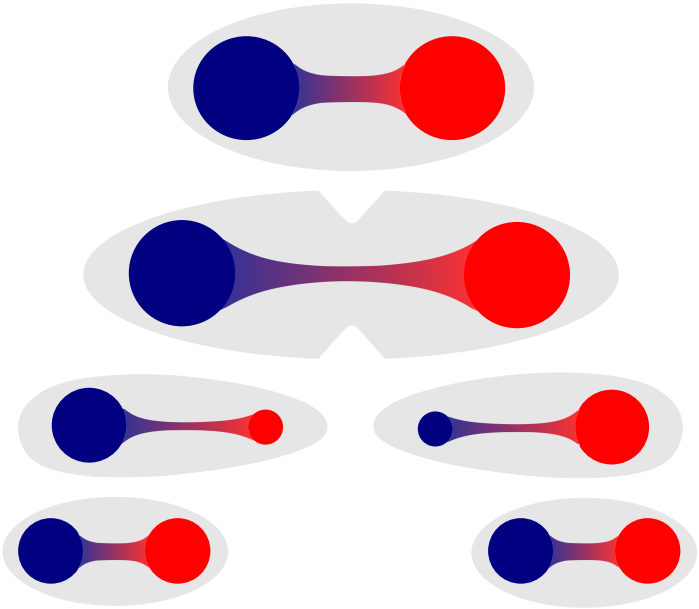
Confinement and String Breaking for the (1+1)-Dimensional Lattice Schwinger Model



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Abstract

This project concerns itself with the understanding of the framework of quantum field theory, and thus gauge theories, both in the continuum limit and on a lattice, such that we in the project are able to derive the (1+1)-dimensional Schwinger model for both the continuum case and on a lattice ourselves. The continuum formulation of the Schwinger model is achieved by imposing local U(1) gauge invariance to Dirac's fermionic Lagrangian density, which results in (3+1)-dimensional quantum electrodynamics, and restricting ourselves to (1+1)-dimensions. Discretizing the space of the found continuum Hamiltonian using an infinite lattice with finite lattice spacing and applying the Jordan-Wigner transformation to the matter fields and a quantum link model to the gauge fields, a spin version of the (1+1)-dimensional lattice Schwinger model is found.

Due to the (1+1)-dimensional lattice Schwinger model exhibiting several of the properties of (3+1)-dimensional quantum chromodynamic it yields an ideal candidate for examining properties of quantum chromodynamic. In this project the two properties colour confinement – the fact that colourful particles, such as quarks, cannot exist freely – and string breaking – that it at some point is energetically favourable to create another meson instead of increasing the interquark distance of the already existing one – is examined both analytically and numerically for the case of mesons.

Resumé

Dette projekt beskæftiger sig med forståelse af kvantefeltteori, og dermed gaugeteorier, både i kontinuumtilfældet og for diskretisering på et gitter, således at vi i dette projekt er klædt på til selv at udlede den (1+1)-dimensionelle Schwingermodel for både kontinuum og gitterdiskretiseringen. Kontinuum Schwingermodellen opnås ved at kræve, at Diracs Lagrangedensitet for fermioner skal være lokalt U(1) gaugeinvariant, hvilket resulterer i (3+1)-dimensionel kvanteelektrodynamik, og begrænser os til at arbejde i (1+1)-dimensioner. Ved dernæst at diskretisere rummet i den fundne Hamiltonfunktion for kontinuummodellen ved brug af et uendeligt gitter med endelig gitterkonstant og benytte Jordan-Wignertransformationen på massefelterne og en quantum link model på gaugefelterne opnår vi en spinversion af den (1+1)-dimensionelle gitter-Scwhingermodel.

Idet at den (1+1)-dimensionelle gitter-Schwingermodel udviser flere de samme egenskaber som (3+1)-dimensionel kvantekromodynamik, da virker den som et ideelt bud på en model, der kan benyttes til at undersøge kvantekromodynamiske egenskaber. I dette projekt undersøger vi de to egenskaber confinement af quarker – idéen om at kun farveløse partikler kan eksistere frit i naturen – og strengbrud – at det på et tidspunkt bliver energifavorabelt at danne en ny meson i stedet for at øge distancen mellem en quark og en antiquark i den allerede eksisterende meson – både analytisk og numerisk for mesoner.

Preface

Before moving on to the project I would like to acknowledge my supervisor associate professor Nikolaj Thomas Zinner, who has always been fast and committed to help when asked about the subject of this project.

I would also like to greatly thank PhD student Simon Panyella Pedersen for his very helpful, and many, email correspondences regarding the understanding of specific parts of his master thesis [S. P. Pedersen, *Realizing Lattice Gauge Theory via Direct Three-Body Interactions in Superconducting Circuits* (2019)] which has laid the foundation for my project, and for sparring regarding the numerical parts of this project. He has been very committed to help and has taken the time necessary for me to understand the phenomenon about which I asked.

Furthermore, I would like to thank the people who took the time to proof-read and comment on this manuscript: Nikolaj Paludan Bjerregaard, Rasmus Strid, and my parents.

On a more personal level I would like to thank my delightful best friends, Anne Kristine Haagen and Louise Kirkeby Hansen, my boyfriend Jacob Emil Ulvedal Rosborg, and my parents for their moral support during the period of writing as well as during my three years at Aarhus University in general, and for their willingness to listen to me talk for hours about my project, while non of them are physicists nor necessarily very interested in the subject of this project.

Contents

Chapter 1

Introduction

Quantum field theory is a theoretical framework extending quantum mechanics by combining it with special relativity and classical field theory. While quantum mechanics is concerned with describing one or few particles and relativistic quantum mechanics extends this focus to include the spin of the particles, quantum field theory is a theory enabling us to describe systems of many particles and also allows the treatment of fields, thus both particles and fields can be treated in the same framework. [1]

The developing the quantum field theory framework began shortly after the development of quantum mechanics, due to physicists wanting to not only describe particles but also fields in quantum mechanics. The foundation of the first quantum field theory, the quantum electrodynamics, is often said to be laid by Dirac for his work on quantisation of the electromagnetic field (combining quantum mechanics and field theory) and the relativistic theory of the the electron, the Dirac equation, (combining quantum mechanics and special relativity) [2]. As quantum electrodynamics in the 1950s became a reliable and non-preliminary theory, physicists began to create analogous theories, for example quantum chromodynamics, and in 1973 the concept of colour charge as the source of the strong nuclear field was developed [3]. There exists a considerably amount of quantum field theories, but the above mentioned theories, quantum electrodynamics and quantum chromodynamics, are two of the best known.

Today one of the most successful theories in physics is the so-called *Standard Model* of elementary particle physics [4] describing three of the four fundamental forces: The strong nuclear force, the weak nuclear force and the electromagnetic force, but not the gravitational force. As it turns out, the Standard Model is

a gauge theory – which is a type of quantum field theory – resulting from the demand of local U(1)⊗SU(2)⊗SU(3) gauge invariance (see Section 2.3.2) [2, 4]. One of the beautiful key mechanisms of the Standard Model is the confinement of colour charge, also known as *colour confinement* or just confinement, which states that colourful particles, such as quarks, cannot exist freely but only exist in hadrons [4–6]. This is a consequence of the Standard Model incorporating quantum chromodynamics (local SU(3) gauge invariance). Wanting to examine the properties of quantum chromodynamics exactly is met with complications, since this quantum field theory is not analytically solvable [7].

Considering quantum electrodynamics and reducing it from the usual (3+1)-dimensions (three spatial and one temporal dimension) to (1+1)-dimensions one obtains the *Schwinger model*. The (1+1)-dimensional Schwinger model is interesting for multiple reasons: Firstly it is analytically solvable, though merely for very light and very heavy fermions [8], and secondly it exhibits several of the desired properties of (3+1)-dimensional quantum chromodynamics including confinement [9]. Thus the (1+1)-dimensional Schwinger model yields an ideal candidate for a "toy model" of quantum chromodynamics.

This project has been concerning itself with the understanding of the theoretical framework of quantum field theory, and thus gauge theories, both in the continuum limit and on a lattice, such that we would be able to derive the lattice Schwinger model ourselves. In Chapter 2 we take as a starting point Dirac's fermionic Lagrangian density and impose local U(1) gauge invariance leading to the derivation of quantum electrodynamics, and thus the continuum Schwinger model. In Chapter 3 the continuum Schwinger model is discretized using lattice gauge theory and then approximated by spin matrices, such that it can be modelled on a computer by a system, for which computations are well known. Lastly, in Chapter 4 we will be studying the properties of confinement and string breaking analytically and numerically using the derived (1+1)-dimensional lattice Schwinger model.

Furthermore, throughout this project we will be using the natural units $\hbar = c = 1$, thus $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$. This choice is conventional when working with quantum field theory, since we are then able to express most dimensionful quantities using only a single unit, and it simplifies our equations since we can omit these constants in every equation.

Chapter 2

Quantum Field Theory and Gauge Theory

In this chapter the basics of quantum field theory and gauge theory will be explained with the goal of deriving the continuum Schwinger model, which is quantum electrodynamics in (1+1)-dimensions. Firstly the relativistic notation of four-vectors and tensors is presented in Section 2.1 and the formalism of the Lagrangian and Hamiltonian density will be presented in Section 2.2. In Section 2.3 the derivation of the continuum Schwinger model will take place starting from the Dirac Lagrangian density (Section 2.3.1), claiming local U(1) gauge invariance[1] (Section 2.3.2), which results in quantum electrodynamics, and converting from the Lagrangian density to the Hamiltonian while limiting ourselves to (1+1)-dimensions (Section 2.3.3).

2.1 Relativistic tensor notation

In this section the notation of relativistic tensors is presented following Refs. [4, 10]. A *four-vector*, which is a tensor of rank 1, is a vector with four entries denoting the coordinates of spacetime, $a = (a_0, a_1, a_2, a_3)$, with index 0 denoting the temporal coordinate and index i ($i = 1, 2, 3$) denoting the spatial coordinates, thus the four-vector might be written as $a = (a_0, \vec{a})$ for $\vec{a}$ being the regular spatial three-vector. We will be using indices to refer to specific entries of the four-vector[2]; Greek indices run

[1] An explanation of U(1) symmetry will be presented in Section 2.3.2, thus the reader does not need to know this in advance.

[2] Indices will also be used to denote the entries of tensors of higher ranks. The number of indices corresponds to the rank of the tensor.

over 0, 1, 2, 3, while Roman indices only run over the spatial coordinates i.e. 1, 2, 3.

In line with the use of indices, we will introduce and employ the Einstein summation convention also known as *Einstein notation*. In this convention repeated indices indicates an implicit summation over these indices, thus

$$a^\mu b_\mu \equiv \sum_\mu a^\mu b_\mu \,. \tag{2.1}$$

One might notice, that the indices in Eq. (2.1) is placed either as a sub- or a superscript. This is due to a tensor[3] being either contravariant, A^μ, or covariant, A_μ. For each contravariant four-vector $a^\mu = (a_0, \vec{a})$ its corresponding covariant four-vector can be calculated as

$$a_\mu = g_{\mu\nu} a^\nu = (a_0, -\vec{a}) \,, \tag{2.2}$$

where the last equality in Eq. (2.2) is due to our choice of the rank 2 metric tensor of spacetime to be the Minkowski metric tensor, $g_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$ [4], since we will be working in a flat spacetime.

Some important four-vectors that will be used later on is the spacetime coordinate vector $x^\mu = (x_0, \vec{x}) = (t, \vec{x})$, which is contravariant, and the covariant derivative

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right) = \left(\partial_0, \vec{\nabla} \right) \,. \tag{2.3}$$

Note that even thought the derivative is covariant, we are actually differentiating with respect to the contravariant spacetime coordinate vector.

We will be using this notation due to it being neat when expressing entrywise equality of equations, and due to it securing covariance of the equations, thus the equations will be independent of the reference frame used [10], which is important, since this is the fundamental principle of relativity.

[3] Tensors of rank 2 or higher is often denoted with a capital letter, thus we here use the capital letter instead of the otherwise non-capital letters used for the four-vectors.

[4] This being the notation of a matrix with the diagonal 1, −1, −1, −1 and 0 elsewhere.

2.2 Lagrangian and Hamiltonian formalism of fields

In quantum field theory the dynamics of a system is described using the Lagrangian formalism [4, 11], but instead of considering the Lagrangian itself one considers the *Lagrangian density* $\mathcal{L}$, which is a function of one or more *fields* $\{\psi_i\}$ and their space- and time-derivatives $\{\partial_\mu \psi_i\}$. From the Lagrangian density the action can be calculated as

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}t \int \mathrm{d}^d x\, \mathcal{L} = \int \mathrm{d}^{d+1} x\, \mathcal{L}\,, \tag{2.4}$$

with L being the Lagrangian and d being the number of spatial dimensions. The action is being presented due to it being defining of symmetries, as we will use in Section 2.3.2.

The Lagrangian formulation is widely used when working with relativistic dynamics, due to the expressions all being explicitly Lorentz invariant [4], hence we will be using this formulation to reach the correct invariance (Section 2.3.2), but due to us working with quantum mechanics, we are going to switch to the Hamiltonian formalism in Section 2.3.3 instead. For this formalism one first needs to set up a Lagrangian density, which can then be used to calculate the *conjugate momentum fields* [4, 11]

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}\,, \tag{2.5}$$

for which $\dot{\psi}_i = \partial_0 \psi_i$. To quantise the equations, the fields and conjugate momentum fields are changed to operators by recasting them in terms of fermionic (or bosonic) creations and annihilation operators [12], also known as ladder operators.

Now the *Hamiltonian density* corresponding to the Lagrangian density can be found using

$$\mathcal{H} = \sum_i \pi_i \dot{\psi}_i - \mathcal{L}\,, \tag{2.6}$$

and from this the *Hamiltonian* is calculated as

$$H = \int \mathrm{d}^d x\, \mathcal{H}\,, \tag{2.7}$$

again with d being the number of spatial dimensions [11].

2.3 The continuum Schwinger model

2.3.1 Dirac field

Due to our interest in examining confinement of quarks, which are spin-½ fermions, we will take as the starting point the most prominent fermionic field, namely the *Dirac field*. For this field the Lagrangian density is

$$\mathcal{L}_D = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \,, \tag{2.8}$$

where $\overline{\psi} = \gamma^0 \psi^\dagger$ is the *Dirac adjoint*, m is the mass of the fermion, ψ is a *Dirac spinor*, and γ^μ are the gamma- or *Dirac-matrices* [4, 13]. For the following descriptions of the above mentioned elements Ref. [4] will be laying the foundation. The Dirac spinor ψ is a four component spinor, with each entry being complex. The upper two entries of ψ corresponds to a particle, while the bottom two corresponds to an antiparticle, and for each of these pairs, the top and bottom entries represent the two possible spin states of the spin-½ particle or antiparticle. Now to defining the Dirac-matrices $\gamma^\mu = (\gamma^0, \gamma^i)$ for $i = 1, 2, 3$. These matrices are 4×4 constant matrices that satisfy the Clifford algebra, thus the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4 \,, \tag{2.9}$$

with $g^{\mu\nu}$ being the Minkowski metric and $\mathbb{1}_4$ the 4×4 identity matrix. In 2×2 block form the Dirac-matrices are

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \,, \tag{2.10}$$

with $\mathbb{1}_2$ being a 2×2 identity matrix and σ^i being the 2×2 Pauli spin matrices.

2.3.2 Local U(1) gauge invariance

When considering systems one may often also consider their symmetries. Generally a *symmetry* is written as an invariance of the action, and thus of the Lagrangian density, with respect to transformation represented by operators [4]. The set of these operators is called the *symmetry group* due to the Lagrangian density being invariant under these operators and combinations

of them [12]. In the following we will consider the unitary gauge U(1), where the name refers to it being the group of unitary 1×1 matrices also known complex numbers with modulus 1 [4]. One of these transformations is the phase of a wave function

$$\psi \longrightarrow U_\alpha \psi = \exp(i\alpha)\, \psi \,, \tag{2.11}$$

with α being a real number. A Lagrangian density being invariant under this transformation is said to be *globally gauge invariant*, thus the wave function's phase will vanish by choosing the correct gauge. Due to U_α being a number and exploiting its unitarity, $U_\alpha^\dagger U_\alpha = \mathbb{1}$, it is trivial to show that the Dirac Lagrangian density $\mathcal{L}_D$ is globally U(1) gauge invariant.

In developing quantum field theory one shall not only impose global gauge invariance but instead demand the more strict *local gauge invariance* [6]. For the transformation in Eq. (2.11) this implies, that the parameter α shall be spacetime-dependent, $\alpha(x^\mu)$. As for most systems, the Dirac Lagrangian density is not locally invariant as it is, thus introduction of new fields for the system to interact with is required [12]. As it happens, the requirement of local gauge invariance supplies a systematic way for determining the equations describing the fundamental interactions: The electromagnetic interaction requires – as we will derive later in this section – local U(1) invariance, the electroweak interaction requires local SU(2)⊗U(1) invariance[5], and the strong nuclear interaction requires local SU(3) invariance [1, 6], thus the Standard Model of elementary particle physics is a gauge theory resulting from demanding local U(1)⊗SU(2)⊗SU(3) symmetry [2].

In the following it will be shown, that the Dirac Lagrangian density, Eq. (2.8), is not locally U(1) invariant, but it will lay the foundation for deriving one that is. The derivation will follow that of Refs. [4, 6, 12]. To lighten the the notation, the parameter $\alpha(x^\mu)$ will be denoted α for the rest of this section. From Eq. (2.11) the conjugated field must transform as[6]

$$\overline{\psi} \longrightarrow \gamma^0(\psi')^\dagger = \gamma^0(U_\alpha\psi)^\dagger = \gamma^0\psi^\dagger U_\alpha^\dagger = \overline{\psi}\exp(-i\alpha) \,, \tag{2.12}$$

for ψ' being the U(1) transformed field, $\psi' = U_\alpha\psi$. Thus Dirac's

[5] SU(N) being the the group of all unitary $N \times N$ matrices with determinant 1 [4].

[6] Hermitian conjugating a product of operators is the product of each operators Hermitian conjugated and in reversed order, i.e. $(XY)^\dagger = Y^\dagger X^\dagger$.

Lagrangian density will transform as

$$\begin{aligned}\mathcal{L}_D = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \longrightarrow &\overline{\psi}\exp(-i\alpha)(i\gamma^\mu\partial_\mu - m)\exp(i\alpha)\,\psi \\ =&\overline{\psi}\exp(-i\alpha)\exp(i\alpha)(i\gamma^\mu[i\partial_\mu\alpha + \partial_\mu] - m)\psi \\ =&\mathcal{L}_D - \overline{\psi}\gamma^\mu\partial_\mu\alpha\psi\end{aligned} \tag{2.13}$$

exploiting again the unitarity of the U(1) symmetry group, $U_\alpha^\dagger U_\alpha = \mathbb{1}$. From Eq. (2.13) it can be seen, that the differential operator is to blame for the lack of local gauge invariance. Thus one must introduce the *gauge covariant derivative*

$$D_\mu = \partial_\mu + iqA_\mu\,, \tag{2.14}$$

which will replace the original derivative. In Eq. (2.14) A_μ is a vector field which under U(1) symmetry transforms as

$$qA_\mu \longrightarrow qA_\mu - \partial_\mu\alpha\,, \tag{2.15}$$

for q being a real number, which in Section 4.2.1 will be shown to be the charge of the particle. Using this new gauge covariant derivative, the Lagrangian density becomes locally U(1) gauge invariant:

$$\begin{aligned}\overline{\psi}(i\gamma^\mu D_\mu - m)\psi \longrightarrow &\overline{\psi}\exp(-i\alpha)(i\gamma^\mu[\partial_\mu + iqA_\mu - i(\partial_\mu\alpha)] - m)\exp(i\alpha)\,\psi \\ =&\overline{\psi}(i\gamma^\mu[\partial_\mu + i\partial_\mu\alpha + iqA_\mu - i\partial_\mu\alpha] - m)\psi \\ =&\overline{\psi}(i\gamma^\mu D_\mu - m)\psi\,.\end{aligned} \tag{2.16}$$

However, there is still one important addition to the Lagrangian density, which we have still not accounted for: For the newly introduced vector field we have only introduced its coupling to the already existing field, ψ, but we also need to include its own "free" term, i.e. its kinetic energy term [6]. The usual form of the kinetic energy is to be a scalar and the square of the derivative of the field, thus we set these as the requirements of A_μ's kinetic energy. Therefore we propose the term $F_{\mu\nu}F^{\mu\nu}$, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.17}$$

being the *gauge field tensor* [14]. Since the gauge field tensor transforms as

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} - \partial_\mu A_\nu\alpha + \partial_\nu A_\mu\alpha\,, \tag{2.18}$$

it is gauge invariant, assuming that α is well-behaved such that the order of the derivatives is irrelevant, $\partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha$, thus they will cancel each other [12].

Combining the above results in the locally U(1) gauge invariant Lagrangian density

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{2.19}$$

where the factor of $-1/4$ is due to convention. This is the Lagrangian density of *quantum electrodynamics*, and for (1+1)D it is known as the Lagrangian density of the *continuum Schwinger model* [15, 16].

2.3.3 Schwinger model Hamiltonian

In this section the arguments will mostly follow that of Ref. [15]. We choose to make the common gauge restriction to work in the temporal gauge, thus the temporal part of the the vector field A_μ is zero, $A_0 = 0$. Using this choice of gauge and working in (1+1)-dimensions, i.e. $\mu, \nu = 0, 1$, one can reduce Eq. (2.19) to

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - \gamma^1 q A_1 - m)\psi + \frac{1}{2} E_1 E^1 , \tag{2.20}$$

where $E_1 = F_{01} = \partial_0 A_1$ is the electric field in the direction of x_1. Computing the last term of Eq. (2.20) requires one to recall, that a contravariant tensor can be found by flipping the sign on the entries in the covariant tensor for which the indices are non-zero, $F^{ij} = -F_{ij}$ due to the choise of metric, and likewise for a vector $A^i = -A_i$, and that the definition of the gauge field tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, thus it's diagonal is 0, hence in (1+1)-dimensions, i.e. for $\mu, \nu = 0, 1$,

$$\begin{aligned} F_{\mu\nu} F^{\mu\nu} &= F_{00} F_{00} + F_{01} F_{01} + F_{10} F_{10} - F_{11} F_{11} \\ &= (\partial_0 A_1 - \partial_1 A_0)^2 + (\partial_1 A_0 - \partial_0 A_1)^2 \\ &= 2\,(\partial_0 A_1)^2 \\ &= -2\,(\partial_0 A_1)\left(\partial^0 A^1\right) \\ &= -2 E_1 E^1 , \end{aligned} \tag{2.21}$$

where we have used the fact, that we are working in the temporal gauge, i.e. $A_0 = 0$.

The continuum Schwinger model Hamiltonian density can now be found using Eqs. (2.5) to (2.7) on Eq. (2.20), and recalling $E_1E^1 = -(E_1)^2 = -(\partial_0 A_1)^2$. The Hamiltonian density therefore becomes

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial(\partial_0 A_1)}\partial_0 A_1 + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial_0\psi - \mathcal{L} \\ &= -\frac{2}{2}E_1E_1 + \overline{\psi}i\gamma^0\partial_0\psi - \overline{\psi}(i\gamma^\mu\partial_\mu - \gamma^1 qA_1 - m)\psi - \frac{1}{2}E_1E^1 \\ &= E_1E^1 + \overline{\psi}i\gamma^0\partial_0\psi - \overline{\psi}i\gamma^0\partial_0\psi - \overline{\psi}(i\gamma^1\partial_1 - \gamma^1 qA_1 - m)\psi - \frac{1}{2}E_1E^1 \\ &= -\overline{\psi}(i\gamma^1 D_1 - m)\psi + \frac{1}{2}E_1E^1 \,,\end{aligned} \tag{2.22}$$

and thus we get the Hamiltonian as the spatial integral of the above density

$$H = \int \mathrm{d}x \left[\overline{\psi}(-i\gamma^1 D_1 + m)\psi + \frac{1}{2}E_1E^1\right] . \tag{2.23}$$

As it shall later be shown, it is important to introduce the *generator* of the U(1) symmetry

$$G(x) = \partial_1\frac{\partial\mathcal{L}}{\partial(\partial_0 A_1)} - q\psi^\dagger\psi = \partial_1 E^1 - q\psi^\dagger\psi \,, \tag{2.24}$$

where the electric field is the conjugate momentum field to the gauge field, and the generator generates the U(1) symmetry since it recovers the gauge transformations of ψ and A_μ by a similarity transformation with the unitary operator $\exp(-i\alpha G/q)$, i.e. $\exp(i\alpha G/q)\,\psi\exp(-i\alpha G(x)/q) = \exp(i\alpha)\,\psi$ [12].

Chapter 3

Quantum Field Theory on a Lattice

In this chapter we will go through the process of discretizing the continuum Schwinger model Hamiltonian. This serves two purposes: Firstly, quantum field theory is well-known for leading to divergences, which manifests itself for particle momenta near zero and infinite due to integration with respect to the momenta, and are are called the *ultraviolet divergence* and the *infrared divergence* respectively [4, 12]. These can be fixed using the lattice gauge theory of spacetime discretization proposed by Wilson [5] in which one considers a finite lattice of spacetime points separated each by a *lattice spacing a*. Restricting ourselves to a finite lattice spacing sets a lower bound for the wavelength, which in turn sets an upper bound for the possible momenta, thus fixes the infrared divergence, while restricting the lattice to be finite fixes the ultraviolet divergence due to it setting an upper bound for the wavelength and thus a lower bound for the momenta [17].

The second purpose of the discretization is to simulate the system using the lattice Schwinger model on a computer. A computer is not able to handle continuous variables, since this would result in infinitely many points to calculate, thus the variables must be discretized for the computer to be able to work with them. Introducing the discretization from start one hopes to be able to analytically solve a larger part of the system before using computer simulations [12].

As mentioned above, this chapter will focus on the discretization process of the continuum Schwinger model Hamiltonian. Firstly, the overall discretization to lattice gauge theory is presented in Section 3.1. In Section 3.2 the discretization is taken one

step further by mapping the operators and fields to spin operators, since spin systems are well known and thus we already know how to manipulate them. This is done in two steps: The matter fields are mapped to spin-½ operators through the Jordan-Wigner transformation (Section 3.2.1) and the gauge field operator is mapped to spin-S operators using quantum link models (Eq. (3.16)). Lastly in Section 3.2.3 the (1+1)-dimensional Schwinger model Hamiltonian is rewritten in terms of the spin operators.

As in Section 2.3.3 we will in this chapter only be concerned with (1+1)-dimensions, since our main goal is to derive the lattice Schwinger model.

3.1 Discretization

In this section we will be discretizing the continuum Schwinger model Hamiltonian, Eq. (2.23). We chose to only discretize space while keeping the time continuous, thus the discretization will follow Refs. [18–20]. The discretization of the space is accomplished by considering the space as an infinite lattice[1], where the matter is restricted to only live on the lattice sites, thus the spacial coordinates are discretized as

$$x \longrightarrow x_n = an\,, \tag{3.1}$$

where a is the lattice spacing and n the lattice site number. Thus the matter field at the n'th lattice site will be denoted $\psi_n = \psi(x_n)$. Integration over space will be replaced by a sum multiplied by the lattice spacing for the correct spacing between matter sites,

$$\int \mathrm{d}x \longrightarrow a \sum_n \,. \tag{3.2}$$

When discretizing the spatial derivative, one might propose the derivative to be the the difference in the field at two nearby sites divided by the lattice spacing, $\partial_1 f(x) \longrightarrow (f_n - f_{n-1})/a$, but this turns out to cause difficulties, namely the *lattice fermion doubling* problem. This is a problem of degeneracy of fermions, which is 2^d for d discretized dimensions, thus the discretized theory will

[1] We choose to not consider a finite lattice, thus not solving the ultraviolet divergence, since it is tedious to consider the boundary conditions and since it is the infrared divergence (and not the ultraviolet), which dominates our calculations and simulations [21].

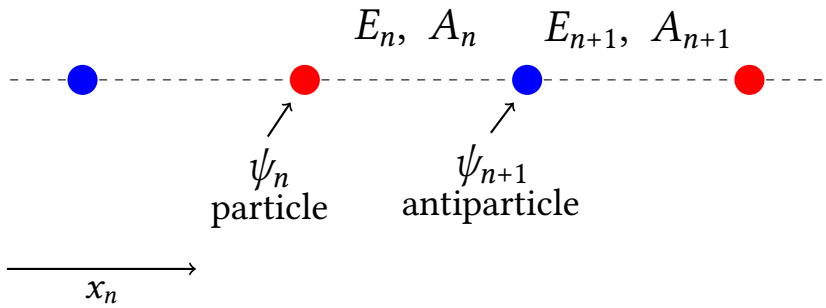

FIGURE 3.1: Segment of a one-dimensional lattice with n even, thus the red and blue matter sites are the matter fields of a particle and an antiparticle respectively, and the gauge fields linking the two nearby matter sites are indicated by dashed lines. Inspired by Ref. [12, Fig. 4.1].

include 2^d fermions for each original (continuum) fermion [22]. Instead of having $2^1 = 2$ fermions in our space discretized (1+1)-dimensional model, we will instead solve the fermion doubling problem by imposing *staggered fermions* also known as Kogut-Susskind fermions [20, 23]: One replaces the four-component spinor ψ with a two-component spinor, ψ_n, which entries corresponds to the two possible spin states, and impose ψ_n at even n to represent the matter field of a particle, while representing an antiparticle for odd n, see Fig. 3.1. By representing either a particle or an antiparticle, the matter fields necessitates a staggered mass term, $(-1)^n \psi_n^\dagger m \psi_n$, to replace the mass term in Eq. (2.23) [23], such that particles are associated with positive mass while antiparticles are associated with negative mass. Now for discretizing the spatial derivative of the matter field, one still employs the difference of two nearby matter fields divided by the distance between them, but the nearby fields are now interpreted as the two nearby either particle or antiparticle matter fields, thus [23]

$$\partial_1 \psi \longrightarrow \frac{\psi_{n+1} - \psi_{n-1}}{2a} \,. \tag{3.3}$$

Having restricted matter fields to live only at the lattice sites it is natural to propose the gauge fields (thus also the electric fields), which couple to the matter fields, to live at the lattice links connecting the lattice sites [15]. We will define A_n (thus also E_n) to be the field at the lattice link between sites n and $n+1$, see Fig. 3.1. Discretizing using only the above stated information will result in a non-locally gauge invariant Hamiltonian, due to the product of fermionic matter fields at different latter sites –

the term containing $\overline{\psi}\partial_1\psi$ – but it can be made locally gauge invariant by compensating for the matter fields by introducing the unitary *link operator*

$$U_n = \exp\left(iaqA_n\right) \tag{3.4}$$

living on the links between matter sites, and restricting the gauge field to only enter the discretized Hamiltonian through this operator [20]. Using the generator of the U(1) symmetry, Eq. (2.24), discretized using the above stated techniques and with the derivative discretized as $\partial_1 f(x) \longrightarrow (f_n - f_{n-1})/a$,

$$G_n = \frac{E_n - E_{n-1}}{a} - q\psi_n^\dagger\psi_n\,, \tag{3.5}$$

one is able to show, that terms of the form

$$E_n^2\,, \quad \psi_n^\dagger\psi_n\,, \quad \text{and} \quad \psi_n^\dagger U_n\psi_{n+1} \tag{3.6}$$

is locally gauge invariant [12, 20].

We have now introduced all the tools needed to discretize the continuum Schwinger model Hamiltonian, Eq. (2.23). This Hamiltonian consists of four terms, which will be considered for small a, due to us wanting the lattice formulation to approach the continuum for which the limit is $a \longrightarrow 0$. Considering first the terms

$$\int \mathrm{d}x \left[\psi^\dagger m\psi + \frac{1}{2}E_1E^1\right] \longrightarrow a\sum_n \left[(-1)^n m\psi_n^\dagger\psi_n + \frac{1}{2}E_n^2\right]\,, \tag{3.7}$$

which is already locally gauge invariant terms according to Eq. (3.6). Taking the staggered mass into account, the first term approaches the mass term of the continuum Hamiltonian, and the second approaches the electric field term, thus Eq. (3.7) shall be used for the discretized Hamiltonian. Considering now the last terms of the continuum Schwinger model Hamiltonian one notices that

$$\begin{aligned}
&\int \mathrm{d}x \left(-i\psi^\dagger\left[\partial_1 + iqA_1\right]\psi\right) \\
&\quad\longrightarrow -ia\sum_n \left(\psi_n^\dagger\frac{\psi_{n+1} - \psi_{n-1}}{2a} + \frac{1}{2}iqA_n\left[\psi_n^\dagger\psi_{n+1} + \psi_{n+1}^\dagger\psi_n\right]\right) \\
&\quad= -\frac{ia}{2a}\sum_n \left(\psi_n^\dagger\left[1 + iaqA_n\right]\psi_{n+1} - \psi_{n+1}^\dagger\left[1 - iaqA_n\right]\psi_n\right) \\
&\quad\simeq -\frac{ia}{2a}\sum_n \left(\psi_n^\dagger U_n\psi_{n+1} - \text{h.c.}\right)\,,
\end{aligned} \tag{3.8}$$

where –h.c. denotes the subtraction of the Hermitian conjugate of the previous term, the indices of some of the terms have been shifted $\psi_n^\dagger \psi_{n-1} \longrightarrow \psi_{n+1}^\dagger \psi_n$, which is allowed due to the sum being over all of the lattice, and a symmetric configuration is chosen for the matter fields $\psi^\dagger \psi \longrightarrow [\psi_n^\dagger \psi_{n+1} + \psi_{n+1}^\dagger \psi_n]/2$ due to us not knowing for which of the lattice sites the matter field shall be Hermitian conjugated. In all of the above the gamma-matrices have disappeared, but this turns out not to be a problem since the disappearance of these corresponds to a specific choise of 2×2 matrices added when performing the continuum limit properly [12].

Combining the locally gauge invariant terms found in Eqs. (3.7) and (3.8) the discretized Hamiltonian of the lattice Schwinger model becomes

$$H = a \sum_n \left[(-1)^n m \psi_n^\dagger \psi_n - \frac{i}{2a} \left(\psi_n^\dagger U_n \psi_{n+\hat{1}} - \text{h.c.} \right) + \frac{1}{2} E_n^2 \right] , \quad (3.9)$$

which in the continuum limit $a \longrightarrow 0$ reproduces the continuum Schwinger model Hamiltonian.

3.2 Mapping to spin system

In the previous section we have gone through the derivation of the lattice Scwhinger model, Eq. (3.9), from the continuum Schwinger model. This discretization allows us to simulate a system using the lattice Schwinger model Hamiltonian, since computers are not able to process continuous variables but need discretized versions of them. In the following we will provided yet another step in the process of making the Hamiltonian easier computable by introducing transformations mapping the operators to that of spin systems. This will be done in two steps: Firstly the matter fields are transformed using the Jordan-Wigner transformation, and secondly the gauge fields are transformed using quantum link models.

3.2.1 Jordan-Wigner transformation

The *Jordan-Wigner transformation* is a transformation mapping fermionic creation and annihilation operators onto spin operators of the spin-½ system for one-dimensional lattices, thus we will use this to transform the fermions on the matters sites to

spin step operators. This section will take its starting point in Refs. [12, 20, 24].

Remembering the definition of the step spin operators [13]

$$\sigma^{\pm} = \frac{\sigma_x \pm i\sigma_y}{2} , \tag{3.10}$$

one may easily find the anticommutator between these for the same lattice site to be $\{\sigma_n^+, \sigma_n^-\} = 1$, as would also be expected from the creation and annihilation operators, $\left\{ \psi_n^\dagger, \psi_n \right\} a = 1$; the a being due to the mass fields each having the unit of $a^{-d/2}$ for d being the number of spatial dimensions [25]. Thus one might be tempted to use the direct mapping $\psi_n^\dagger = a^{-1/2}\sigma_n^+$ and $\psi_n = a^{-1/2}\sigma_n^-$, but then the operators would commute for different lattice sites ($n \neq m$) [23], $\left[\psi_n^\dagger, \psi_m\right] = [\sigma_n^+, \sigma_m^-] = 0$ [2], which is not the case for fermionic operators, for which these must always anticommute [13], $\left\{ \psi_n^\dagger, \psi_m \right\} = 0$.

To accommodate the above stated problem we introduce the Jordan-Wigner transformation

$$\begin{aligned} \psi_n &= \frac{1}{\sqrt{a}} \prod_{m=0}^{n-1} (i\sigma_m^z)\sigma_n^- , \\ \psi_n^\dagger &= \frac{1}{\sqrt{a}} \prod_{m=0}^{n-1} (-i\sigma_m^z)\sigma_n^+ . \end{aligned} \tag{3.11}$$

for which the added product part corresponds to all the lattice sites from one end of the lattice to the site concerned, thus ensuring the anticommutator, which the previous stated (direct) transformation did not satisfy. It can be shown fairly easy, that the Jordan-Wigner transformation ensures the correct anticommutators for $n \neq m$

$$\{\psi_n, \psi_m\} = \frac{(-1)^n i}{a} \{\sigma_n^-, \sigma_n^z\} \prod_{l=n+1}^{m-1} (i\sigma_l^z)\sigma_m^- = 0 , \tag{3.12a}$$

$$\left\{ \psi_n^\dagger, \psi_m^\dagger \right\} = \frac{(-1)^n i}{a} \{\sigma_n^+, \sigma_n^z\} \prod_{l=n+1}^{m-1} (-i\sigma_l^z)\sigma_m^+ = 0 , \quad \text{and} \tag{3.12b}$$

$$\left\{ \psi_n^\dagger, \psi_m \right\} = \frac{i}{a} \{\sigma_n^+, \sigma_n^z\} \prod_{l=n+1}^{m-1} (i\sigma_l^z)\sigma_m^- = 0 , \tag{3.12c}$$

[2] The spin operators only operate on their designated lattice due to them being connecting using the tensor product $A_n B_m = A_n \otimes B_m = (A_n \otimes \mathbb{1})(\mathbb{1} \otimes B_m)$, thus operators with $n \neq m$ won't interfere with each other.

by splitting the product parts[3] and using the anticommutators $\{\sigma_n^{\pm}, \sigma_n^z\} = 0$ and $\{\sigma_n^{\pm}, \sigma_m^z\} = 0$, while the transformation still preserves the anticommutator

$$\left\{ \psi_n^\dagger, \psi_n \right\} a = \{\sigma_n^+, \sigma_n^-\} = 1, \tag{3.12d}$$

due to the added product part in this case cancels with itself. Due to the possibility of inverting the Jordan-Wigner transformation [24] the two formulations – the spin formulation and the creation and annihilation operator formalism – are equivalent, hence exactly the same information is contained in the system in both formalisms [12].

3.2.2 Quantum link models

In Section 3.1 we made the first approximations to our continuum theory by discretizing it, and in this section we will further approximate the discretized theory for it to be expressed by spins [26], which is a system well suited for quantum computation, since we are familiar with manipulating it. This further approximation is done using *quantum link models*, which restricts the Hilbert space of the gauge fields to be finite instead of infinite [27].

In the quantum link models the link operator will no longer be connected to the continuum theory by $U = \exp(iaqA_n)$ but instead we let go of the unitarity of the operator and impose U to be an independent operator with commutation relation [12]

$$\left[U_n, U_m^\dagger\right] = \frac{2}{q} E_n \delta_{n,m} \,, \tag{3.13}$$

which conserves the gauge symmetry and do not interfere with commutation relation of the electric field and the link operator from Section 3.1 [12] [4]

$$[E_n, U_n] = iaq[E_n, A_n]U_n = qU_n \,, \tag{3.14}$$

[3] The product parts shall be split into into three parts: $[0, \dots, n-1]$, n and $[n+1, \dots, m-1]$, thus requiring $m > n$, but due to the definition of anticommutators $\{A, B\} = \{B, A\}$, thus one can switch all the indices ($n \leftrightarrow m$) and while the equations remain true. Therefore all cases of $n \neq m$ are covered.

[4] Here one use the commutator rule $[B, \exp(\lambda C)] = \lambda c \exp(\lambda C)$, with λ being a number and $[B, C] = c$, which is found from series expanding the exponential and using the commutator identities.

where $[A_n, E_m] = i\delta_{n,m}a^{-1}$ since E_n is the conjugate momentum field of A_n (see Section 2.3.3). The commutators Eqs. (3.13) and (3.14) are reminiscent of the SU(2) algebra of angular momentum [27, 28], thus stating U_n and $U_n^\dagger$ to be the step-up and -down operators respectively and E_n be the z-component of the angular momentum – with the correct proportionality constant – one obtains the correct commutations relations

$$\left[L_i, L_j\right] = i\epsilon_{ijk}L_k \,. \tag{3.15}$$

Here $i, j, k = x, y, z$ and ϵ_{ijk} is the Levi-Civita symbol defined such that even and odd permutations of ijk yields $+1$ and -1 respectively and no permutation, i.e. repeated indices, yields 0.

Now letting the angular momentum be that of the spin formalism, E_n, U_n and $U_n^\dagger$ can be represented by finite spin S using the spin operators S^i $(i = z, +, -)$ as

$$E_n = qSS_n^z \,, \quad U_n = S_n^+ \,, \quad \text{and} \quad U_n^\dagger = S_n^- \,. \tag{3.16}$$

3.2.3 Spin model of the lattice Schwinger model

Equipped with the tools of transforming matter fields and gauge fields to be dependent of the spin operators, we can now remodel the Hamiltonian of the lattice Schwinger model in terms of spin. Substituting the operators of Eq. (3.9) according to Eqs. (3.11) and (3.16) results in the Hamiltonian

$$\begin{aligned} H &= a\sum_n \left[\frac{(-1)^n m}{a}\sigma_n^+\sigma_n^- - \frac{i}{2a^2}\left(\sigma_n^+ S_n^+ i\sigma_n^z\sigma_{n+1}^- - \text{h.c.}\right) + \frac{1}{2}\left(qSS_n^z\right)^2\right] \\ &= \sum_n \left[(-1)^n\frac{m}{2}\sigma_n^z - \frac{1}{2a}\left(\sigma_n^+ S_n^+\sigma_{n+1}^- + \text{h.c.}\right) + \frac{aq^2S^2}{2}\left(S_n^z\right)^2\right] \,, \end{aligned} \tag{3.17}$$

where we have used the fact that the multiplication of the step up and down spin operators is $\sigma_n^+\sigma_n^- = (1 + \sigma_n^z)/2$, where the offset constant has been removed due to being physically irrelevant, that the Hermitian conjugate of the imaginary unit is $i^\dagger = -i$, and that $\sigma_n^+\sigma_n^z = -\sigma_n^+$ [5] thus $(-\sigma_n^+)^\dagger = -(\sigma_n^+)^\dagger$.

[5] This can be used since the operators on different sites and links are connected using the tensor product, i.e. $A_nB_mC_n = (A_n \otimes \mathbb{1})(\mathbb{1} \otimes B_m)(C_n \otimes \mathbb{1}) = (A_nC_n) \otimes B_m$ for $n \neq m$, thus σ_n^z can be "pulled through" S_n^+ due to the identity in the tensor product.

Chapter 4

Confinement in the Lattice Schwinger Model

In this chapter one of the uses of the (1+1)-dimensional lattice Schwinger model is examined, namely the use of the Schwinger model as a "toy model" for quantum chromodynamics. More specifically we will be examining the properties of confinement and string breaking. In Section 4.1 the two above mentioned phenomenons are described, and in Section 4.2 the properties are examined analytically and numerically (Section 4.2.2) after the lattice Schwinger model Hamiltonian is converted to fit the requirements of a spin-1 gauge field and the state configurations are found (Section 4.2.1). Lastly further examination possibilities of confinement and string breaking is presented in Section 4.3.

4.1 Confinement and string breaking

The contents of this section is mostly found in Refs. [6, 19]. *Confinement*, also known as colour confinement, is one of the most fundamental properties of quantum chromodynamics, the quantum field theory describing strong interactions of quarks and gluons. Confinement is the experimentally observed fact that colourful particles, such as quarks, cannot exist freely but only in hadrons as mesons and baryons. This section will consider the theory of confinement using mesons as example, but it could as well have been using baryons.

The confinement of quarks is due to the shape of the potential of quantum chromodynamics. Most of the potentials that we normally encounter, i.e. the electromagnetic potential and the

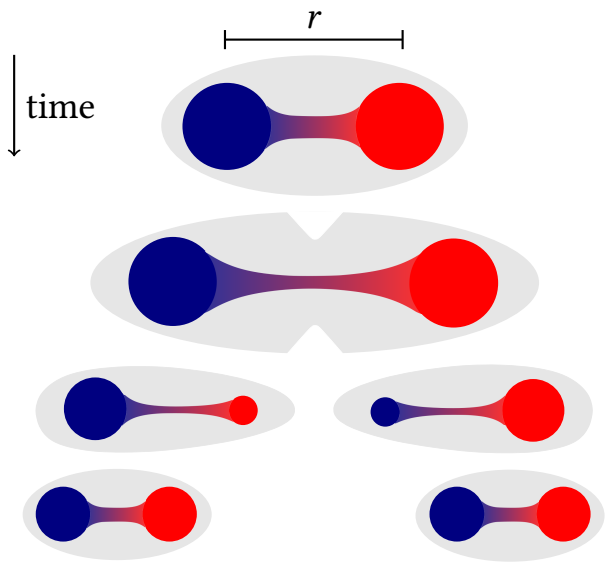

FIGURE 4.1: As a quark (red) and an antiquark (blue) in a meson are pulled away from each other (distance r increases), the energy needed increases until the string tension of the flux tube (blue to red gradient) between the particles is sufficient enough, thus the flux tube break, known as string breaking, and the energy stored in the flux tube creates a new quark and antiquark such that there is now two mesons. That a quark and an antiquark can never exist on its own is know as colour confinement. Inspired by Ref. [30].

gravitational potential, are decaying due to a negative power of the interparticle distance, i.e. r^{-1} for the two before mentioned potentials, but the potential of quantum chromodynamics is *linearly growing* as a function of the interquark distance. In other words, the quark-antiquark pair cannot escape each other; they are *confined*. It might be worth noting that at short interquark distances the behaviour of the potential is actually slightly of the Coulomb form [29], but it relatively quickly turns linear, thus the linear approximation is the commonly used for confinement.

A possible mechanism resulting in the linearly growing potential is that the gluon field of the strong interaction creates a narrow colour *flux tube* (also known as a string) between the quark and antiquark [31]. Thus the potential becomes $V(r) \approx \sigma r$ for r being the interquark distance and σ being the string tension. At some point the amount of energy in the flux tube becomes sufficient enough for it to be energetically favourable to break the tube, which is known as *string breaking*, see Fig. 4.1.

As an analogy we consider an elastic band: As it is stretched the energy stored in the band grows and at some point the band snaps. Instead of one elastic band with two end points you are now left with two bands with each two ends. Thus new ends are created due to the release of stored energy. Translating this to a system of a meson, Fig. 4.1, it can be seen that at some point it becomes energetically favourable for the string to break and create a quark-antiquark pair, thus resulting in two mesons.

4.2 Lattice Schwinger model exhibiting confinement

In Section 3.2.3 we remodelled the (1+1)-dimensional lattice Schwinger model into containing spin operators, Eq. (3.17). For this we have chosen to define the spin states as $|0\rangle = (1,\ 0)^T$ and $|1\rangle = (0,\ 1)^T$, i.e. $|0\rangle$ is the ground state and $|1\rangle$ is the excited state, for the spin-½ system, since this is easy to generalise to larger spin systems as spin-1 by expanding the existing states with a new 0-row and stating a new state with only a 1-row at the last row (the $(2S + 1)^{\text{th}}$ row), i.e. for spin-1: $|0\rangle = (1,\ 0,\ 0)^T$, $|1\rangle = (0,\ 1,\ 0)^T$ and $|2\rangle = (0,\ 0,\ 1)^T$. This definition of the states for the spin-½ system results in the following σ^z Pauli matrix

$$\sigma^z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.1}$$

and using that the matrix elements of $\sigma^{\pm}$ are found as [13]

$$\left\langle s\middle|\sigma^{\pm}\middle|s'\right\rangle = \sqrt{(S \mp m_s)(S \pm m_s + 1)}\,\delta_{m_s+1,m_{s'}} \tag{4.2}$$

with s and s' being two states with spin projection m_s and $m_{s'}$ respectively and S being the spin of the system, the spin step operators attains the form

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

In Eq. (3.17) S^z and S^+ are the spin-S analogies of the Pauli matrix σ^z and the spin step operator σ^+ respectively. Due to us wanting to simulate confinement, the gauge field spin operators is required to be that of the spin-1 system[1], since the possibility of

[1] They are required to have integer spin, since this enables us to have a state of zero field. Spin-1 is just the lowest non-trivial integer spin, thus it is easier to work with.

zero field between the mesons is necessary. The spin operators S^z and S^+ of a spin-1 system can be calculated by first coupling the spin projections to the states: $S^z|0\rangle = -1|0\rangle$, $S^z|1\rangle = 0|1\rangle$ and $S^z|2\rangle = 1|2\rangle$, which becomes the diagonal of S^z [2], and then use Eq. (4.2) with $S = 1$ for S^+, hence

$$S^z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad S^+ = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{4.4}$$

Letting the gauge field be represented by spin-1 operators, thus setting $S = 1$, results in the Hamiltonian

$$H = \sum_n \left[(-1)^{n+1} \frac{m}{2} \sigma_n^z - \frac{1}{2a} \left(\sigma_n^+ S_n^+ \sigma_{n+1}^- + \text{h.c.} \right) + \frac{aq^2}{2} \left(S_n^z \right)^2 \right], \tag{4.5}$$

with the spin operators defined in Eqs. (4.1), (4.3) and (4.4).

4.2.1 Computing the states

To be able to simulate confinement it is necessary to know the interpretation of the states, i.e. for matter states the above mentioned definition of $|0\rangle$ and $|1\rangle$ necessitates $|1_n\rangle$ to be interpreted as the presence of a particle if n is even and a vacant site if n is odd, while $|0_n\rangle$ means the site is vacant for even n and that an antiparticle is present for odd n.

We add to the generator, Eq. (3.5), a constant of $q[(-1)^n - 1]/(2a)$ which ensures particles and antiparticles an intuitive charge [12],

$$G_n = \frac{E_n - E_{n-1}}{a} - q\psi_n^\dagger \psi_n - \frac{(-1)^n - 1}{2a}. \tag{4.6}$$

As with the Hamiltonian in Section 3.2.3 the generator is remodelled using the spin formalism with $S = 1$, thus

$$G_n = \frac{q}{a} \left(S_n^z - S_{n-1}^z - \frac{1}{2} \left[\sigma_n^z + (-1)^n \right] \right). \tag{4.7}$$

Calculating now the eigenvalues of the the generator for the 18 different configurations of two spin-1 gauge fields surrounding a spin-½ matter field results in Table 4.1. Examining the table one may notice that for even sites $\left| 0_g\, 1_m\, 1_g \right\rangle$ and $\left| 0_g\, 0_m\, 1_g \right\rangle$ (m and g is here denoting matter fields and gauge fields respectively) only

[2] The matrix elements of S^z can be found as $\left\langle s \middle| S^z \middle| s' \right\rangle = \frac{m_s}{S} \delta_{m_s, m_{s'}}$ [13].

State	aG_n/q	
	Even n	Odd n
$\lvert 0_g\, 0_m\, 0_g \rangle$	0	1
$\lvert 0_g\, 1_m\, 0_g \rangle$	−1	0
$\lvert 0_g\, 0_m\, 1_g \rangle$	1	2
$\lvert 0_g\, 1_m\, 1_g \rangle$	0	1
$\lvert 0_g\, 0_m\, 2_g \rangle$	2	3
$\lvert 0_g\, 1_m\, 2_g \rangle$	1	2
$\lvert 1_g\, 0_m\, 0_g \rangle$	−1	0
$\lvert 1_g\, 1_m\, 0_g \rangle$	−2	−1
$\lvert 1_g\, 0_m\, 1_g \rangle$	0	1
continued to the right		

State	aG_n/q	
	Even n	Odd n
continued from the left		
$\lvert 1_g\, 1_m\, 1_g \rangle$	−1	0
$\lvert 1_g\, 0_m\, 2_g \rangle$	1	2
$\lvert 1_g\, 1_m\, 2_g \rangle$	0	1
$\lvert 2_g\, 0_m\, 0_g \rangle$	−2	−1
$\lvert 2_g\, 1_m\, 0_g \rangle$	−3	−2
$\lvert 2_g\, 0_m\, 1_g \rangle$	−1	0
$\lvert 2_g\, 1_m\, 1_g \rangle$	−2	−1
$\lvert 2_g\, 0_m\, 2_g \rangle$	0	1
$\lvert 2_g\, 1_m\, 2_g \rangle$	−1	0

TABLE 4.1: The eigenvalues of the generator from Eq. (4.7) for the 18 configurations of two gauge fields (denoted with a g) with spin-1 and a matter field (denoted with an m) with spin-½ between them.

differ by the presence of a particle and of the value q. The same can be shown for odd sites $\lvert 1_g\, 0_m\, 0_g \rangle$ and $\lvert 1_g\, 1_m\, 0_g \rangle$, which only differ by the presence of an antiparticle and by the values $-q$. Thus by understanding the eigenvalue of aG_n as a static background (anti)particle [12], it is shown that q is in fact, as stated in Section 2.3.2, the charge of a particle and likewise $-q$ the charge of an antiparticle.

Now the direction of the electric field is to be found. For this we postulate $\lvert 1_g \rangle$ to be the zero field, which is in agreement with $G_n \lvert 1_g\, 0_m\, 1_g \rangle = 0 \lvert 1_g\, 0_m\, 1_g \rangle$ for even n since this indicates neither a field nor a particle, i.e. vacuum, which is expected to have zero charge. Using now the gauge interaction operator $\sigma_n^+ S_n^+ \sigma_{n+1}^-$ from the Hamiltonian, Eq. (4.5), on the matter site and thus right hand link of $\lvert 1_g\, 0_m\, 1_g \rangle$ one gets $\sigma_n^+ S_n^+ \sigma_{n+1}^- \lvert 1_g\, 0_m\, 1_g \rangle = \lvert 1_g\, 1_m\, 2_g \rangle$, thus we have created a particle and changed the right hand gauge field. We would like the field to point from particles to antiparticles, thus $\lvert 2_g \rangle$ must be interpreted as an electric field pointing right. The two observations of $\lvert 1_g \rangle$ and $\lvert 2_g \rangle$ happens to be consistent with the rest of the states, thus the interpretations becomes $\lvert 0_g \rangle = \lvert \leftarrow \rangle$, $\lvert 2_g \rangle = \lvert \rightarrow \rangle$ and $\lvert 1_g \rangle = \lvert 0 \rangle$, the latter indicating zero field.

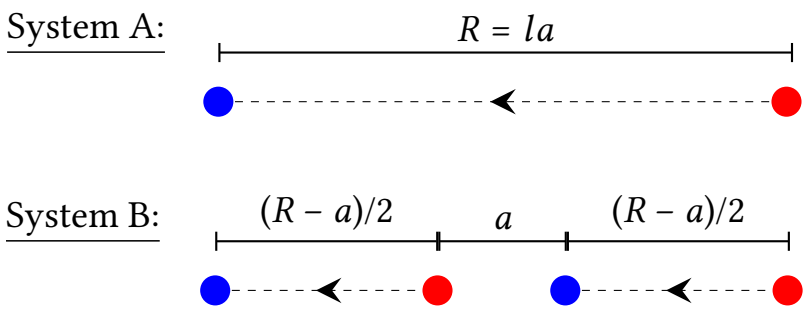

FIGURE 4.2: The two states for which the study of analytical string breaking is performed. The energy of the two systems is calculated and it is found, that for $R < R_0$ the energy of System A, a meson with interquark distance $R = la$, is smallest, but at the string breaking length R_0 this changes, and it is energy efficient to break the flux tube and create a new quark-antiquark pair, such that there is now two mesons with interquark distance $(R - a)/2$ each.

4.2.2 Confinement and string breaking

Analytical study: Variables favourable for string breaking

For the analytical study of confinement and string breaking the systems shown in Fig. 4.2 are used. System A illustrates a meson with interquark distance $R = la$ for which a is the lattice spacing and l is an odd integer fulfilling $(l-1)/2$ also being an odd integer[3]. When R becomes sufficiently large, R_0, string breaking occurs, thus we will now be in system B instead, where there is now two mesons with interquark distance $(R - a)/2$ each. In this section the connection between the values of lattice spacing, mass and charge favourable for string breaking will be found by computing and comparing the energy of the two systems.

The energy of the systems is found by computing the expectation value of the Hamiltonian Eq. (4.5) with respect to the two states respectively. Let us consider the three terms of the Hamiltonian, for convenience defined as $H = \sum_n [H_m + H_I + H_E]$, individually. Firstly we will consider the interaction term H_I, which consists of step operators and their Hermitian conjugates. It is known that $\langle s'|\sigma^{\pm}|s\rangle \propto \delta_{s',s+1}$ for states $|s\rangle$ and $|s'\rangle$, and equally for $S^{\pm}$, thus $\langle H_I\rangle = \langle\Psi|H_I|\Psi\rangle = 0$ for $|\Psi\rangle$ being the state of either System A or B. Now let us consider the mass term H_m, by con-

[3]Both requirements is due to there being an odd number of lattice spacings between matter and antimatter sites (the staggering effect).

sidering the expectation value with a (anti)quark present and at a vacant site. Recall that $\sigma^z |0_m\rangle = 1$ and $\sigma^z |1_m\rangle = -1$, thus for even n we have

$$\langle H_m \rangle_{1_m} = \left\langle 1_m \middle| (-1)^{n+1} \frac{m}{2} \sigma_n^z \middle| 1_m \right\rangle = \left\langle 1_m \middle| -1 \frac{m}{2} (-1) \middle| 1_m \right\rangle = \frac{m}{2}\,, \quad \text{and}$$
$$\langle H_m \rangle_{0_m} = \left\langle 0_m \middle| (-1)^{n+1} \frac{m}{2} \sigma_n^z \middle| 0_m \right\rangle = \left\langle 0_m \middle| -1 \frac{m}{2} 1 \middle| 0_m \right\rangle = -\frac{m}{2}\,, \tag{4.8}$$

thus by inserting a quark the energy has increased with $\langle H_m \rangle_{1_m} - \langle H_m \rangle_{0_m} = m$. The same can be shown for the antiquark where $(-1)^{n+1}$ changes the sign such that the energy from the presence of an antiquark results in the energy increase $\langle H_m \rangle_{0_m} - \langle H_m \rangle_{1_m} = m$ (for odd n). Thus the energy from the mass term is $E_m = km$ for k being the number of quarks and antiquarks in the system. Lastly we are considering the electric field term H_E. For this term it is trivial to observe that $(S^z)^2 \left|0_g\right\rangle = 1 \left|0_g\right\rangle$, $(S^z)^2 \left|1_g\right\rangle = 0 \left|1_g\right\rangle$ and $(S^z)^2 \left|2_g\right\rangle = 1 \left|2_g\right\rangle$, thus for every link l with a non-zero field the energy is $aq^2/2$.

From the above observations the expectation value of the Hamiltonian becomes[4]

$$\langle H \rangle = km + \frac{aq^2}{2} l \tag{4.9}$$

with k being the number of quarks and antiquarks and l satisfying $L = la$ as before. Using Eq. (4.9) the energy of the two systems in Fig. 4.2 becomes

$$E_A = 2m + \frac{aq^2}{2} l\,, \quad \text{and} \quad E_B = 4m + \frac{aq^2}{2}(l-1)\,. \tag{4.10}$$

Equating the two energies yields the following relation between the properties of the system,

$$a = \frac{4m}{q^2}\,, \tag{4.11}$$

for which string breaking is energetically favourable.

Numerical study: Confinement and string breaking

For the numerical study of confinement and string breaking the system will consist of the two states shown in Fig. 4.3. These

[4] Actually, the expectation value of the Hamiltonian lacks a constant offset, but since we will be comparing the energy of two systems the offset would disappear anyways, thus it have been omitted in Eq. (4.9).

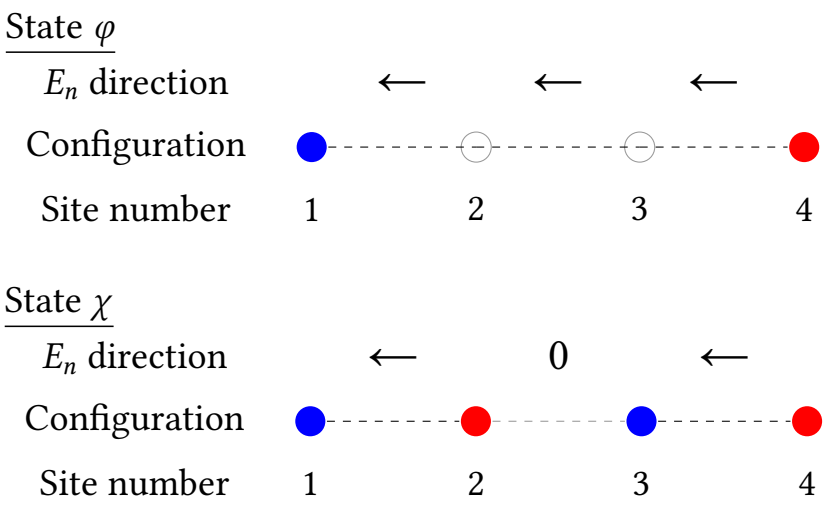

FIGURE 4.3: The two states for which the numerical study of string breaking is performed. The system starts in state φ with the variables set to be favourable for string breaking, thus the system will over time go to state χ. The red and blue circles are quarks and antiquarks respectively, the grey outlined circles are sites with neither a quark nor an antiquark, and the dashed lines is the gauge field (electric field). The direction of the electric field between the quarks and antiquarks is noted above the lattice links, and the grey dashed line reflects the electric field not being present.

are actually the same as for the analytical study, Fig. 4.2, with $l = 3$, thus we work with three links and four matter sites ($N = 4$). Instead of examining the energy of the two systems we will here examine the *fidelity* of the two states with the time evolving initial state, $\varphi(t)$. Starting from the definition of fidelity of two states $|\Phi\rangle$ and $|\Psi\rangle$, the norm square of the inner product of the two states, it can be shown that this equals the expectation value with respect to one of the states of the *density operator*[5] of the other state,

$$|\langle\Phi|\Psi\rangle|^2 = \langle\Phi|\Psi\rangle\langle\Psi|\Phi\rangle = \Big\langle\Phi\Big|\big(\,|\Psi\rangle\langle\Psi|\,\big)\Big|\Phi\Big\rangle\ . \tag{4.12}$$

For the numerical study we will be using the PYTHON module QUTIP[6] and its master equation solver mesolve, which time evolves its input state relative to the constructed Hamiltonian,

[5] The density operator contains all the physically significant information that is obtainable for an ensemble [13].

[6] For this the QUTIP version 4.5.0 is used.

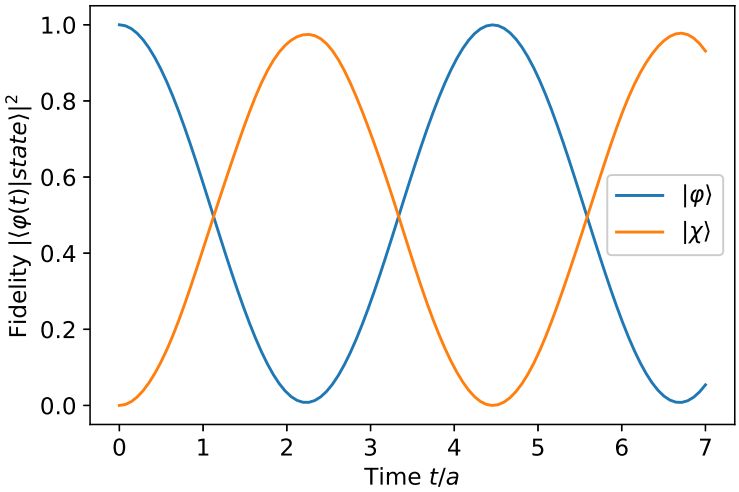

FIGURE 4.4: The fidelity of the time evolved initial state, $|\varphi(t)\rangle$ for the system and the initial state, $|\varphi\rangle$, and the state expected after string breaking, $|\chi\rangle$. Here the string breaking favourable values $m = 2$, $q = 2\sqrt{2}$ and $a = 4m/q^2 = 1$ (found in Eq. (4.11)) is used.

Eq. (4.5), and give as output the expectation value of the density operators of $|\varphi\rangle$ and $|\chi\rangle$ provided the solver with respect to the time evolved initial state. Fig. 4.4 is the resulting graph of the fidelity, for the string breaking favourable values $m = 2$, $q = 2\sqrt{2}$ and $a = 4m/q^2 = 1$ (found in Eq. (4.11)), and the two states

$$\varphi = \left| 1_m\, 0_g\, 1_m\, 0_g\, 0_m\, 0_g\, 0_m \right\rangle \quad \text{and} \quad \chi = \left| 1_m\, 0_g\, 0_m\, 1_g\, 1_m\, 0_g\, 0_m \right\rangle . \tag{4.13}$$

As seen at Fig. 4.4 the system is initiated in state $|\psi\rangle$, i.e. the fidelity $|\langle\varphi(t=0)|\varphi\rangle|^2 = 1$, and then proceeds to be almost fully in state $|\chi\rangle$ at $t = 3.5$, i.e. $|\langle\varphi(t=0)|\chi\rangle|^2 \approx 1$, as was expected of the system. Now we would expect the system to stay in the new state – the energy of the new state should be less than that of the initial according to the theory – but looking at the graph this is not the case. Instead the fidelity looks like a periodic function, thus the system periodically over time changes from state $|\varphi\rangle$ to state $|\chi\rangle$ and back. This periodic recurrence in the time evolution is due to the so-called *quantum revival* effect [32], which can be found in all systems, but is most prominent in small systems as the one used; only the two states pictured in Fig. 4.2 are contained in the calculations, thus as the initial state distributes in the only

other possible state it quickly returns. Extending the number of states should increase the revival time, thus making the string breaking even clearer when graphed.

4.3 Outlook

In this chapter the string breaking distance R_0 was introduced but neither the analytical nor the numerical study reveals its value. This is therefore proposed as a further study of the confinement and string breaking using the (1+1)-dimensional lattice Schwinger model. The string breaking distance can then be compared to other studies using the Schwinger model [16] and that of numerical quantum chromodynamic models [29, 33], which states the string breaking distance to be in the interval $R_0 \in [15, 20]a$ with a still denoting the lattice spacing.

An interesting further development of this chapter would be to examine the system of the numerical study (Fig. 4.3) but with a higher number of lattice sites and with an additional number of possible states, thus hoping to show string breaking with less quantum revival effect. Even though increasing the number of lattice sites might result in a more precise results it has a negative effect on the computability of the system, due to its higher complexity thus the functions become more time consuming to compute. The calculations of confinement use a lattice of N sites, with two spins states, and thus $N-1$ links, with three spin states, resulting in the dimension of the problem being $2^N \cdot 3^{N-1}$, i.e. the dimension of the problem increases exponentially. For the case of the numerical study above a lattice with $N = 4$ lattice sites have been used, thus the dimensions became 432, which was computable on a regular laptop. Considering instead $N = 20$ lattice sites would yield a problem dimension of 1.2×10^{15}, which is numerically infeasible without extensive computational power, for example a supercomputer.

Lastly, one may want to search for signatures of confinement beyond the capacities of classical computers. For this a simulation of the dynamics of two free particles and a meson respectively using a quantum spin chain can be used and run on a few-qubit quantum computer as in Ref. [34], and whereas the free particles have constant velocity away from each other the meson velocity can be found to decrease exponentially as a function of time, hence indicating confinement.

Chapter 5

Conclusion

The aim of this project has been to develop an understanding of the theoretical framework of quantum field theory in both the continuum limit and on a lattice, and to use this to derive the (1+1)-dimensional lattice Schwinger model ourselves, which could then be used as a "toy model" for quantum chromodynamics to show confinement and string breaking.

Starting from Dirac's fermionic Lagrangian density, we have derived the continuum (1+1)-dimensional Schwinger model Hamiltonian, Eq. (2.23), by imposing local U(1) gauge invariance and restricting the developed equation for quantum electrodynamics to (1+1)-dimensions. Then, discretizing space using an infinite lattice with finite lattice spacing and mapping the matter fields to spin-½ operators, using the Jordan-Wigner transformation, and the gauge fields to spin-1 operators, using quantum link models, we obtained the (1+1)-dimensional lattice Schwinger model, Eq. (3.17). For this $\sigma^{\pm}$ and σ^{z} generated from the spin-½ states $|0\rangle = (1, 0)^T$, and S^{+} and S_z are generated from the spin-1 states $|0\rangle = (1, 0, 0)^T$, $|1\rangle = (0, 1, 0)^T$ and $|2\rangle = (0, 0, 1)^T$, where $|0\rangle$ is the ground state and $|1\rangle$ (and $|2\rangle$ for spin-1) is the excited state(s).

Utilising the derived lattice Schwinger model, we have analytically shown the relation between the properties of the Hamiltonian, for which string breaking is energetically favourable, to be $a = 4m/q^2$ for a being the lattice spacing, and m and q being the mass and charge of a quark respectively. Also a numerical study of confinement and string breaking has been performed, in which the expectation value of two density operators, one describing the system before string breaking and one describing it afterwards, with respect to the time evolved initial state is found. It is shown that the system does in fact migrate from the initial

state of one meson with interquark distance $L = la$ to a state of two mesons with each interquark distance $(L - a)/2$, despite the quantum revival effect also pictured on the graph, Fig. 4.4, due to the system only containing few possible states. This change of states illustrates the effect of confinement, namely that colourful particles, such as quarks, cannot exist freely but exists instead only in hadrons.

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