Affine Inflation in Polynomial Affine Gravity in $3+1$ dimensions with torsion

Jose Perdiguero Garate

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Abstract

The Polynomial Affine Gravity its a purely affine model that mediates gravitational interactions solely and exclusive through the affine connection instead of the metric tensor. In this paper we study solutions to the field equations including the presence of the torsion. Then, we study the cosmological consequence of using the Ricci tensor and Poplaswki's torison-metric tensor under the cosmological solutions.

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The Polynomial Affine Gravity model its a purely affine model on which we endowed the manifold only with an affine connection $(\mathcal{M},\Gamma)$. This allow us to define the notion of parallelism by the covariant derivative ∇. Since we only have an affine connection Γ we can only deffine the following chain of geometric objects

$$\Gamma_\mu{}^\sigma{}_\nu \to \nabla_\mu \to \mathcal{R}_{\mu\sigma}{}^\tau{}_\nu \to \mathcal{R}_{\mu\nu} \tag{1}$$

Notice that in the absence of the metric tensor it is not possible to define the $\mathcal{R}$.

In order to built the action of the Polynomial Affine Gravity we use the irreducible fields of the affine connection, by separating the connection into its symmetric and antisymmetric part

$$\hat{\Gamma}_\mu{}^\sigma{}_\nu = \Gamma_\mu{}^\sigma{}_\nu + \mathcal{B}_\mu{}^\sigma{}_\nu + \delta^\sigma_{[\mu}\mathcal{A}_{\nu]} \tag{2}$$

where $\Gamma_{\mu}{}^{\sigma}{}_{\nu}$ correspond to the symmetric part of the connection, $\mathcal{B}_{\mu}{}^{\sigma}{}_{\nu}$ its the traceless part of the torsion tensor and $\mathcal{A}_{\mu}$ its the vectorial part of the torsion tensor. Additionally, we need to define the volume form, which can be written using only the wedge product

$$\mathrm{d}V^{\alpha\beta\gamma\delta} = J(x)\mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\delta} \tag{3}$$

The action must preserv the invariance under diffemorphism, which is why the symmetric part of the connection goes indirectly throught the covariant derivative. The fundamental fields to build the action are $\nabla, \mathcal{A}, \mathcal{B}, \mathrm{d}V$. Then we perform a sort of *dimensional structural analysis technique* studying everysingle possible non-trivial contribution to the action.

THen, the most general action in $3+1$ dimension up to boundary terms is

$$\begin{aligned} S = \int \mathrm{d}V^{\alpha\beta\gamma\delta} \Big[& B_1 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho} \mathcal{B}_{\alpha}{}^{\nu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + B_3 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} \mathcal{A}_{\delta} + B_4 \mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} \mathcal{A}_{\sigma} \\ & + B_5 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{A}_{\sigma} + C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} + C_2 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \nabla_{\beta} \mathcal{R}_{\gamma}{}^{\lambda}{}_{\delta} \\ & + D_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} + D_3 \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \nabla_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\delta} + D_4 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \nabla_{\lambda} \mathcal{A}_{\sigma} + D_5 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\sigma} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\ & + D_6 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\gamma} \nabla_{\lambda} A_{\delta} + D_7 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\lambda} \nabla_{\gamma} A_{\delta} + E_1 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + E_2 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\gamma} \mathcal{A}_{\delta} \\ & + F_1 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\rho} \mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda} \mathcal{B}_{\delta}{}^{\lambda}{}_{\rho} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + F_3 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \mathcal{A}_{\delta} + F_4 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \Big]. \end{aligned}$$

In previous works we have mentioned some of the features of the action: (i) Its rigidity, since contains all possible combinations of the fields and their derivatives; (ii) All the coupling constants are dimensionless, which might be a sign of conformal invariance, and also ensure that the model is power-counting renormalisable; (iii) The field equations are second order differential equations, and the Einstein spaces are a subset of their solu- tions; (iv) The supporting symmetry group is the group of diffeomorphisms, desirable for the background independence of the model; (v) Even though there is no fundamental metric, it is possible to obtain emergent (connection- descendent) metric tensors; (vi) The cosmological constant appears in the solutions as an integration constant, changing the paradigm concerning its interpretation; (vii) The model can be extended to be coupled with a scalar field, and the field equations are equivalent to those of General Relativity interacting with a massless scalar field.

In order to solve the field equations, one need to build an ansatz, since we want to do cosmology, we need to build an ansazts compatible with the symmetries of the cosmological principle, which are rotation and translations. It is possible to build an ansatz for our fundamental geometric objects using the Lie derivative along the Killing vector fields. The most general ansatz for the symmetric part of the connection $\Gamma_{\mu}{}^{\sigma}{}_{\nu}$ is

$$\Gamma_{t}{}^{t}{}_{t} = f(t), \quad \Gamma_{i}{}^{t}{}_{j} = g(t) S_{ij} \tag{4}$$

$$\Gamma_{t}{}^{i}{}_{j} = h(t)\delta^{i}_{j}, \quad \Gamma_{i}{}^{j}{}_{k} = \gamma_{i}{}^{j}{}_{k} \tag{5}$$

Notice that $f(t)$ affine function can vanishes completely by a time reparametrization.

The same procedure applies to build the ansatz for the torsion tensor. The traceless part of the torsion tensor $\mathcal{B}_{\mu}{}^{\sigma}{}_{\nu}$ is completely define by only one time depending function

$$\mathcal{B}_{\theta}{}^{r}{}_{\varphi} = \psi(t) r^2 \sin\theta \sqrt{1-\kappa r^2} \qquad \mathcal{B}_{r}{}^{\theta}{}_{\varphi} = \frac{\psi(t)\sin\theta}{\sqrt{1-\kappa r^2}}$$

$$\mathcal{B}_{r}{}^{\varphi}{}_{\theta} = \frac{\psi(t)}{\sqrt{1-\kappa r^2}\sin\theta}$$

and finally, the vectorial torsion tensor $\mathcal{A}_{\mu}$ is given by

$$\mathcal{A}_{t} = \eta(t) \tag{6}$$

3 Cosmological solutions in the abscence of torsion

The field equation its given by

$$\nabla_{[\alpha} \mathcal{R}_{\beta]\gamma} = 0 \tag{7}$$

from here we distinguish three types of families solutions

$$\mathcal{R}_{\beta\gamma} = 0 \qquad \nabla_{\alpha}\mathcal{R}_{\beta\gamma} = 0 \qquad \nabla_{[\alpha]}\mathcal{R}_{\beta]\gamma} = 0 \tag{8}$$

Here we briefly review all the possible solutions to the field equations under the cosmological ansatz.

3.1 $\mathcal{R}_{\beta\gamma} = 0$

Here we have two differential equations named

$$\dot{h} + h^2 = 0 \qquad \dot{g} + gh + 2\kappa = 0 \tag{9}$$

The two equations can be solved exactly as

$$h(t) = \frac{1}{t - h_0} \qquad g(t) = \frac{t\kappa\,(2h_0 - t) - g_0}{t - h_0} \tag{10}$$

Under the assumption $h(t) = \frac{\dot{a}}{a}$ and $g(t) = \dot{a}a$ the equations are

$$\frac{\ddot{a}}{a} = 0 \qquad \ddot{a}a + 2\left(\kappa + \dot{a}^2\right) = 0 \tag{11}$$

then the solutions are

$$a(t) = t\sqrt{-\kappa} + a_0 \qquad h(t) = \frac{\sqrt{-\kappa}}{t\sqrt{-\kappa} + a_0} \qquad g(t) = \sqrt{-\kappa}\left(t\sqrt{-\kappa} + a_0\right) \tag{12}$$

Notice that under the above assumptions, we have restrictions for the geometry factor κ, meaning that it can only take values as 0 and -1 to have a real geometry, whereas in the pure affine geometry, there is no constraint for κ.

3.2 $\nabla_{\alpha}\mathcal{R}_{\beta\gamma} = 0$

Here we have three differential equations named

$$\ddot{h} + 2h\dot{h} = 0 \tag{13}$$

$$2gh^2 - 2\kappa h - h\dot{g} + 3g\dot{h} = 0 \tag{14}$$

$$2gh^2 + 4\kappa h + h\dot{g} - g\dot{h} - \ddot{g} = 0 \tag{15}$$

The system of differential equations can be solve analytically by the functions

$$h(t) = \sqrt{c_0}\tanh\left(\sqrt{c_0}t\right) \qquad g(t) = \frac{\kappa\sinh 2t\sqrt{c_0}}{2\sqrt{c_0}} \tag{16}$$

Notice that for a flat space-time requirement $\kappa = 0$, then the $g(t)$ affine functions vanishes, while the $h(t)$ remains un changed.

Under the assumption $h(t) = \frac{\dot{a}}{a}$ and $g(t) = \dot{a}a$ the equations are

$$\dddot{a}a - \dot{a}\ddot{a} = 0 \qquad \kappa + \dot{a}^2 - a\ddot{a} = 0 \qquad \frac{4\dot{a}\left(\kappa + \dot{a}^2\right)}{a} - 3\dot{a}\ddot{a} - a\dddot{a} = 0 \tag{17}$$

there are to type of solutions

$$a(t) = e^{-c_1}\tanh\left(e^{-c_1}(t + c_2)\right)\sqrt{\frac{\kappa}{\frac{-1}{\cosh^2(e^{c_1}(t+c_2))}}} \tag{18}$$

Notice that to have a real scale factor, it is necessary to have $\kappa = -1$.

3.3 $\nabla_{[\alpha}\mathcal{R}_{\beta]\gamma} = 0$

Here we have one differential equation named

$$4gh^2 + 2\kappa h + 2g\dot{h} - \ddot{g} = 0 \tag{19}$$

Notice that here, we have one differential equation for the two affine functions, we can not find analytically expressions for $h(t)$ and $g(t)$ functions.

Under the assumptions that $h(t) = \frac{\dot{a}}{a}$ and $g(t) = \dot{a}a$ the equation is

$$\frac{2\dot{a}\left(\kappa + \dot{a}^2\right)}{a} - \dot{a}\ddot{a} - a\dddot{a} = 0 \tag{20}$$

which can only be solved numerically.

4 Cosmological solutions with torsion

5 Conclusions

Some key points about this work

- Polynomial Affine Gravity formulation.
- Briefly review of the vacuum solutions.
- Extensive analysis of field equations with torsion.

6 Appendix A: Dimensional Analysis

In this section we briefly show how to build the action and coupling the scalar field in the absence of the metric tensor, by using a sort of *dimensional analysis* technique.

In order to build the most general action while preserving the invariance under diffeormophism, we perform a *dimensional analysis* technique. First, we define an operator $\mathcal{N}$ to count the number of free index and a second operator $\mathcal{W}$ to define the weight density of the object. Applying both operators to the fundamental fields leads to

$$\mathcal{N}(\mathcal{A}_\mu) = -1 \qquad \mathcal{N}(\mathcal{B}_\mu{}^\lambda{}_\nu) = -1 \qquad \mathcal{N}(\Gamma_\mu{}^\lambda{}_\nu) = -1 \qquad \mathrm{d}V^{\alpha\beta\gamma\delta} = 4 \tag{21}$$

$$\mathcal{W}(\mathcal{A}_\mu) = 0 \qquad \mathcal{W}(\mathcal{B}_\mu{}^\lambda{}_\nu) = 0 \qquad \mathcal{W}(\Gamma_\mu{}^\lambda{}_\nu) = 0 \qquad \mathrm{d}V^{\alpha\beta\gamma\delta} = 1 \tag{22}$$

A generic term will have the following form

$$\mathcal{O} = \mathcal{A}^m\mathcal{B}^n\Gamma^p\mathrm{d}V^q \tag{23}$$

Applying the operators defined above, yield the equations

$$\mathcal{N}(\mathcal{O}) = 4q - m - n - p \qquad \mathcal{W}(\mathcal{O}) = q \tag{24}$$

Notice that we are interested in building scalar densities, meaning that the number of free index must be zero and the weight density must be equal to the unity. Therefore, we have two constraints

$$m + n + p = 4 \qquad q = 1 \tag{25}$$

The terms contributing to the action are shown in the below table

From the above table, one use the symmetries of the tensor to see which terms will have non trivial contribution to the action. For example, the term with four $\mathcal{A}$ does not contribute to the action since its contraction with the volume element is identically zero. Whenever two covariant derivatives are contracted with the volume form they give a curvature tensor, and since the curvature is defined for the symmetric part of the connection, such curvature satisfy the torsion-free Bianchi identities, which relate some of the several possible contractions of the indices. An additional argument that helps to drop contraction of indices is that $\mathcal{B}$ is traceless.

$\mathcal{A}^m$	$\mathcal{B}^n$	Γ^p	Type of configuration	Action term
4	0	0	$\mathcal{AAAA}$	0
3	1	0	$\mathcal{AAAB}$	0
3	0	1	$\mathcal{AAA}\nabla$	0
2	2	0	$\mathcal{AABB}$	F_4
2	1	1	$\mathcal{AAB}\nabla$	D_6, D_7
2	0	2	$\mathcal{AA}\nabla\nabla$	0
1	3	0	$\mathcal{ABBB}$	F_3
1	2	1	$\mathcal{ABB}\nabla$	D_4, D_5
1	1	2	$\mathcal{AB}\nabla\nabla$	B_3, B_4, B_5, E_2
1	0	3	$\mathcal{A}\nabla\nabla\nabla$	0
0	4	0	$\mathcal{BBBB}$	F_1, F_2
0	3	1	$\mathcal{BBB}\nabla$	D_1, D_2, D_3
0	2	2	$\mathcal{BB}\nabla\nabla$	B_1, B_2, E_1
0	1	3	$\mathcal{B}\nabla\nabla\nabla$	C_1, C_2
0	0	4	$\nabla\nabla\nabla\nabla$	0

Table 1: Possible terms contributing to the action of Polynomial Affine gravity

7 Appendix B: Time coordinate reparametrization

Under a coordinate transformation, the connection's coefficients changes as

$$\frac{\partial^2 x^i}{\partial x'^a \partial x'^b} + \Gamma_j{}^i{}_k \frac{\partial x^j}{\partial x'^a} \frac{\partial x^k}{\partial x'^b} = \Gamma'_a{}^c{}_b \frac{\partial x^i}{\partial x'^c} \tag{26}$$

Next, consider a transformation of the form

$$t' = t'(t) \qquad r' = r \qquad \theta' = \theta \qquad \phi' = \phi \tag{27}$$

Then, we can find a *coordinate system* where the connection's coefficients $\Gamma'_0{}^0{}_0 = 0$ vanishes completely

$$\frac{\partial^2 t}{\partial t'^2} + f \left(\frac{\partial t}{\partial t'} \right)^2 = 0 \tag{28}$$

which can be written as a total derivative

$$\frac{1}{X} \partial_{t'} \left(X \partial_{t'} t \right) = 0 \tag{29}$$

for $f = \frac{1}{X} \partial_t X$. From here, one have that

$$t^{'} = \int \mathrm{d}t e^F(t) \tag{30}$$

where $X = e^{\int \mathrm{d}t f(t)} = e^{F(t)}$. With the above transformation, it can be checked with ease that the effect of the time reparametrisation is a scaling of the other time functions entering in the connection $g(t) \to g(t')$ and $h(t) \to h(t')$.

8 Appendix C: Complete set of field equations